Dr Oliver Mathematics
Advanced Level: Pure Mathematics 2
November 2021: Calculator
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. In an arithmetic series

   - the first term is 16 and
   - the 21st term is 24.

   (a) Find the common difference of the series. (2)

   Solution
   Let a be the first term (which is 16) and let d be the common difference. Then

   $$16 + 20d = 24 \Rightarrow 20d = 8$$
   $$\Rightarrow \underline{\underline{d = \tfrac{2}{5}}}.$$

   (b) Hence find the sum of the first 500 terms of the series. (2)

   Solution
   $S = \frac{1}{2}n[2a + (n-1)d]$:

   $$S_{500} = \tfrac{1}{2} \times 500 \times [2 \times 16 + 499 \times \tfrac{2}{5}]$$
   $$= \underline{\underline{57\,900}}.$$

2. The functions f and g are defined by

   $$\mathrm{f}(x) = 7 - 2x^2,\ x \in \mathbb{R},$$
   $$\mathrm{g}(x) = \frac{3x}{5x-1},\ x \in \mathbb{R},\ x \neq \tfrac{1}{5}.$$

   (a) State the range of f. (1)

Solution

$\underline{\underline{f(x) \leqslant 7}}$.

(b) Find g f(1.8). (2)

Solution

$$\begin{aligned} \mathrm{g}\,\mathrm{f}(1.8) &= \mathrm{g}(\mathrm{f}(1.8)) \\ &= \mathrm{g}(7 - 2(1.8)^2) \\ &= \mathrm{g}(0.52) \\ &= \underline{\underline{0.975}}. \end{aligned}$$

(c) Find $\mathrm{g}^{-1}(x)$. (2)

Solution

$$\begin{aligned} y = \frac{3x}{5x-1} &\Rightarrow y(5x-1) = 3x \\ &\Rightarrow 5xy - y = 3x \\ &\Rightarrow 5xy - 3x = y \\ &\Rightarrow x(5y-3) = y \\ &\Rightarrow x = \frac{y}{5y-3}; \end{aligned}$$

hence,

$$\underline{\underline{\mathrm{g}^{-1}(x) = \frac{x}{5x+3},\ x \neq -\tfrac{3}{5}}}.$$

3. Using the laws of logarithms, solve the equation (3)

$$\log_3(12y+5) - \log_3(1-3y) = 2.$$

Solution

$$\begin{aligned}
\log_3(12y+5) - \log_3(1-3y) = 2 &\Rightarrow \log_3\left(\frac{12y+5}{1-3y}\right) = 2 \\
&\Rightarrow \frac{12y+5}{1-3y} = 3^2 \\
&\Rightarrow \frac{12y+5}{1-3y} = 9 \\
&\Rightarrow 12y+5 = 9(1-3y) \\
&\Rightarrow 12y+5 = 9-27y \\
&\Rightarrow 39y = 4 \\
&\Rightarrow \underline{\underline{y = \tfrac{4}{39}}}.
\end{aligned}$$

4. Given that θ is small and measured in radians, use the small angle approximations to show that (3)

$$4\sin\tfrac{1}{2}\theta + 3\cos^2\theta \approx a + b\theta + c\,\theta^2,$$

where a, b, and c are integers to be found.

Solution

Well,

$$\begin{aligned}
\cos 2\theta = 2\cos^2\theta - 1 &\Rightarrow cos 2\theta + 1 = 2\cos^2\theta \\
&\Rightarrow \cos^2\theta = \tfrac{1}{2}(\cos 2\theta + 1)
\end{aligned}$$

and

$$\begin{aligned}
4\sin\tfrac{1}{2}\theta + 3\cos^2\theta &= 4\sin\tfrac{1}{2}\theta + \tfrac{3}{2}(\cos 2\theta + 1) \\
&= 4(\tfrac{1}{2}\theta + \ldots) + \tfrac{3}{2}[1 - \tfrac{1}{2}(2\theta)^2 + \ldots + 1] \\
&= 4(\tfrac{1}{2}\theta + \ldots) + \tfrac{3}{2}[2 - 2\theta^2 + \ldots] \\
&= 2\theta + 3 - 3\theta^2 + \ldots \\
&= \underline{\underline{3 + 2\theta - 3\theta^2 + \ldots}}.
\end{aligned}$$

5. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11,\ x \in \mathbb{R}.$$

(a) Find (3)

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$,

Solution

$$\underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 20x^3 - 72x^2 + 84x - 32.}}$$

(ii) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Solution

$$\underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 60x^2 - 144x + 84.}}$$

(b) (i) Verify that C has a stationary point at $x = 1$. (4)

Solution

$$x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 20 - 72 + 84 - 32$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 0;$$

so, C has a stationary point at $\underline{\underline{x = 1}}$.

