Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2010 June Paper 2 Variant 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

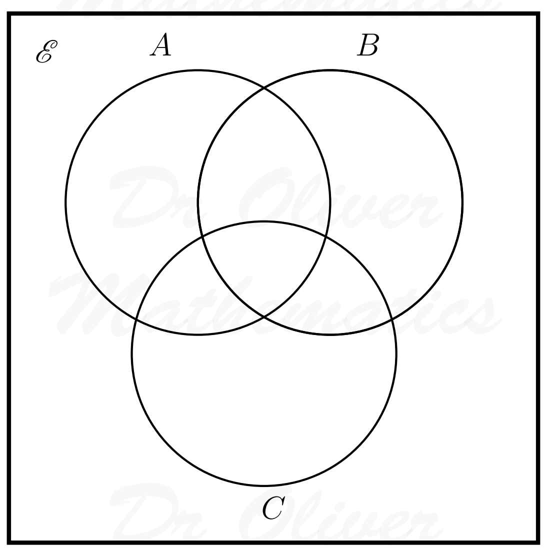
1. Find (3)

$$\int \left[2 + 5x - \frac{1}{(x-2)^2}\right] \, \mathrm{d}x.$$

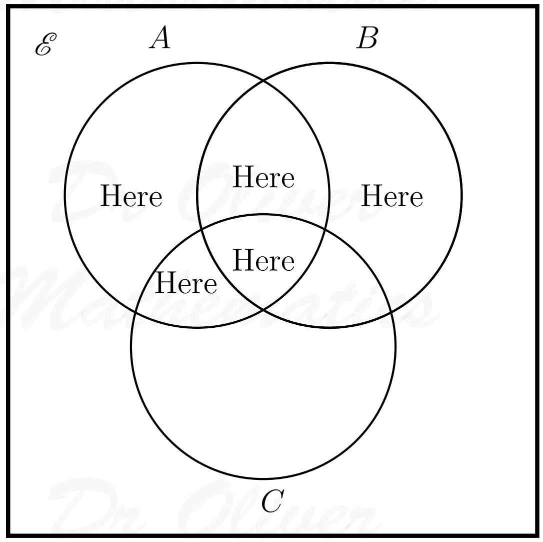
Solution

$$\begin{aligned} \int \left[2 + 5x - \frac{1}{(x-2)^2}\right] \, \mathrm{d}x &= \int \left[2 + 5x - (x-2)^{-2}\right] \, \mathrm{d}x \\ &= \underline{\underline{2x + \tfrac{5}{2}x^2 + (x-2)^{-1} + c}}. \end{aligned}$$

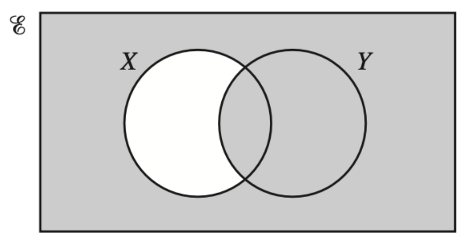
2. (a) Copy the diagram and shade the region which represents the set $A \cup (B \cap C')$. (1)



Solution



(b) Express, in set notation, the set represented by the shaded region. (1)



Solution

$X' \cup Y$.

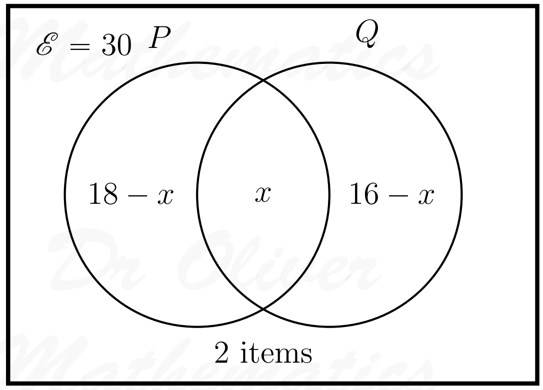
(c) The universal set $\mathscr{E}$ and the sets P and Q are such that (2)

- $n(\mathscr{E}) = 30$,
- $n(P) = 18$, and
- $n(Q) = 16$

Given that $n(P \cup Q)' = 2$, find $n(P \cap Q)$.

