Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2006 June Paper 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. A curve has the equation (4)

$$y = (x-1)(2x-3)^8.$$

Find the gradient of the curve at the point where $x = 2$.

Solution

Well,

$$u = x - 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = (2x-3)^8 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 16(2x-3)^7$$

so

$$y = (x-1)(2x-3)^8 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (x-1)[16(2x-3)^7] + (1)(2x-3)^8$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 16(x-1)(2x-3)^7 + (2x-3)^8.$$

Finally,

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 16(2-1)(4-3)^7 + (4-3)^8$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 16 + 1$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 17}}.$$

2. The line (6)

$$y + 4x = 23$$

intersects the curve

$$xy + x = 20$$

at two points, A and B.

Find the equation of the perpendicular bisector of the line AB.

Solution

Now,

$$y + 4x = 23 \Rightarrow y = 23 - 4x$$

and

$$\begin{aligned} xy + x = 20 &\Rightarrow x(23 - 4x) + x = 20 \\ &\Rightarrow 23x - 4x^2 + x = 20 \\ &\Rightarrow 4x^2 - 24x + 20 = 0 \\ &\Rightarrow 4(x^2 - 6x + 5) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -6 \\ \text{multiply to:} & +5 \end{array}\right\} -5, -1$$

$$\begin{aligned} &\Rightarrow 4(x - 5)(x - 1) = 0 \\ &\Rightarrow x = 5 \text{ or } x = 1 \\ &\Rightarrow y = 3 \text{ or } y = 19; \end{aligned}$$

so, say, $A(1, 19)$ and $B(5, 3)$. Next,

$$\begin{aligned} m_{AB} &= \frac{19 - 3}{1 - 5} \\ &= \frac{16}{-4} \\ &= -4 \end{aligned}$$

and so

$$m_{\text{normal}} = \tfrac{1}{4}.$$

Finally, the midpoint of AB is $(3, 11)$ and the equation of the perpendicular bisector of the line AB is

$$\begin{aligned} y - 11 = \tfrac{1}{4}(x - 3) &\Rightarrow y - 11 = \tfrac{1}{4}x - \tfrac{3}{4} \\ &\Rightarrow \underline{\underline{y = \tfrac{1}{4}x + \tfrac{41}{4}}}. \end{aligned}$$

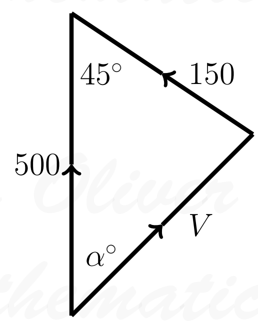
3. A plane flies due north from A to B, a distance of 1 000 km, in a time of 2 hours.

During this time a steady wind, with a speed of 150 km h^{-1}, is blowing from the south-east.

Find

(a) the speed of the plane in still air, (4)

Solution
Let the speed of the aircraft be V km h^{-1}.



Cosine rule:

$$\begin{aligned} & V^2 = 500^2 + 150^2 - 2 \times 500 \times 150 \times \cos 45^\circ \\ \Rightarrow\ & V = 407.963\,212\,6 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{V = 408 \text{ (3 sf)}}}. \end{aligned}$$

(b) the direction in which the plane must be headed. (2)

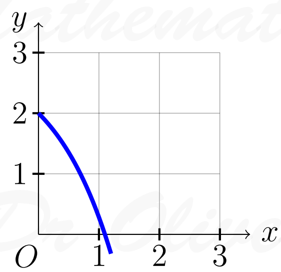
Solution
Sine rule:

$$\frac{\sin \alpha^\circ}{150} = \frac{\sin 45^\circ}{407.963\ldots} \Rightarrow \sin \alpha^\circ = \frac{150 \sin 45^\circ}{407.963\ldots}$$
$$\Rightarrow \alpha = 15.069\,419\,55 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\alpha = 15.1 \text{ (3 sf)}}}.$$

4. The diagram shows part of the curve $y = \text{f}(x)$, where

$$\text{f}(x) = p - \text{e}^x$$

and p is a constant.