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

Solution

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 60x^2 - 144x + 84 \Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 120x - 144$$

and

$$x = 1 \Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 120 - 144 = -24 \neq 0;$$

hence, $x = 1$ is a <u>point of inflection</u>.

6. The shape $OABCDEFO$ shown in Figure 1 is a design for a logo.

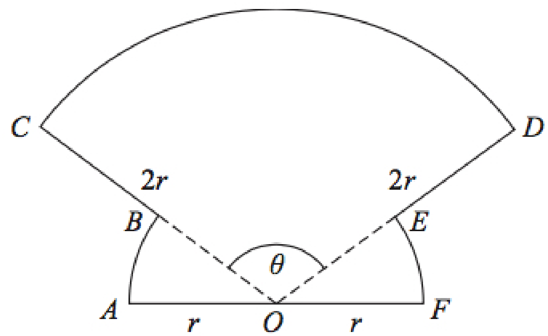


Figure 1: a design for a logo

In the design

- OAB is a sector of a circle centre O and radius r,
- sector OFE is congruent to sector OAB,
- ODC is a sector of a circle centre O and radius $2r$, and
- AOF is a straight line.

Given that the size of angle COD is θ radians,

(a) write down, in terms of θ, the size of angle AOB. (1)

> **Solution**
> $\angle\, AOB = \frac{1}{2}(\pi - \theta)$.

(b) Show that the area of the logo is (2)

$$\tfrac{1}{2}r^2(3\theta + \pi).$$

> **Solution**
> Moving from left to right:
>
> $$\begin{aligned} \text{Area} &= \tfrac{1}{2}(r^2)[\tfrac{1}{2}(\pi - \theta)] + \tfrac{1}{2}(2r)^2(\theta) + \tfrac{1}{2}(r^2)[\tfrac{1}{2}(\pi - \theta)] \\ &= \tfrac{1}{4}r^2(\pi - \theta) + 2r^2\theta + \tfrac{1}{4}r^2(\pi - \theta) \\ &= \tfrac{1}{2}r^2(\pi - \theta) + 2r^2\theta \\ &= \tfrac{1}{2}r^2[(\pi - \theta) + 4\theta] \\ &= \tfrac{1}{2}r^2(3\theta + \pi), \end{aligned}$$
>
> as required.

(c) Find the perimeter of the logo, giving your answer in simplest form in terms of r, θ, and π. (2)

Solution

Moving from O:

$$\begin{aligned}\text{Perimeter} &= 2(OA + AB + BC) + CD \\ &= 2[r + r(\tfrac{1}{2}(\pi - \theta)) + r] + 2r\theta \\ &= 2[r + r(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta) + r] + 2r\theta \\ &= 2[2r + \tfrac{1}{2}r\pi - \tfrac{1}{2}r\theta] + 2r\theta \\ &= 4r + r\pi - r\theta + 2r\theta \\ &= 4r + r\pi + r\theta \\ &= \underline{\underline{r(4 + \pi + \theta)}}.\end{aligned}$$

7. Figure 2 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + 27x - 23.$$

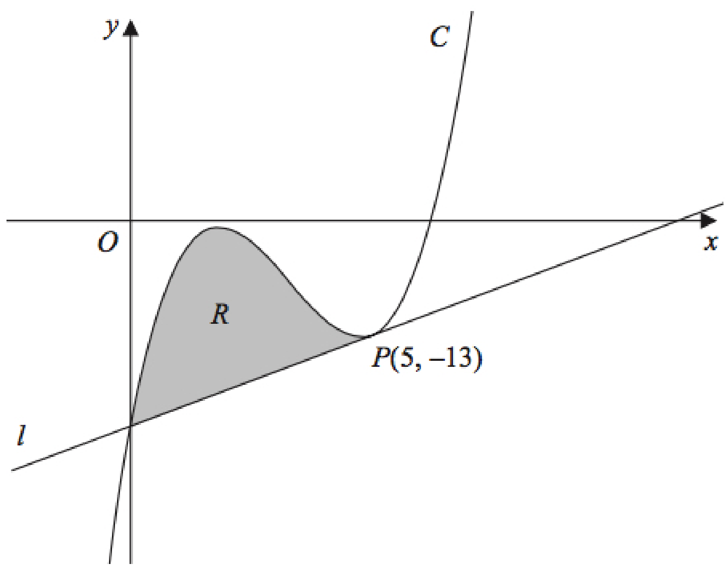


Figure 2: $y = x^3 - 10x^2 + 27x - 23$

The point $P(5, -13)$ lies on C.