Solution

Well,



Now,

$$(18 - x) + x + (16 - x) + 2 = 30 \Rightarrow 36 - x = 30$$
$$\Rightarrow \underline{\underline{x = 6}}.$$

3. The volume, V cm^3 of a spherical ball of radius r cm is given by (4)

$$V = \tfrac{4}{3}\pi r^3.$$

Given that the radius is increasing at a constant rate of $\frac{1}{\pi}$ cm s^{-1}, find the rate at which the volume is increasing when $V = 288\pi$.

Solution

Now,

$$V = \tfrac{4}{3}\pi r^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$$

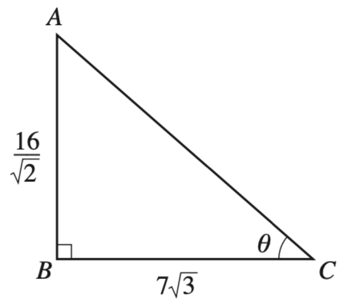
and

$$288\pi = \tfrac{4}{3}\pi r^3 \Rightarrow r^3 = 216$$
$$\Rightarrow r = 6.$$

Next,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$= 4\pi \times 6^2 \times \frac{1}{\pi}$$
$$= \underline{\underline{144 \text{ cm s}^{-1}}}.$$

4. The diagram shows a right-angled triangle ABC in which



- the length of AB is $\dfrac{16}{\sqrt{2}}$,
- the length of BC is $7\sqrt{3}$, and
- angle BCA is θ.

(a) Find $\tan\theta$ in the form $\dfrac{a\sqrt{6}}{b}$, where a and b are integers. (2)

Solution

Well,

$$\tan\theta = \frac{\frac{16}{\sqrt{2}}}{7\sqrt{3}} = \underline{\underline{\frac{8\sqrt{6}}{21}}};$$

hence, $\underline{\underline{a = 8}}$ and $\underline{\underline{b = 21}}$.

(b) Calculate the length of AC, giving your answer in the form $c\sqrt{d}$, where c and d are integers and d is as small as possible. (3)

Solution

$$
\begin{aligned}
AC &= \sqrt{AB^2 + BC^2} \\
&= \sqrt{\left(\frac{16}{\sqrt{2}}\right)^2 + (7\sqrt{3})^2} \\
&= \sqrt{128 + 147} \\
&= \sqrt{275} \\
&= \sqrt{25 \times 11} \\
&= \sqrt{25} \times \sqrt{11} \\
&= \underline{\underline{5\sqrt{11}}};
\end{aligned}
$$

hence, $\underline{\underline{c = 5}}$ and $\underline{\underline{d = 11}}$.

5. Solve the equation (6)

$$2x^3 - 3x^2 - 11x + 6 = 0.$$

Solution

Let

$$\mathrm{f}(x) = 2x^3 - 3x^2 - 11x + 6.$$

Then,

$$
\begin{aligned}
\mathrm{f}(1) &= 2 - 3 - 11 + 6 = -6 \\
\mathrm{f}(-1) &= -2 - 3 + 11 + 6 = 12 \\
\mathrm{f}(2) &= 16 - 12 - 22 + 6 = -12 \\
\mathrm{f}(-2) &= -16 - 12 + 22 + 6 = 0,
\end{aligned}
$$

and $(x + 2)$ is a factor.

We use synthetic division:

$$
\begin{array}{r|rrrr}
-2 & 2 & -3 & -11 & 6 \\
 & \downarrow & -4 & 14 & -6 \\
\hline
 & 2 & -7 & 3 & 0
\end{array}
$$

So,

$$\mathrm{f}(x) = (x + 2)(2x^2 - 7x + 3).$$

Now,

$$\begin{array}{ll} \text{add to:} & -7 \\ \text{multiply to:} & (+2)\times(+3)=+6 \end{array}\Bigg\}\,-1,\,-6$$

and

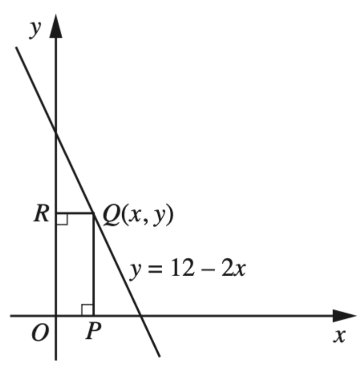
$$\begin{aligned} 2x^2-7x+3=0 &\Rightarrow 2x^2-6x-x+3=0 \\ &\Rightarrow 2x(x-3)-1(x-3)=0 \\ &\Rightarrow (2x-1)(x-3)=0 \\ &\Rightarrow 2x-1=0 \text{ or } x-3=0 \\ &\Rightarrow x=\tfrac{1}{2} \text{ or } x=3. \end{aligned}$$