The curve crosses the y-axis at $(0, 2)$.

(a) Find the value of p. (2)

Solution

$$x = 0, y = 2 \Rightarrow 2 = p - e^0$$
$$\Rightarrow 2 = p - 1$$
$$\Rightarrow \underline{\underline{p = 3}}.$$

(b) Find the coordinates of the point where the curve crosses the x-axis. (2)

Solution

$$f(x) = 0 \Rightarrow 3 - e^x = 0$$
$$\Rightarrow e^x = 3$$
$$\Rightarrow \underline{\underline{x = \ln 3}}.$$

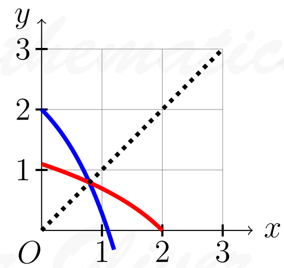
(c) Copy the diagram above and on it sketch the graph of $y = f^{-1}(x)$. (2)

Solution

Well,

$$y = 3 - e^x \Rightarrow e^x = 3 - y$$
$$\Rightarrow x = \ln(3 - y)$$

and



5. The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} -2 & -1 \\ 6 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 4 & 3 \end{pmatrix}.$$

Find matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

(a) $\mathbf{P} = \mathbf{B}^2 - 2\mathbf{A}$, (3)

Solution

$$\begin{aligned} \mathbf{P} &= \mathbf{B}^2 - 2\mathbf{A} \\ &= \begin{pmatrix} 0 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 4 & 3 \end{pmatrix} - 2\begin{pmatrix} -2 & -1 \\ 6 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -4 & -3 \\ 12 & 5 \end{pmatrix} - \begin{pmatrix} -4 & -2 \\ 12 & 4 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}}}. \end{aligned}$$

(b) $\mathbf{Q} = \mathbf{B}(\mathbf{A}^{-1})$. (4)

Solution

Well,

$$\det \mathbf{A} = -4 - (-6) = 2$$

and

$$\mathbf{A}^{-1} = \tfrac{1}{2}\begin{pmatrix} 2 & 1 \\ -6 & -2 \end{pmatrix}.$$

Finally,

$$\begin{aligned}\mathbf{Q} &= \mathbf{B}(\mathbf{A}^{-1}) \\ &= \begin{pmatrix} 0 & -1 \\ 4 & 3 \end{pmatrix} \cdot \tfrac{1}{2} \begin{pmatrix} 2 & 1 \\ -6 & -2 \end{pmatrix} \\ &= \tfrac{1}{2} \begin{pmatrix} 6 & 2 \\ -10 & -2 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 3 & 1 \\ -5 & -1 \end{pmatrix}}}.\end{aligned}$$

6. The cubic polynomial $\mathrm{f}(x)$ is such that the coefficient of x^3 is 1 and the roots of $\mathrm{f}(x) = 0$ are -2, $1 + \sqrt{3}$, and $1 - \sqrt{3}$.

   (a) Express $\mathrm{f}(x)$ as a cubic polynomial in x with integer coefficients. (3)

Solution

We want

$$\begin{aligned}\mathrm{f}(x) &= (x+2)[x-(1+\sqrt{3})][x-(1-\sqrt{3})] \\ &= (x+2)(x-1-\sqrt{3})(x-1+\sqrt{3})\end{aligned}$$

$\times$	x	-1	$-\sqrt{3}$
x	x^2	$-x$	$-\sqrt{3}x$
-1	$-x$	$+1$	$+\sqrt{3}$
$+\sqrt{3}$	$+\sqrt{3}x$	$-\sqrt{3}$	-3

$$= (x+2)(x^2 - 2x - 2)$$

$\times$	x^2	$-2x$	-2
x	x^3	$-2x^2$	$-2x$
$+2$	$+2x^2$	$-4x$	-4

$$= \underline{\underline{x^3 - 6x - 4}}.$$

(b) Find the remainder when $f(x)$ is divided by $(x - 3)$. (2)

Solution
We use synthetic division:

$$\begin{array}{c|cccc} 3 & 1 & 0 & -6 & -4 \\ & \downarrow & 3 & 9 & 9 \\ \hline & 1 & 3 & 3 & 5 \end{array}$$

and the remainder is $\underline{\underline{5}}$.