The line l is the tangent to C at P

(a) Use differentiation to find the equation of l, giving your answer in the form $y = mx + c$, where m and c are integers to be found. (4)

Solution

$$y = x^3 - 10x^2 + 27x - 23 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 20x + 27$$

and

$$x = 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 75 - 100 + 27 = 2.$$

So, the tangent at $P(5, -13)$ is $y = 2x + c$. Now,

$$\begin{aligned} x = 5, y = -13 &\Rightarrow -13 = 2 \times 5 + c \\ &\Rightarrow c = -23 \end{aligned}$$

and the tangent is

$$\underline{\underline{y = 2x - 23}}.$$

(b) Hence verify that l meets C again on the y-axis. (1)

Solution

From the curve,

$$x = 0 \Rightarrow y = -23,$$

and from the tangent,

$$x = 0 \Rightarrow y = -23.$$

Hence, l meets C again on the $\underline{\underline{y\text{-axis}}}$.

The finite region R, shown shaded in Figure 2, is bounded by the curve C and the line l.

(c) Use algebraic integration to find the exact area of R. (4)

Solution

$$\begin{aligned} \text{Area} &= \int_0^5 [(x^3 - 10x^2 + 27x - 23) - (2x - 23)]\,\mathrm{d}x \\ &= \int_0^5 (x^3 - 10x^2 + 25x)\,\mathrm{d}x \\ &= \left[\tfrac{1}{4}x^4 - \tfrac{10}{3}x^3 + \tfrac{25}{2}x^2\right]_{x=0}^5 \\ &= (156\tfrac{1}{4} - 416\tfrac{2}{3} + 312\tfrac{1}{2}) - (0 - 0 + 0) \\ &= \underline{\underline{52\tfrac{1}{12}}}. \end{aligned}$$

8. The curve C has equation

$$px^3 + qxy + 3y^2 = 26,$$

where p and q are constants.

(a) Show that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{apx^2 + bqy}{qx + cy},$$

where a, b, and c are integers to be found.

Solution

Implicit differentation:

$$\begin{aligned} px^3 + qxy + 3y^2 = 26 &\Rightarrow 3px^2 + \left(qy + qx\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 6y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ &\Rightarrow qx\frac{\mathrm{d}y}{\mathrm{d}x} + 6y\frac{\mathrm{d}y}{\mathrm{d}x} = -3px^2 - qy \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}(qx + 6y) = -3px^2 - qy \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-3px^2 - qy}{qx + 6y}}}; \end{aligned}$$

hence, $\underline{\underline{a = -3}}$, $\underline{\underline{b = -1}}$, and $\underline{\underline{c = 6}}$.

Given that

- the point $P(-1, -4)$ lies on C and
- the normal to C at P has equation

$$19x + 26y + 123 = 0,$$

(b) find the value of p and the value of q. (5)

Solution

The normal:

$$\begin{aligned} 19x + 26y + 123 = 0 &\Rightarrow 26y = -19x - 123 \\ &\Rightarrow y = -\tfrac{19}{26}x - \tfrac{123}{26} \end{aligned}$$

and the tangent is

$$-\frac{1}{-\tfrac{19}{26}} = \tfrac{26}{19}.$$

Now,

$$x = -1, y = -5 \Rightarrow \frac{-3p[(-1)^2] - q(-4)}{q(-1) + 6(-4)} = \tfrac{26}{19}$$
$$\Rightarrow \frac{-3p + 4q}{-q - 24} = \tfrac{26}{19}$$
$$\Rightarrow 19(-3p + 4q) = 26(-q - 24)$$
$$\Rightarrow -57p + 76q = -26q - 624$$
$$\Rightarrow -57p + 102q = -624 \quad (1).$$

and

$$x = -1, y = -4 \Rightarrow p[(-1)^3] + q(-1)(4) + 3[(-4)^2] = 26$$
$$\Rightarrow -p + 4q + 48 = 26$$
$$\Rightarrow -p + 4q = -22 \quad (2).$$