Hence, the solutions are

$$\underline{\underline{x=-2,\ x=\tfrac{1}{2},\ \text{or } x=3.}}$$

6. The diagram shows part of the line

$$y = 12 - 2x.$$



The point $Q(x, y)$ lies on this line and the points P and R lie on the coordinate axes such that $OPQR$ is a rectangle.

(a) Write down an expression, in terms of x, for the area A of the rectangle $OPQR$. (2)

Solution

Well,

$$\begin{aligned} A &= xy \\ &= x(12 - 2x) \\ &= \underline{\underline{12x - 2x^2}}. \end{aligned}$$

(b) Given that x can vary, find the value of x for which A has a stationary value. (3)

Solution

Now,

$$A = 12x - 2x^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}x} = 12 - 4x$$

and

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}x} = 0 &\Rightarrow 12 - 4x = 0 \\ &\Rightarrow 4x = 12 \\ &\Rightarrow \underline{\underline{x = 3}}. \end{aligned}$$

(c) Find this stationary value of A and determine its nature. (2)

Solution

Well,

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -4 < 0$$

and so $x = 3$ is a <u>maximum</u> and this maximum value is

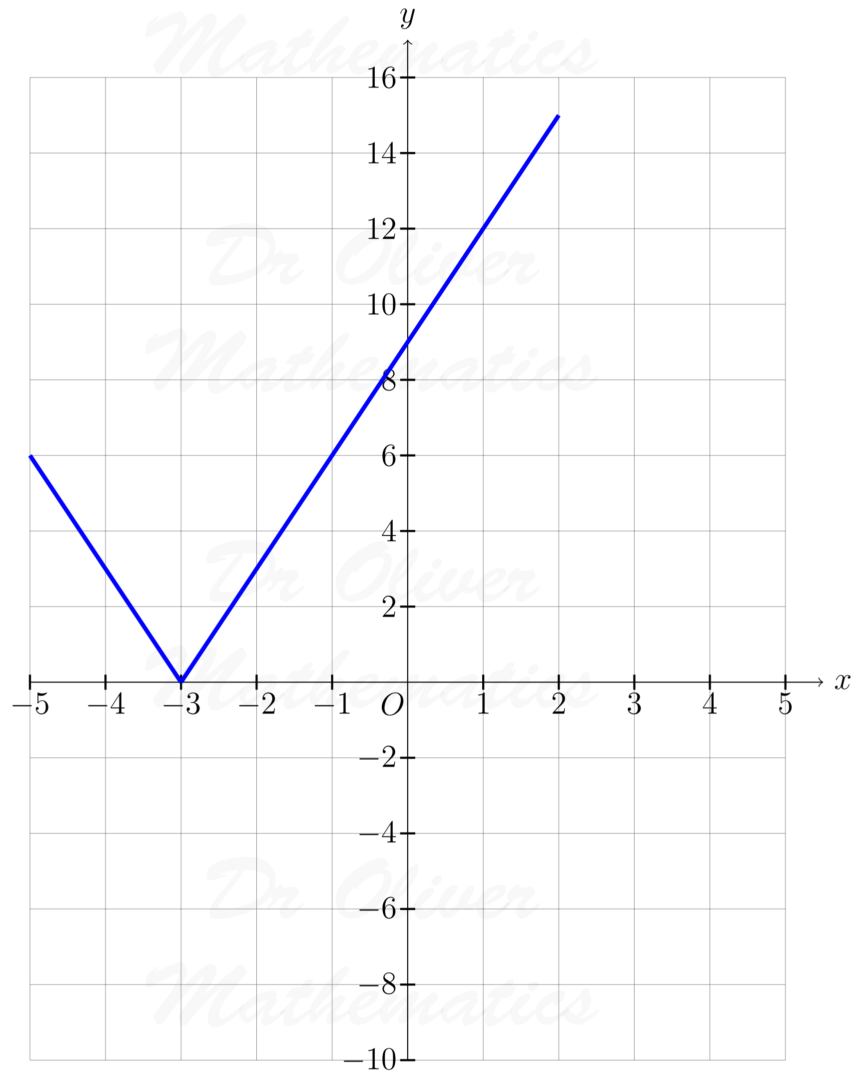
$$\underline{\underline{A = 18}}.$$

7. (a) Sketch the graph of (3)

$$y = |3x + 9| \text{ for } -5 < x < 2,$$

showing the coordinates of the points where the graph meets the axes.

