# Dr Oliver Mathematics
# Further Mathematics
# Method of Differences
# Past Examination Questions

This booklet consists of 23 questions across a variety of examination topics.
The total number of marks available is 185.

1. (a) Express $\dfrac{1}{r(r+2)}$ in partial fractions. (2)

   (b) Hence prove, by the method of differences, that (5)

$$\sum_{r=1}^{n} \frac{4}{r(r+2)} = \frac{n(3n+5)}{(n+1)(n+2)}.$$

   (c) Find the value of $\displaystyle\sum_{r=50}^{100} \frac{4}{r(r+2)}$, to 4 decimal places. (3)

2. (a) By expressing (2)

$$\frac{2}{4r^2-1}$$

   in partial fractions, or otherwise, prove that

$$\sum_{r=1}^{n} \frac{2}{4r^2-1} = 1 - \frac{1}{2n+1}.$$

   (b) Hence find the exact value of $\displaystyle\sum_{r=1}^{20} \frac{2}{4r^2-1}$. (3)

3. Given that for all real values of $r$,

$$(2r+1)^3 - (2r-1)^3 \equiv Ar^2 + B,$$

   where $A$ and $B$ are constants,

   (a) find the value of $A$ and find the value of $B$. (2)

   (b) Hence, or otherwise, prove that (5)

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

(c) Calculate $\sum_{r=1}^{40}(3r-1)^2$. (3)

4. (a) Show that (3)

$$\frac{r^3-r+1}{r(r+1)} \equiv r-1+\frac{1}{r}-\frac{1}{r+1}.$$

(b) Find (6)

$$\sum_{r=1}^{n}\frac{r^3-r+1}{r(r+1)},$$

expressing your answer as a single fraction in its simplest form.

5. (a) Show that (2)

$$(r+1)^3-(r-1)^3 \equiv 6r^2+2.$$

(b) Hence show that (5)

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

(c) Show that (4)

$$\sum_{r=n}^{2n} r^2 = \tfrac{1}{6}n(n+1)(an+b),$$

where $a$ and $b$ are constants to be found

6. (a) Express (4)

$$\frac{5r+4}{r(r+1)(r+2)}$$

in partial fractions.

(b) Hence, or otherwise, show that (5)

$$\sum_{r=1}^{n}\frac{5r+4}{r(r+1)(r+2)} = \frac{7n^2+11n}{2(n+1)(n+2)}.$$

7. (a) Express (2)

$$\frac{2}{(r+1)(r+3)}$$

in partial fractions.

(b) Hence prove, by the method of differences, that (6)

$$\sum_{r=1}^{n}\frac{2}{(r+1)(r+3)} = \frac{n(an+b)}{6(n+2)(n+3)},$$

where $a$ and $b$ are constants to be found.

(c) Find the value of (3)

$$\sum_{r=21}^{30} \frac{2}{(r+1)(r+3)},$$

to 5 decimal places.

8. (a) Show, using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, that (5)

$$\sum_{r=1}^{n} (6r^2 + 4r - 1) = n(n+2)(2n+1).$$

(b) Hence, or otherwise, find the value of (2)

$$\sum_{r=11}^{20} (6r^2 + 4r - 1).$$

9. (a) Show that (4)

$$\sum_{r=1}^{n} (r^2 - r - 1) = \tfrac{1}{3}n(n^2 - 4).$$

(b) Hence, or otherwise, find the value of $\sum_{r=10}^{20} (r^2 - r - 1)$. (2)

10. (a) Show that (5)

$$\sum_{r=1}^{n} r(r+2)(r+4) = \frac{1}{4}n(n+1)(n+4)(n+5).$$

(b) Hence evaluate (2)

$$\sum_{r=21}^{30} r(r+2)(r+4).$$

11. (a) Using the formulae for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$, and $\sum_{r=1}^{n} r^3$, show that (7)

$$\sum_{r=1}^{n} r(r+1)(r+3) = \frac{1}{12}n(n+1)(n+2)(3n+k),$$

where $k$ is a constant to be found.