(c) Solve the equation $f(-x) = 0$. (2)

Solution

$$\begin{aligned} f(-x) = 0 &\Rightarrow -x = -2,\ -x = 1 + \sqrt{3},\ \text{or}\ -x = 1 - \sqrt{3} \\ &\Rightarrow \underline{\underline{x = 2,\ x = -1 - \sqrt{3},\ \text{or}\ x = -1 + \sqrt{3}}}. \end{aligned}$$

7. A particle moves in a straight line, so that, t s after leaving a fixed point O, its velocity, v ms^{-1}, is given by (7)

$$v = pt^2 + qt + 4,$$

where p and q are constants.

When $t = 1$ the acceleration of the particle is 8 ms^{-2}.

When t = 2 the displacement of the particle from O is 22 m.

Find the value of p and of q.

Solution
Well,

$$v = pt^2 + qt + 4 \Rightarrow a = 2pt + q$$

and

$$\begin{aligned} t = 1, a = 8 &\Rightarrow 8 = 2p(1) + q \\ &\Rightarrow 2p + q = 8 \quad (1). \end{aligned}$$

Next,

$$v = pt^2 + qt + 4 \Rightarrow s = \tfrac{1}{3}pt^3 + \tfrac{1}{2}qt^2 + 4t + c,$$

for some constant c. Now,

$$t = 0, s = 0 \Rightarrow c = 0$$

and we are left with

$$s = \tfrac{1}{3}pt^3 + \tfrac{1}{2}qt^2 + 4t.$$

So,

$$\begin{aligned} t = 2, s = 22 \Rightarrow 22 &= \tfrac{1}{3}p(2^3) + \tfrac{1}{2}q(2^2) + 4(2) \\ \Rightarrow 22 &= \tfrac{8}{3}p + 2q + 8 \\ \Rightarrow \tfrac{8}{3}p &+ 2q = 14 \\ \Rightarrow \tfrac{4}{3}p &+ q = 7 \quad (2). \end{aligned}$$

Do (1) − (2):

$$\begin{aligned} \tfrac{2}{3}p = 1 &\Rightarrow \underline{\underline{p = \tfrac{3}{2}}} \\ &\Rightarrow 2(\tfrac{3}{2}) + q = 8 \\ &\Rightarrow 3 + q = 8 \\ &\Rightarrow \underline{\underline{q = 5}}. \end{aligned}$$

8. (a) Given that (5)

$$y = \frac{1 + \sin x}{\cos x},$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 - \sin x}.$$

Solution

Well,

$$\begin{aligned} u = 1 + \sin x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \\ v = \cos x &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = -\sin x \end{aligned}$$

and

$$y = \frac{1 + \sin x}{\cos x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\cos x)(\cos x) - (1 + \sin x)(-\sin x)}{(\cos x)^2}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + \sin x}{\cos^2 x}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + \sin x}{1 - \sin^2 x}$$

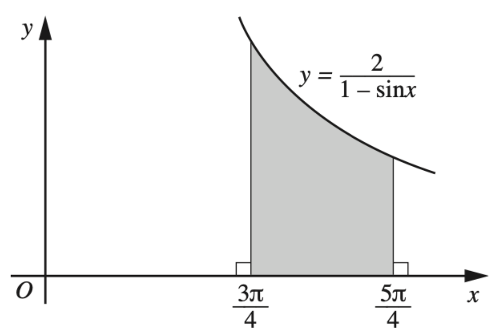
difference of two squares:

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + \sin x}{(1 - \sin x)(1 + \sin x)}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 - \sin x}}},$$

as required.