Solution



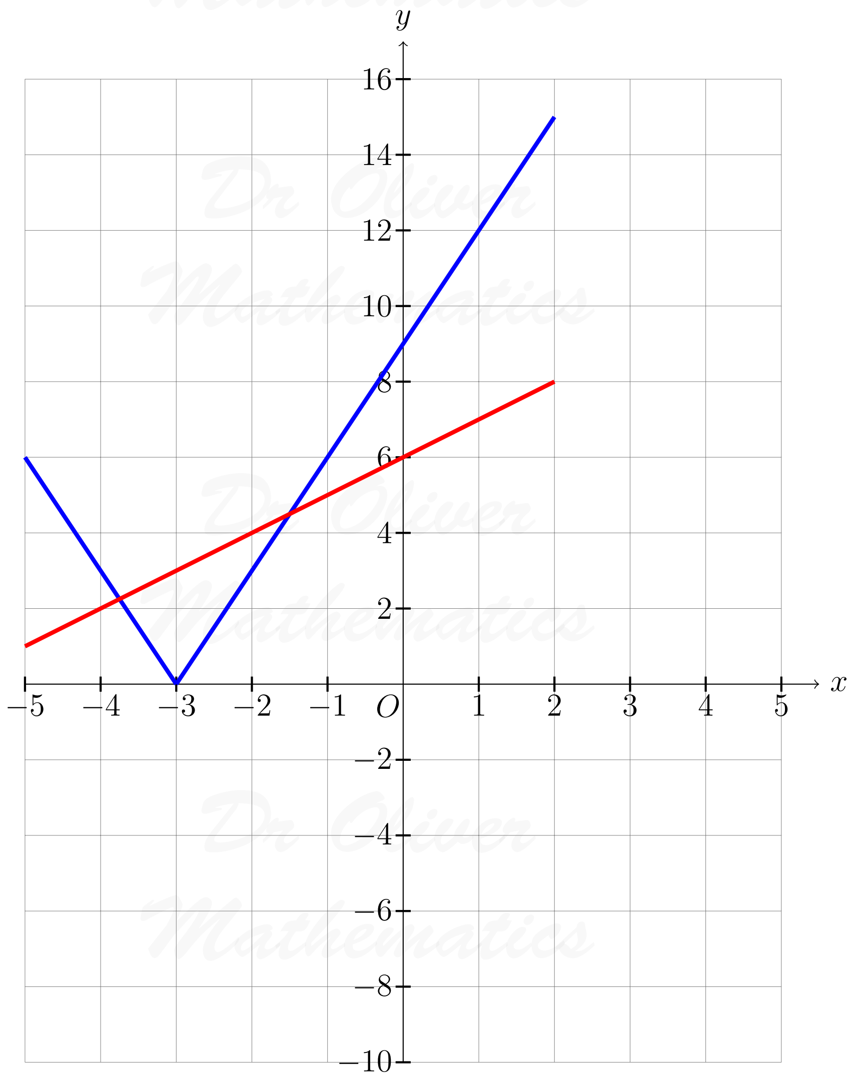
The coordinates of the points where the graph meets the axes are

$$\underline{\underline{(-3, 0) \text{ and } (0, 9)}}.$$

(b) On the same diagram, sketch the graph of (1)

$$y = x + 6.$$

Solution



(c) Solve the equation (3)

$$|3x + 9| = x + 6.$$

Solution

<u>$3x + 9 = x + 6$</u> :

$$\begin{aligned} 3x + 9 = x + 6 &\Rightarrow 2x = -3 \\ &\Rightarrow x = -1\tfrac{1}{2}. \end{aligned}$$