Do $57 \times (2)$:

$$-57p + 228q = -1\,254 \quad (3)$$

and do $(3) - (2)$:

$$126q = -630 \Rightarrow \underline{\underline{q = -5}}$$
$$\Rightarrow -p + 4(-5) = -22$$
$$\Rightarrow -p - 20 = -22$$
$$\Rightarrow -p = -2$$
$$\Rightarrow \underline{\underline{p = 2}}.$$

9. Show that (3)

$$\sum_{n=2}^{\infty} (\tfrac{3}{4})^n \cos(180n)^\circ = \tfrac{9}{28}.$$

Solution

Well, we expand the sum:

$$\begin{aligned}
& \sum_{n=2}^{\infty} (\tfrac{3}{4})^n \cos(180n)^\circ \\
= \quad & (\tfrac{3}{4})^2 \cos(360)^\circ + (\tfrac{3}{4})^3 \cos(540)^\circ + (\tfrac{3}{4})^4 \cos(720)^\circ + (\tfrac{3}{4})^5 \cos(900)^\circ + \ldots \\
= \quad & (\tfrac{3}{4})^2 - (\tfrac{3}{4})^3 + (\tfrac{3}{4})^4 - (\tfrac{3}{4})^5 + \ldots \\
= \quad & (\tfrac{3}{4})^2 \left[1 - (\tfrac{3}{4}) + (\tfrac{3}{4})^2 - (\tfrac{3}{4})^3 + \ldots\right] \\
= \quad & (\tfrac{3}{4})^2 \times \frac{1}{1 - (-\tfrac{3}{4})} \\
= \quad & (\tfrac{3}{4})^2 \times \frac{1}{\tfrac{7}{4}} \\
= \quad & \tfrac{9}{16} \times \tfrac{4}{7} \\
= \quad & \underline{\underline{\tfrac{9}{28}}},
\end{aligned}$$

as required.

10. The time, T seconds, that a pendulum takes to complete one swing is modelled by the formula

$$T = al^b,$$

where l metres is the length of the pendulum and a and b are constants.

(a) Show that this relationship can be written in the form (2)

$$\log_{10} T = b \log_{10} l + \log_{10} a.$$

Solution

$$\begin{aligned}
T = al^b &\Rightarrow \log_{10} T = \log_{10}(al^b) \\
&\Rightarrow \log_{10} T = \log_{10} a + \log_{10} l^b \\
&\Rightarrow \underline{\underline{\log_{10} T = b \log_{10} l + \log_{10} a}},
\end{aligned}$$

as required.

A student carried out an experiment to find the values of the constants a and b.

The student recorded the value of T for different values of l.

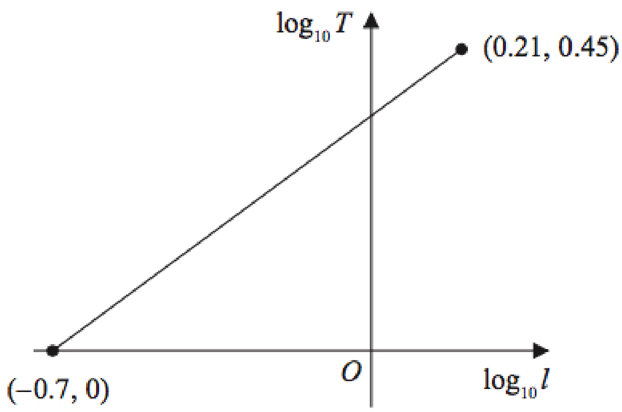


Figure 3: a graph of $\log_{10} l$ and $\log_{10} T$

Figure 3 shows the linear relationship between $\log_{10} l$ and $\log_{10} T$ for the student's data. The straight line passes through the points $(-0.7, 0)$ and $(0.21, 0.45)$.

Using this information,

(b) find a complete equation for the model in the form (3)

$$T = al^b,$$

giving the value of a and the value of b, each to 3 significant figures.

Solution

$$\begin{aligned}\text{Gradient} &= \frac{0.45 - 0}{0.21 - (-0.7)} \\ &= \frac{0.45}{0.91} \\ &= \tfrac{45}{91} \text{ or } 0.494\,505\,495 \text{ (FCD)}\end{aligned}$$

and so the line is

$$\log_{10} T = \tfrac{45}{91} \log_{10} l + \log_{10} a.$$

Now,

$$\begin{aligned}\log_{10} l = -0.7, \log_{10} T = 0 &\Rightarrow 0 = \tfrac{45}{91}(-0.7) + \log_{10} a \\ &\Rightarrow 0 = -\tfrac{9}{26} + \log_{10} a \\ &\Rightarrow \log_{10} a = \tfrac{9}{26} \\ &\Rightarrow a = 10^{\frac{9}{26}} \\ &\Rightarrow a = 2.218\,982\,341 \text{ (FCD)}.\end{aligned}$$

Hence,

$$\underline{\underline{T = 2.22\, l^{0.495} \text{ (3 sf)}}}.$$

(c) With reference to the model, interpret the value of the constant a. (1)

Solution

E.g., the time taken for one swing of a pendulum of length 1 m.