(b) Hence evaluate $\sum_{r=21}^{40} r(r+1)(r+3)$. (2)

12. (a) Express (2)

$$\frac{1}{(2r+1)(2r+5)}$$

in partial fractions.

(b) Hence show, using the method of differences, that (6)

$$\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+5)} = \frac{n(8n+17)}{15(2n+3)(2n+5)}.$$

13. (a) Prove by induction that, for any positive integer $n$, (5)

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$

(b) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$, show that (5)

$$\sum_{r=1}^{n}(r^3+3r+2) = \frac{1}{4}n(n+2)(n^2+7).$$

(c) Hence evaluate (2)

$$\sum_{r=15}^{25}(r^3+3r+2).$$

14. (a) Express (2)

$$\frac{3}{(3r-1)(3r+2)}$$

in partial fractions.

(b) Using the result in part (a) and the method of differences, show that (3)

$$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{3n}{2(3n+2)}.$$

(c) Evaluate (2)

$$\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)},$$

giving your answer to 3 significant figures.

15. Given that

$$(2r+1)^3 = Ar^3 + Br^2 + Cr + 1,$$

(a) find the values of the constants $A$, $B$, and $C$. (2)

(b) Show that (2)

$$(2r+1)^3 - (2r-1)^3 = 24r^2 + 2.$$

(c) Using the result in part (b) and the method of differences, show that (5)

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

16. (a) Express (2)

$$\frac{1}{r(r+2)}$$

in partial fractions.

(b) Hence prove, by the method of differences, that (6)

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(an+b)}{4(n+1)(n+2)},$$

where $a$ and $b$ are constants to be found.

(c) Hence show that (3)

$$\sum_{r=n+1}^{2n} \frac{1}{r(r+2)} = \frac{n(4n+5)}{4(n+1)(n+2)(2n+1)}.$$

17. (a) Express (2)

$$\frac{2}{(2r+1)(2r+3)}$$

in partial fractions.

(b) Using your answer to part (a), find, in terms of $n$, (3)

$$\sum_{r=1}^{n} \frac{3}{(2r+1)(2r+3)}.$$

Give your answer as a single fraction in its simplest form.

18. (a) Express (2)

$$\frac{2}{(r+1)(r+3)}$$

in partial fractions.

(b) Hence show that (4)

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}.$$

(c) Evaluate (2)

$$\sum_{r=10}^{100} \frac{2}{(r+1)(r+3)},$$

giving your answer to 3 significant figures.

19. (a) Express (2)

$$\frac{2}{4r^2-1}$$

in partial fractions.

(b) Hence use the method of difference to show that (3)

$$\sum_{r=1}^{n} \frac{2}{4r^2-1} = \frac{n}{2n+1}.$$

20. (a) Express (1)

$$\frac{2}{(r+2)(r+4)}$$

in partial fractions.

(b) Hence show that (4)

$$\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \frac{n(7n+25)}{12(n+3)(n+4)}.$$

21. (a) Show that (3)

$$r^2(r+1)^2 - (r-1)^2r^2 \equiv 4r^3.$$

Given that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$,

(b) use the identity in part (a) and the method of differences to show that (4)

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2.$$

22. (a) Show that, for $r > 0$, (2)

$$r - 3 + \frac{1}{r+1} - \frac{1}{r+2} = \frac{r^3 - 7r - 5}{(r+1)(r+2)}.$$

(b) Hence prove, using the method of differences, that (5)

$$\sum_{r=1}^{n} \frac{r^3 - 7r - 5}{(r+1)(r+2)} = \frac{n(n^2 + an + b)}{2(n+2)},$$

where $a$ and $b$ are constants to be found.

23. (a) Show that, for $r > 0$, (1)

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} \equiv \frac{2r+1}{r^2(r+1)^2}.$$

(b) Hence prove that, for $n \in \mathbb{N}$, (3)

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{n(n+2)}{(n+1)^2}.$$

(c) Show that, $n \in \mathbb{N}$, $n > 1$, (3)

$$\sum_{r=n}^{3n} \frac{6r+3}{r^2(r+1)^2} = \frac{an^2 + bn + c}{n^2(3n+1)^2},$$

where $a$, $b$, and $c$ are constants to be found.