$\underline{-(3x+9) = x+6}$:

$$\begin{aligned} -(3x+9) = x+6 &\Rightarrow -3x-9 = x+6 \\ &\Rightarrow -15 = 4x \\ &\Rightarrow x = -3\tfrac{3}{4}. \end{aligned}$$

Hence,

$$\underline{\underline{x = -3\tfrac{3}{4} \text{ or } x = -1\tfrac{1}{2}}}.$$

8. (a) (i) Write down the first 4 terms, in ascending powers of x, of the expansion of (3)

$$(1-3x)^7.$$

Solution

$$\begin{aligned} (1-3x)^7 &= 1^7 + \binom{7}{1}(1)^6(-3x) + \binom{7}{2}(1)^5(-3x)^2 + \binom{7}{1}(1)^4(-3x)^3 + \ldots \\ &= \underline{\underline{1 - 21x + 189x^2 - 945x^3 + \ldots}}. \end{aligned}$$

(ii) Find the coefficient of x^3 in the expansion of (2)

$$(5+2x)(1-3x)^7.$$

Solution

Well,

$\times$	5	$+2x$
$189x^2$	$\ldots$	$+378x^3$
$-945x^3$	$-4\,725x^3$	$\ldots$

so the coefficient of x^3 is

$$378 - 4\,725 = \underline{\underline{-4\,347}}.$$

(b) Find the term which is independent of x in the expansion of (3)

$$\left(x^2 + \frac{2}{x}\right)^9.$$

Solution

The general term is

$$\begin{aligned}\binom{9}{r}(x^2)^r(2x^{-1})^{9-r} &= \binom{9}{r}2^{9-r}x^{2r}x^{-(9-r)} \\ &= \binom{9}{r}2^{9-r}x^{2r-(9-r)} \\ &= \binom{9}{r}2^{9-r}x^{2r-9+r} \\ &= \binom{9}{r}2^{9-r}x^{3r-9}.\end{aligned}$$

Now, the term which is independent of x is

$$3r - 9 = 0 \Rightarrow r = 3$$

and the specific term is

$$\binom{9}{3}2^6 = \underline{\underline{5\,376}}.$$

9. (a) Given that (5)

$$y = \frac{x+2}{(4x+12)^{\frac{1}{2}}},$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k(x+4)}{(4x+12)^{\frac{3}{2}}},$$

where k is a constant to be found.

Solution

Quotient rule:

$$\begin{aligned}u = x + 2 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \\ v = (4x+12)^{\frac{1}{2}} &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2(4x+12)^{-\frac{1}{2}}\end{aligned}$$

and so

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{[(4x+12)^{\frac{1}{2}}][1] - [2(4x+12)^{-\frac{1}{2}}][x+2]}{4x+12} \\
&= \frac{(4x+12)^{-\frac{1}{2}}\left[(4x+12)-(2)(x+2)\right]}{4x+12} \\
&= \frac{4x+12-2x-4}{(4x+12)^{\frac{3}{2}}} \\
&= \frac{2x+8}{(4x+12)^{\frac{3}{2}}} \\
&= \underline{\underline{\frac{2(x+4)}{(4x+12)^{\frac{3}{2}}}}};
\end{aligned}$$

hence, $\underline{\underline{k=2}}$.

(b) Hence evaluate (3)

$$\int_1^{13} \frac{x+4}{(4x+12)^{\frac{3}{2}}}\,\mathrm{d}x.$$

Solution

Now,

$$\begin{aligned}
\int_1^{13} \frac{x+4}{(4x+12)^{\frac{3}{2}}}\,\mathrm{d}x &= \tfrac{1}{2}\int_1^{13} \frac{2(x+4)}{(4x+12)^{\frac{3}{2}}}\,\mathrm{d}x \\
&= \tfrac{1}{2}\left[\frac{x+2}{(4x+12)^{\frac{1}{2}}}\right]_{x=1}^{13} \\
&= \tfrac{1}{2}(\tfrac{15}{8} - \tfrac{3}{4}) \\
&= \underline{\underline{\tfrac{9}{16}}}.
\end{aligned}$$

10. (a) Given that

$$\log_p X = 6 \text{ and } \log_p Y = 4,$$

find the value of

(i) $\log_p\left(\dfrac{X^2}{Y}\right)$, (2)