11. Figure 4 shows a sketch of the graph with equation

$$y = |2x - 3k|,$$

where k is a positive constant.

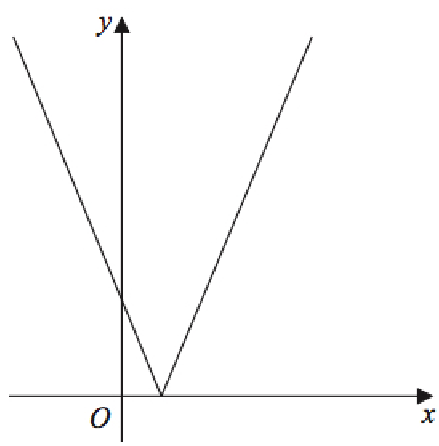


Figure 4: $y = |2x - 3k|$

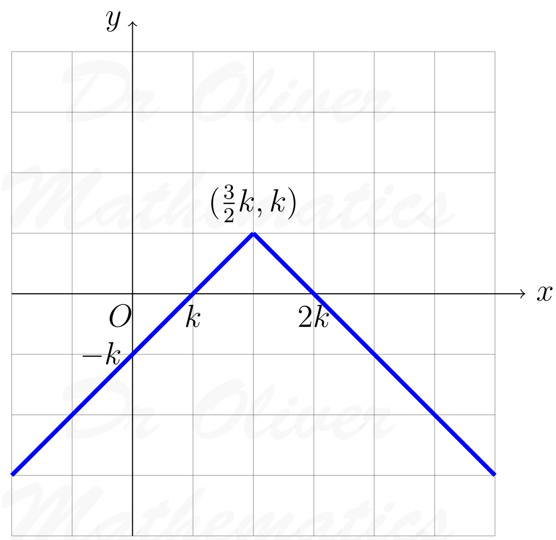
(a) Sketch the graph with equation $y = \mathrm{f}(x)$ where (4)

$$\mathrm{f}(x) = k - |2x - 3k|,$$

stating

- the coordinates of the maximum point and
- the coordinates of any points where the graph cuts the coordinate axes.

Solution



Maximum point: $\underline{\underline{(0, k)}}$.

Coordinates:

$$\begin{aligned} y = 0 &\Rightarrow 0 = k - (2x - 3k) \\ &\Rightarrow 0 = 4k - 2x \\ &\Rightarrow x = 2k \end{aligned}$$

and

$$\begin{aligned} y = 0 &\Rightarrow 0 = k - (3k - 2x) \\ &\Rightarrow 0 = -2k + 2x \\ &\Rightarrow x = k; \end{aligned}$$

$\underline{\underline{(k, 0)}}$ and $\underline{\underline{(2k, 0)}}$.

(b) Find, in terms of k, the set of values of x for which (4)

$$k - |2x - 3k| > x - k,$$

giving your answer in set notation.

Solution

$$\begin{aligned} k-(2x-3k) > x-k &\Rightarrow k-2x+3k > x-k \\ &\Rightarrow 5k > 3x \\ &\Rightarrow x < \tfrac{5}{3}k \end{aligned}$$

and

$$\begin{aligned} k-(3k-2x) > x-k &\Rightarrow k-3k+2x > x-k \\ &\Rightarrow x > k; \end{aligned}$$

hence,

$$\underline{\underline{\left\{x \in \mathbb{R} | k < x < \tfrac{5}{3}k\right\}}}.$$

(c) Find, in terms of k, the coordinates of the minimum point of the graph with equation (2)

$$y = 3 - 5\,\mathrm{f}(\tfrac{1}{2}x).$$

Solution

Well, the maximum of $\mathrm{f}(x)$ is at $(\frac{3}{2}k, k)$ and so the the maximum of $\mathrm{f}(\frac{1}{2}x)$ is at $(3k, k)$ and

$$x = 3k \Rightarrow y = 3 - 5k;$$

so, the coordinates of the minimum point is $\underline{\underline{(3k, 3-5k)}}$.