The diagram shows part of the curve

$$y = \frac{2}{1 - \sin x}.$$



(b) Using the result given in part (i), find the area of the shaded region bounded by the curve, the x-axis and the lines $x = \frac{3}{4}\pi$ and $x = \frac{5}{4}\pi$. (3)

Solution

$$\begin{aligned}\text{Area} &= \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} \left(\frac{2}{1-\sin x}\right) \mathrm{d}x \\ &= 2\int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} \left(\frac{1}{1-\sin x}\right) \mathrm{d}x \\ &= 2\left[\frac{1+\sin x}{\cos x}\right]_{x=\frac{3}{4}\pi}^{\frac{5}{4}\pi} \\ &= 2\left[(1-\sqrt{2})-(-1-\sqrt{2})\right] \\ &= \underline{\underline{4}}.\end{aligned}$$

9. (a) Given that (5)

$$u = \log_4 x,$$

find, in simplest form in terms of u,

(i) x,

Solution

$$u = \log_4 x \Rightarrow \underline{\underline{x = 4^u}}.$$

(ii) $\log_4 \left(\frac{16}{x}\right)$,

Solution

$$\begin{aligned}\log_4 \left(\frac{16}{x}\right) &= \log_4 16 - \log_4 x \\ &= \log_4 4^2 - u \\ &= \underline{\underline{2-u}}.\end{aligned}$$

(iii) $\log_x 8$.

Solution

$$
\begin{aligned}
\log_x 8 &= \frac{\log_4 8}{\log_4 x} \\
&= \frac{\log_4 2^3}{u} \\
&= \frac{\log_4 (2^2)^{\frac{3}{2}}}{u} \\
&= \frac{\log_4 4^{\frac{3}{2}}}{u} \\
&= \underline{\underline{\frac{3}{2u}}}.
\end{aligned}
$$

(b) Solve the equation (4)

$$(\log_3 y)^2 + \log_3(y^2) = 8.$$

Solution

Now,

$$(\log_3 y)^2 + \log_3(y^2) = 8 \Rightarrow (\log_3 y)^2 + 2(\log_3 y) - 8 = 0$$

let $a = \log_3 y$:

$$\Rightarrow a^2 + 2a - 8 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & +2 \\ \text{multiply to:} & -8 \end{array}\right\} -2,\ +4$$

$$
\begin{aligned}
&\Rightarrow (a+4)(a-2) = 0 \\
&\Rightarrow a = -4 \text{ or } a = 2 \\
&\Rightarrow \log_3 y = -4 \text{ or } \log_3 y = 2 \\
&\Rightarrow y = 3^{-4} \text{ or } y = 3^2 \\
&\Rightarrow \underline{\underline{y = \tfrac{1}{81} \text{ or } y = 9}}.
\end{aligned}
$$

10. The function f is defined, for $0^\circ \leqslant x \leqslant 180^\circ$, by

$$\mathrm{f}(x) = 3\cos 4x - 1.$$

(a) Solve the equation $\mathrm{f}(x) = 0$. (3)

Solution

$$\begin{aligned} & \mathrm{f}(x) = 0 \\ \Rightarrow\ & 3\cos 4x - 1 = 0 \\ \Rightarrow\ & 3\cos 4x = 1 \\ \Rightarrow\ & \cos 4x = \tfrac{1}{3} \end{aligned}$$

$0^\circ \leqslant x \leqslant 180^\circ \Rightarrow 0^\circ \leqslant 4x \leqslant 720^\circ$:

$$\begin{aligned} \Rightarrow\ & 4x = 70.528\,779\,37,\ 289.471\,220\,6,\ 430.528\,779\,37,\ 649.471\,220\,6 \text{ (FCD)} \\ \Rightarrow\ & x = 17.632\,194\,84,\ 72.367\,805\,16,\ 107.632\,194\,8,\ 162.367\,805\,2 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{x = 17.6,\ 72.4,\ 108,\ 162 \text{ (3 sf)}}}. \end{aligned}$$

(b) State the amplitude of f. (1)

Solution

$\underline{\underline{3}}$.

