Dr Oliver Mathematics
Further Mathematics
Conic Sections: Hyperbolas
Past Examination Questions

This booklet consists of 13 questions across a variety of examination topics.
The total number of marks available is 128.

1. Figure 1 shows the curve C which is part of the hyperbola with parametric equations

$$x = a \cosh t, y = 2a \sinh t,$$

where a is a positive constant and $x \geqslant a$.

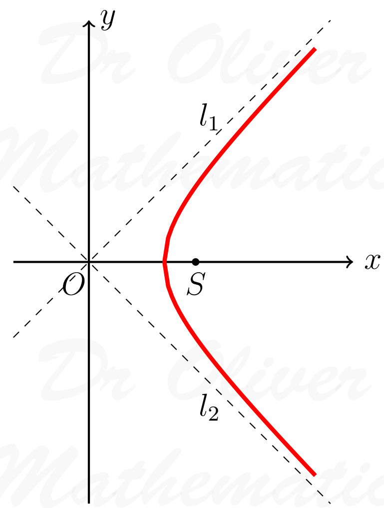

Figure 1: $x = a \cosh t, y = 2a \sinh t$

The lines l_1 and l_2 are asymptotes to C.

(a) Show that an equation for the tangent to C at a point $P(a \cosh p, 2a \sinh t)$ is (4)

$$2x \cosh p - y \sinh p = 2a.$$

The tangent to the curve C at P meets the asymptote l_1 at Q. Given that QS is parallel to the y-axis, where S is the focus,

(b) show that $p = \frac{1}{2} \ln 5$. (8)

2. The hyperbola C has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

(a) Show that an equation for the normal to C at a point $P(a\sec t, b\tan t)$ is (5)

$$ax\sin t + by = (a^2 + b^2)\tan t.$$

The normal to C at P cuts the x-axis at the point A and S is a focus of C. Given that the eccentricity of C is $\frac{3}{2}$, and that $OA = 3OS$, where O is the origin,

(b) determine the possible values of t, for $0 \leqslant t < 2\pi$. (8)

3. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{4} = 1.$$

Find

(a) the value of the eccentricity of H, (2)

(b) the distance between the foci of H. (2)

The ellipse E has equation

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$

(c) Sketch H and E on the same diagram, showing the coordinates of the points where each curve crosses the axes. (3)

4. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

(a) Show that an equation for the normal to H at a point $P(4\sec t, 3\tan t)$ is (6)

$$4x\sin t + 3y = 25\tan t.$$

The point S, which lies on the positive x-axis, is a focus of H. Given that PS is parallel to the y-axis and the the y-coordinate of P is positive,

(b) find the values of the coordinates of P. (5)

Given that the normal to H at this point P intersects the x-axis at the point R,

(c) find the area of triangle PRS. (3)

5. The hyperbola H has equation

$$x^2 - 4y^2 = 4a^2, a > 0.$$

(a) Find the eccentricity of H. (3)

Given that $x = 10$ is an equation of a directrix of H,

(b) find the value of a. (2)

6. A hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where a and b are a positive constants. The line L has equation

$$y = mx + c,$$

where m and c are constants.

(a) Given that L and H meet, show that the x-coordinates of the points of intersection are the roots of the equation (2)

$$(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(c^2 + b^2) = 0.$$

Hence, given that L is a tangent to H,

(b) show that (2)

$$a^2m^2 = b^2 + c^2.$$

The hyperbola H' has equation

$$\frac{x^2}{25} - \frac{y^2}{16} = 1.$$

(c) Find the equations of the tangents to H' which pass through the point $(1, 4)$. (7)

7. A hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{4} = 1.$$

The line l_1 is the tangent to H at the point $P(4\sec t, 2\tan t)$.

(a) Use calculus to show that an equation of l_1 is (5)

$$2y \sin t = x - 4\cos t.$$

The line l_2 passes through the origin and is perpendicular to l_1. The lines l_1 and l_2 intersect at the point Q.

(b) Show that, at t varies, an equation of the locus of Q is (8)

$$(x^2 + y^2)^2 = 16x^2 - 4y^2.$$

8. A hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

(a) Use calculus to show that an equation of the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$ may be written in the form (4)

$$xb\cosh\theta - ya\sinh\theta = ab.$$

The line l_1 is the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$, $\theta \neq 0$. Given that l_1 meets the x-axis at the point P,

(b) find, in terms of a and θ, the coordinates of P. (2)

The line l_2 is the tangent to H at the point $(a, 0)$. Given that l_1 and l_2 meet at the point Q,

(c) find, in terms of a, b, and θ, the coordinates of Q. (2)

(d) Show that, at θ varies, the locus of the mid-point of PQ has equation (6)

$$x(4y^2 + b^2) = ab^2.$$

9. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

Find

(a) the coordinates of the foci of H, (3)

(b) the equations of the directrices of H. (2)

10. A hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{25} = 1,$$

where a is a positive constant. The foci of H are at the points with coordinates $(13, 0)$ and $(-13, 0)$. Find

(a) the value of the constant a, (3)

(b) the equation of the directrices of H. (3)

11. The hyperbola H has foci at $(5, 0)$ and $(-5, 0)$ and directrices with equations $x = \pm\frac{9}{5}$. Find a cartesian equation for H. (7)

12. The hyperbola H is given by the equation $x^2 - y^2 = 1$.

(a) Write down the equations of the two asymptotes of H. (1)

(b) Show that an equation of the tangent to H at the point $P(\cosh t, \sinh t)$ is (3)

$$y\sinh t = x\cosh t - 1.$$

The tangent at P meets the asymptotes of H at the points Q and R.

(c) Show that P is the midpoint of QR. (3)

(d) Show that the area of the triangle OQR, where O is the origin, is independent of t. (3)

13. The hyperbola H has the equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

The point $P(4\sec\theta, 3\tan\theta)$, $0 < \theta < \frac{\pi}{2}$, lies on H.

(a) Show that an equation of the normal to H at the point P is (5)

$$3y + 4x\sin\theta = 25\tan\theta.$$

The line l is the directrix of H for which $x > 0$. The normal to H at P crosses the line l at the point Q. Given that $\theta = \frac{\pi}{4}$,

(b) find the y-coordinate of Q, giving your answer in the form $a + b\sqrt{2}$, where a and b are rational numbers to be found. (6)