12. (a) Use the substitution $u = 1 + \sqrt{x}$ to show that (3)

$$\int_0^{16} \left(\frac{x}{1+\sqrt{x}}\right) \mathrm{d}x = \int_p^q \frac{2(u-1)^3}{u}\,\mathrm{d}u,$$

where p and q are constants to be found.

Solution

Well,

$$\begin{aligned}
u = 1 + \sqrt{x} &\Rightarrow u = 1 + x^{\frac{1}{2}} \\
&\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \tfrac{1}{2}x^{-\frac{1}{2}} \\
&\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2x^{\frac{1}{2}}} \\
&\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2(u-1)} \\
&\Rightarrow 2(u-1)\,\mathrm{d}u = \mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
x = 0 &\Rightarrow u = 1 \\
x = 16 &\Rightarrow u = 5.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^{16} \left(\frac{x}{1+\sqrt{x}}\right) \mathrm{d}x &= \int_1^5 \frac{(u-1)^2}{u} \cdot 2(u-1)\,\mathrm{d}u \\
&= \underline{\underline{\int_1^5 \frac{2(u-1)^3}{u}\,\mathrm{d}u}};
\end{aligned}$$

hence, $\underline{\underline{p = 1}}$ and $\underline{\underline{q = 5}}$.

(b) Hence show that (4)

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,\mathrm{d}x = A - B\ln 5,$$

where A and B are constants to be found.

Solution

$$\begin{aligned}
\int_0^{16} \left(\frac{x}{1+\sqrt{x}} \right) \mathrm{d}x &= \int_1^5 \frac{2(u-1)^3}{u} \, \mathrm{d}u \\
&= \int_1^5 \frac{2(u^3 - 3u^2 + 3u - 1)}{u} \, \mathrm{d}u \\
&= \int_1^5 \frac{(2u^3 - 6u^2 + 6u - 2)}{u} \, \mathrm{d}u \\
&= \int_1^5 \left(2u^2 - 6u + 6 - \frac{2}{u} \right) \mathrm{d}u \\
&= \left[\tfrac{2}{3}u^3 - 3u^2 + 6u - 2\ln|u| \right]_{u=1}^5 \\
&= (83\tfrac{1}{3} - 75 + 30 - 2\ln 5) - (\tfrac{2}{3} - 3 + 6 - 0) \\
&= \underline{\underline{34\tfrac{2}{3} - 2\ln 5}};
\end{aligned}$$

hence, $\underline{\underline{A = 32\tfrac{2}{3}}}$ and $\underline{\underline{B = 2}}$.

13. The curve C has parametric equations

$$x = \sin 2\theta, \; y = \operatorname{cosec}^3 \theta, \; 0 < \theta < \tfrac{1}{2}\pi.$$

(a) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ. (3)

Solution

Now,

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\theta} &= 2\cos 2\theta \\
\frac{\mathrm{d}y}{\mathrm{d}\theta} &= 3\operatorname{cosec}^2 \theta \cdot (-\operatorname{cosec}\theta \cot\theta) \\
&= -3\cot\theta \operatorname{cosec}^3 \theta
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} \\
&= \underline{\underline{\frac{-3\cot\theta \operatorname{cosec}^3\theta}{2\cos 2\theta}}}.
\end{aligned}$$

(b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$. (3)

Solution

Now,

$$\begin{aligned} y = 8 &\Rightarrow \operatorname{cosec}^3\theta = 8 \\ &\Rightarrow \operatorname{cosec}\theta = 2 \\ &\Rightarrow \sin\theta = \tfrac{1}{2} \\ &\Rightarrow \theta = \tfrac{1}{6}\pi \end{aligned}$$

and

$$\begin{aligned} \text{gradient of the tangent} &= \frac{-3\cot(\frac{1}{6}\pi)\operatorname{cosec}^3(\frac{1}{6}\pi)}{2\cos 2(\frac{1}{6}\pi)} \\ &= \frac{-3\cot(\frac{1}{6}\pi)\operatorname{cosec}^3(\frac{1}{6}\pi)}{2\cos(\frac{1}{3}\pi)} \\ &= \underline{\underline{-24\sqrt{3}}}. \end{aligned}$$

14. Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

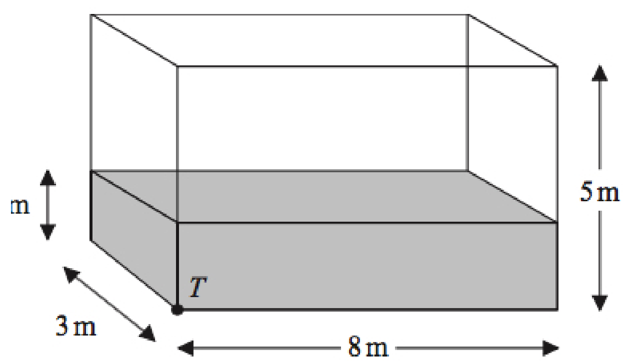


Figure 5: a large tank

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres,

- water is flowing into the tank at a constant rate of 0.48 m^3 per minute, and
- water is modelled as leaving the tank through the tap at a rate of $0.1h$ m^3 per minute.

(a) Show that, according to the model, (4)

$$1\,200\,\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h.$$

Solution

Let V be the volume of water at time t. Now,

$$V = 3 \times 8 \times h = 24h$$

and

$$\frac{\mathrm{d}V}{\mathrm{d}h} = 24.$$

Next,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 0.48 - 0.1h$$

so

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} \\ &= \frac{1}{\frac{\mathrm{d}V}{\mathrm{d}h}} \times (0.48 - 0.1h) \\ &= \frac{(0.48 - 0.1h)}{24}.\end{aligned}$$

Finally,

$$24\frac{\mathrm{d}h}{\mathrm{d}t} = 0.48 - 0.1h \Rightarrow \underline{\underline{1\,200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h}},$$

as required.

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model, (6)

$$h = A + B\mathrm{e}^{-kt},$$

where A, B, and k are constants to be found.

Solution

$$\begin{aligned}
1\,200\,\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h &\Rightarrow \frac{1\,200}{24-5h}\,\mathrm{d}h = \mathrm{d}t \\
&\Rightarrow \int \frac{1\,200}{24-5h}\,\mathrm{d}h = \int \mathrm{d}t \\
&\Rightarrow -240\ln(24-5h) = t + c \\
&\Rightarrow \ln(24-5h) = -\tfrac{1}{240}t + c \\
&\Rightarrow 24 - 5h = \mathrm{e}^{-\frac{1}{240}t+c} \\
&\Rightarrow -5h = \mathrm{e}^{-\frac{1}{240}t+c} - 24 \\
&\Rightarrow h = \tfrac{24}{5} - \tfrac{1}{5}\mathrm{e}^{-\frac{1}{240}t+c}.
\end{aligned}$$

Now,

$$\begin{aligned}
t = 0, h = 2 &\Rightarrow 2 = \tfrac{24}{5} - \tfrac{1}{5}\mathrm{e}^{c} \\
&\Rightarrow \tfrac{1}{5}\mathrm{e}^{c} = \tfrac{14}{5} \\
&\Rightarrow \mathrm{e}^{c} = 14 \\
&\Rightarrow c = \ln 14
\end{aligned}$$

and

$$\begin{aligned}
h &= \tfrac{24}{5} - \tfrac{1}{5}\mathrm{e}^{-\frac{1}{240}t+\ln 14} \\
&= \tfrac{24}{5} - \tfrac{1}{5}\mathrm{e}^{-\frac{1}{240}t}\cdot \mathrm{e}^{\ln 14} \\
&= \tfrac{24}{5} - \tfrac{1}{5}\mathrm{e}^{-\frac{1}{240}t}\cdot 14 \\
&= \underline{\underline{\tfrac{24}{5} - \tfrac{14}{5}\mathrm{e}^{-\frac{1}{240}t}}};
\end{aligned}$$

hence, $\underline{\underline{A = \frac{24}{5}}}$, $\underline{\underline{B = -\frac{14}{5}}}$, and $\underline{\underline{k = -\frac{1}{240}}}$.

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer. (2)

Solution

Well,

$$t \to \infty \Rightarrow \tfrac{14}{5}\mathrm{e}^{-\frac{1}{240}t} \to 0$$

so the maximum that h can be is less than 4.8. Hence, it can <u>never become full</u>.

15. (a) Express (3)

$$2\cos\theta - \sin\theta$$

in the form

$$R\cos(\theta + \alpha),$$

where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$.

Give the exact value of R and the value of α in radians to 3 decimal places.