(c) State the period of f. (1)

Solution

$$\frac{360}{4} = \underline{\underline{90^\circ}}.$$

(d) State the maximum and minimum values of f. (2)

Solution

The maximum is

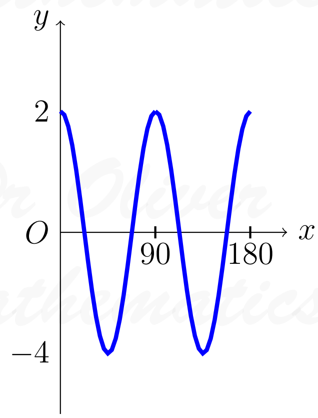
$$3 - 1 = \underline{\underline{2}}$$

and the minimum is

$$-3 - 1 = \underline{\underline{-4}}.$$

(e) Sketch the graph of $y = \mathrm{f}(x)$. (3)

Solution



EITHER

11. The table below shows values of the variables x and y which are related by the equation

$$y = \frac{a}{x+b},$$

where a and b are constants.

x	0.1	0.4	1.0	2.0	3.0
y	8.0	6.0	4.0	2.6	1.9

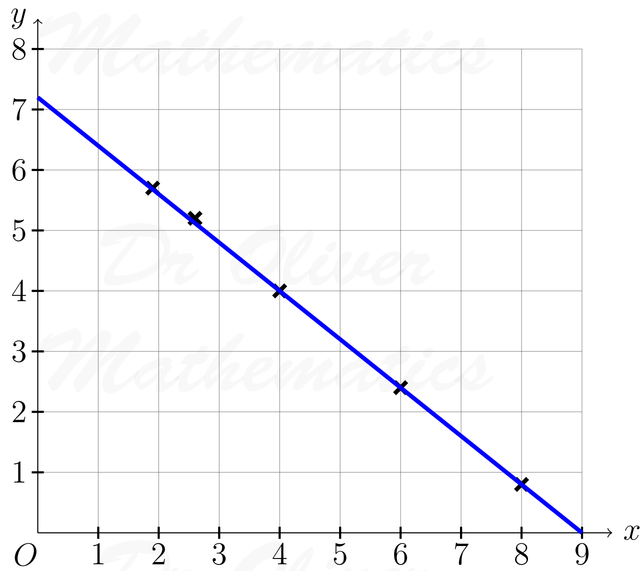
(a) Using graph paper, plot y against xy and draw a straight line graph. (3)

Solution

Well,

y	8.0	6.0	4.0	2.6	1.9
xy	0.8	2.4	4	5.2	5.7

and we plot the graph:



(b) Use your graph to estimate the value of a and of b. (4)

Solution

Now,

$$m = \frac{7.2 - 0}{0 - 9} = 0.8$$

and the equation of the line is

$$\begin{aligned} xy - 7.2 = 0.8(y - 0) &\Rightarrow xy = 0.8y + 7.2 \\ &\Rightarrow xy - 0.8y = 7.2 \\ &\Rightarrow y(x - 0.8) = 7.2 \\ &\Rightarrow \underline{\underline{y = \frac{7.2}{x - 0.8}}}; \end{aligned}$$

hence, $\underline{\underline{a = 7.2}}$ and $\underline{\underline{b = -0.8}}$.

An alternative method for obtaining a straight line graph for the equation

$$y = \frac{a}{x + b},$$

is to plot x on the vertical axis and $\frac{1}{y}$ on the horizontal axis.