Solution

$$\begin{aligned}\log_p\left(\frac{X^2}{Y}\right) &= \log_p X^2 - \log_p Y \\ &= 2\log_p X - \log_p Y \\ &= 2 \times 6 - 4 \\ &= \underline{\underline{8}}.\end{aligned}$$

(ii) $\log_Y X$. (2)

Solution

$$\begin{aligned}\log_Y X &= \frac{\log_p X}{\log_p Y} \\ &= \frac{6}{4} \\ &= \underline{\underline{\tfrac{3}{2}}}.\end{aligned}$$

(b) Find the value of (3)

$$2^z,$$

where

$$z = 5 + \log_2 3.$$

Solution

$$\begin{aligned}2^z &= 2^{5+\log_2 3} \\ &= 2^5 \times 2^{\log_2 3} \\ &= 32 \times 3 \\ &= \underline{\underline{96}}.\end{aligned}$$

(c) Express (2)

$$\sqrt{512}$$

as a power of 4.

Solution

Now,

$$512 = 256 \times 2 = 4^4 \times 2$$

and

$$\begin{aligned}\sqrt{512} &= \sqrt{4^4 \times 2} \\ &= \sqrt{4^4} \times \sqrt{2} \\ &= 4^2 \times \sqrt{2} \\ &= 4^2 \times 4^{\frac{1}{4}} \\ &= \underline{\underline{4^{\frac{9}{4}}}}.\end{aligned}$$

11. (a) Solve, for $0 < x < 3$ radians, the equation (3)

$$4 \sin x - 3 = 0,$$

giving your answers correct to 2 decimal places.

Solution

$$\begin{aligned}4 \sin x - 3 = 0 &\Rightarrow 4 \sin x = 3 \\ &\Rightarrow \sin x = \tfrac{3}{4} \\ &\Rightarrow x = 0.848\,062\,079,\ 2.293\,530\,575 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 0.848,\ 2.29 \text{ (3 sf)}}}.\end{aligned}$$

(b) Solve, for $0^\circ < y < 360^\circ$, the equation (6)

$$4 \operatorname{cosec} y = 6 \sin y + \cot y.$$

Solution

Well,

$$
\begin{aligned}
4\operatorname{cosec} y = 6\sin y + \cot y &\Rightarrow \frac{4}{\sin y} = 6\sin y + \frac{\cos y}{\sin y} \\
&\Rightarrow \frac{4}{\sin y} = \frac{6\sin^2 y}{\sin y} + \frac{\cos y}{\sin y} \\
&\Rightarrow \frac{4}{\sin y} - \frac{6\sin^2 y}{\sin y} - \frac{\cos y}{\sin y} = 0 \\
&\Rightarrow \frac{1}{\sin y}[4 - 6\sin^2 y - \cos y] = 0 \\
&\Rightarrow \frac{1}{\sin y}[4 - 6(1-\cos^2 y) - \cos y] = 0 \\
&\Rightarrow \frac{1}{\sin y}[4 - 6 + 6\cos^2 y - \cos y] = 0 \\
&\Rightarrow \frac{1}{\sin y}[6\cos^2 y - \cos y - 2] = 0
\end{aligned}
$$

$$
\left.\begin{array}{ll} \text{add to:} & -1 \\ \text{multiply to:} & (+6)\times(-2) = -12 \end{array}\right\} -4,\ +3
$$

e.g.,

$$
\begin{aligned}
&\Rightarrow \frac{1}{\sin y}[6\cos^2 y - 4\cos y + 3\cos y - 2] = 0 \\
&\Rightarrow \frac{1}{\sin y}[2\cos y(3\cos y - 2) + 1(3\cos y - 2)] = 0 \\
&\Rightarrow \frac{1}{\sin y}(2\cos y + 1)(3\cos y - 2) = 0 \\
&\Rightarrow \cos y = -\tfrac{1}{2} \text{ or } \cos y = \tfrac{2}{3}.
\end{aligned}
$$

$\underline{\cos y = -\frac{1}{2}}$:

$$\cos y = -\tfrac{1}{2} \Rightarrow \underline{\underline{y = 120, 240}}.$$

$\underline{\cos y = \frac{2}{3}}$:

$$
\begin{aligned}
\cos y = \tfrac{2}{3} &\Rightarrow y = 48.189\,685\,1,\ 311.810\,314\,9 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{y = 48.2,\ 312 \text{ (3 sf)}}}.
\end{aligned}
$$

EITHER

12. It is given that

$$f(x) = 4x^2 + kx + k.$$

(a) Find the set of values of k for which the equation (5)

$$f(x) = 3$$

has no real roots.