Solution

$$\begin{aligned}2\cos\theta - \sin\theta &\equiv R\cos(\theta + \alpha)\\ &\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha\end{aligned}$$

which means

$$\begin{aligned}R\cos\alpha &\equiv 2 \quad (1)\\ R\sin\alpha &\equiv 1 \quad (2).\end{aligned}$$

Now,

$$\begin{aligned}R &= \sqrt{R^2}\\ &= \sqrt{R^2(\sin^2\alpha + cos^2\alpha)}\\ &= \sqrt{(R\sin\alpha)^2 + (R\cos\alpha)^2}\\ &= \sqrt{1^2 + 2^2}\\ &= \sqrt{5}\end{aligned}$$

and

$$\begin{aligned}\tan\alpha &= \frac{R\sin\alpha}{R\cos\alpha}\\ &= \tfrac{1}{2}\end{aligned}$$

and

$$\alpha = 0.463\,647\,609 \text{ (FCD)}.$$

Hence,

$$2\cos\theta - \sin\theta = \underline{\underline{\sqrt{5}\cos(\theta + 0.464) \text{ (3 sf)}}}.$$

Figure 6 shows the cross-section of a water wheel.

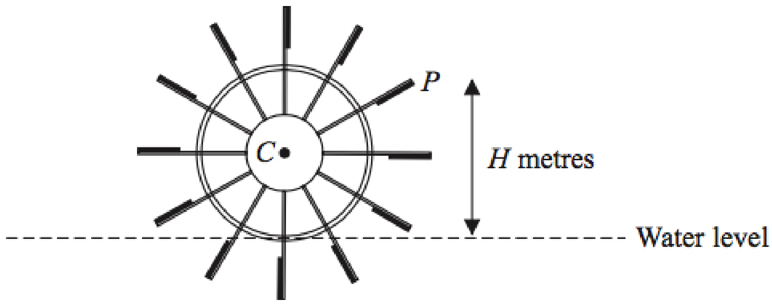


Figure 6: the cross-section of a water wheel

The wheel is free to rotate about a fixed axis through the point C.

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t),$$

where t is the time in seconds after the wheel starts rotating. Using the model, find

(b) (i) the maximum height of P above the water level, (3)

Solution

$$\begin{aligned} H &= 3 + 4\cos(0.5t) - 2\sin(0.5t) \\ &= 3 + 2(2\cos(0.5t) - \sin(0.5t)) \\ &= 3 + 2\sqrt{5}\cos(0.5\theta + 0.463\ldots) \end{aligned}$$

and its maximum value is

$$\underline{\underline{3 + 2\sqrt{5} = 7.47 \text{ m}}}.$$

(ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

Solution

$$\begin{aligned}\cos(0.5\theta + 0.463\ldots) = 1 &\Rightarrow 0.5\theta + 0.463\ldots = 0, 2\pi \\ &\Rightarrow 0.5\theta = 2\pi - 0.463\ldots \\ &\Rightarrow \theta = 2(2\pi - 0.463\ldots) \\ &\Rightarrow \theta = 11.639\,075\,4 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\theta = 11.6 \text{ (FCD)}}}.\end{aligned}$$

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

(c) find the value of T giving your answer to 3 significant figures. (4)

Solution

$$\begin{aligned}H = 0 &\Rightarrow 3 + 2\sqrt{5}\cos(0.5\theta + 0.463\ldots) = 0 \\ &\Rightarrow 2\sqrt{5}\cos(0.5\theta + 0.463\ldots) = -3 \\ &\Rightarrow \cos(0.5\theta + 0.463\ldots) = -\frac{3}{2\sqrt{5}} \\ &\Rightarrow \cos(0.5\theta + 0.463\ldots) = -\frac{3\sqrt{5}}{10} \\ &\Rightarrow 0.5\theta + 0.463\ldots = 2.306\,110\,78,\ 3.977\,074\,528 \text{ (FCD)} \\ &\Rightarrow 0.5\theta = 1.842\,463\,172,\ 3.513\,426\,919 \text{ (FCD)} \\ &\Rightarrow \theta = 3.684\,926\,341,\ 7.026\,853\,837 \text{ (FCD)}\end{aligned}$$

and

$$\begin{aligned}\text{time} &= 7.026\ldots - 3.684\ldots \\ &= 3.341\,927\,496 \text{ (FCD)} \\ &= \underline{\underline{3.34 \text{ (3 sf)}}}.\end{aligned}$$

In reality, the water level may not be of constant height.

(d) Explain how the equation of the model should be refined to take this into account. (1)

Solution

E.g., the '3' in the question would need to vary.