(c) Without drawing a second graph, use your values of a and b to estimate the gradient and the intercept on the vertical axis of the graph of x plotted against $\frac{1}{y}$. (3)

Solution

Now,

$$y = \frac{7.2}{x - 0.8} \Rightarrow \frac{1}{y} = \frac{x - 0.8}{7.2}$$
$$\Rightarrow \frac{1}{y} = \tfrac{5}{36}x - \tfrac{1}{9};$$

so,

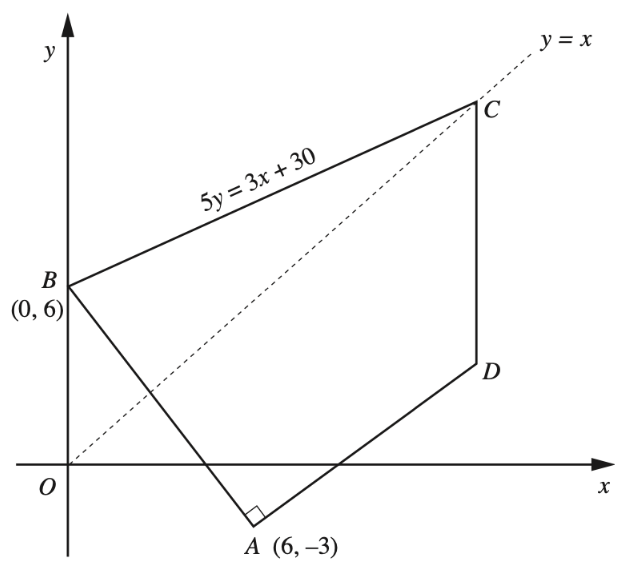
$$\text{gradient} = \underline{\underline{\tfrac{5}{36}}}$$

and

$$\text{intecept} = \underline{\underline{-\tfrac{1}{9}}}.$$

OR

12. The diagram, which is not drawn to scale, shows a quadrilateral $ABCD$ in which A is $(6, -3)$, B is $(0, 6)$, and angle BAD is $90°$.



The equation of the line BC is

$$5y = 3x + 30$$

and C lies on the line $y = x$.

The line CD is parallel to the y-axis.

(a) Find the coordinates of C and of D. (6)

Solution

Well,

$$5y = 3x + 30 \Rightarrow y = \tfrac{3}{5}x + 6$$

and $C(x, x)$ so

$$\begin{aligned} x = \tfrac{3}{5}x + 6 &\Rightarrow \tfrac{2}{5}x = 6 \\ &\Rightarrow x = 15; \end{aligned}$$

so $\underline{\underline{C(15, 15)}}$.

Now,

$$\begin{aligned} m_{AB} &= \frac{6 - (-3)}{0 - 6} \\ &= -\tfrac{3}{2} \end{aligned}$$

so

$$m_{AD} = -\frac{1}{-\frac{3}{2}} = \tfrac{2}{3}.$$

Next, the equation of AD is

$$\begin{aligned} y + 3 = \tfrac{2}{3}(x - 6) &\Rightarrow y + 3 = \tfrac{2}{3}x - 4 \\ &\Rightarrow y = \tfrac{2}{3}x - 7. \end{aligned}$$

Finally,

$$\begin{aligned} x = 15 &\Rightarrow y = \tfrac{2}{3}(15) - 7 \\ &\Rightarrow y = 10 - 7 \\ &\Rightarrow y = 3 \end{aligned}$$

so $\underline{\underline{D(15, 3)}}$.

(b) Show that triangle BAD is isosceles and find its area. (4)

Solution

Well,

$$\begin{aligned} AB &= \sqrt{(6-0)^2 + (-3-6)^2} \\ &= \sqrt{36+81} \\ &= \sqrt{117} \\ &= 3\sqrt{13} \end{aligned}$$

and

$$\begin{aligned} AD &= \sqrt{(6-15)^2 + (-3-3)^2} \\ &= \sqrt{81+36} \\ &= 3\sqrt{13}; \end{aligned}$$

so, the triangle BAD is isosceles and

$$\begin{aligned} \text{area} &= \tfrac{1}{2} \times AB \times AD \\ &= \tfrac{1}{2} \times 3\sqrt{13} \times 3\sqrt{13} \\ &= \underline{\underline{58\tfrac{1}{2}}}. \end{aligned}$$