Solution

Well,

$$4x^2 + kx + k = 3 \Rightarrow 4x^2 + kx + (k - 3) = 0.$$

Now, $a = 4$, $b = k$, and $c = k - 3$:

$$\begin{aligned} b^2 - 4ac < 0 &\Rightarrow k^2 - 4(4)(k-3) < 0 \\ &\Rightarrow k^2 - 16k + 48 < 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -16 \\ \text{multiply to:} & +48 \end{array}\right\} -4, -12$$

$$\begin{aligned} &\Rightarrow (k-4)(k-12) < 0 \\ &\Rightarrow \underline{\underline{4 < k < 12}}. \end{aligned}$$

In the case where $k = 10$,

(b) express $f(x)$ in the form (3)

$$(ax + b)^2 + c,$$

Solution

$$\begin{aligned} 4x^2 + 10x + 10 &= 4[x^2 + \tfrac{5}{2}x] + 10 \\ &= 4[(x^2 + \tfrac{5}{2}x + \tfrac{25}{16}) - \tfrac{25}{16}] + 10 \\ &= 4(x + \tfrac{5}{4})^2 - \tfrac{25}{4} + 10 \\ &= \underline{\underline{(2x + \tfrac{5}{2})^2 + \tfrac{15}{4}}}. \end{aligned}$$

(c) find the least value of $f(x)$ and the value of x for which this least value occurs. (2)

Solution

Well, the least value of $\mathrm{f}(x)$ is $\underline{\underline{\frac{15}{4}}}$ and the value of x for which this least value occurs is

$$2x + \tfrac{5}{2} = 0 \Rightarrow \underline{\underline{x = -\tfrac{5}{4}}}.$$

OR

13. The functions f, g, and h are defined, for $x \in \mathbb{R}$, by

$$\begin{aligned} \mathrm{f}(x) &= x^2 + 1, \\ \mathrm{g}(x) &= 2x - 5, \text{ and} \\ \mathrm{h}(x) &= 2^x. \end{aligned}$$

(a) Write down the range of f. (1)

Solution

$\underline{\underline{\mathrm{f}(x) \geqslant 1}}$.

(b) Find the value of $\mathrm{g}\,\mathrm{f}(3)$. (2)

Solution

$$\begin{aligned} \mathrm{g}\,\mathrm{f}(3) &= \mathrm{g}(\mathrm{f}(3)) \\ &= \mathrm{g}(10)) \\ &= \underline{\underline{15}}. \end{aligned}$$

(c) Solve the equation (5)

$$\mathrm{f}\,\mathrm{g}(x) = \mathrm{g}^{-1}(15).$$

Solution

Well,

$$\begin{aligned} \mathrm{f}\,\mathrm{g}(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \mathrm{f}(2x - 5) \\ &= (2x - 5)^2 + 1. \end{aligned}$$

Now,

$$\begin{aligned} y = 2x - 5 &\Rightarrow y + 5 = 2x \\ &\Rightarrow \frac{y+5}{2} = x, \end{aligned}$$

so

$$\mathrm{g}^{-1}(x) = \frac{x+5}{2}.$$

Next,

$$\begin{aligned} \mathrm{f}\,\mathrm{g}(x) = \mathrm{g}^{-1}(15) &\Rightarrow (2x-5)^2 + 1 = 10 \\ &\Rightarrow (2x-5)^2 = 9 \\ &\Rightarrow 2x - 5 = -3 \text{ or } 2x - 5 = 3 \\ &\Rightarrow 2x = 2 \text{ or } 2x = 8 \\ &\Rightarrow \underline{\underline{x = 1 \text{ or } x = 4}}. \end{aligned}$$

(d) On the same axes, sketch the graph of $y = \mathrm{h}(x)$ and the graph of the inverse function $y = \mathrm{h}^{-1}(x)$, indicating clearly which graph represents h and which graph represents $\mathrm{h}^{-1}(x)$. (2)

Solution