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$$a \sin x + b \cos x = c$$

30 June 2014

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(a) Express

$$3\sin x + 2\cos x$$

in the form $R\sin(x+\alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(b) Hence find the greatest value of

$$(3\sin x + 2\cos x)^4.$$

(c) Solve, for $0 < x < 2\pi$, the equation

$$3\sin x + 2\cos x = 1,$$

giving your answers to 3 decimal places.

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Part (a)

$$3\sin x + 2\cos x \equiv R\sin(x+\alpha)$$
$$\equiv R\sin x\cos\alpha + R\sin\alpha\cos x$$

Hence

$$R = \sqrt{3^2+2^2} = \sqrt{13}$$

and

$$R\sin\alpha = 2,\ R\cos\alpha = 3$$

which leads to

$$\tan\alpha = \tfrac{2}{3} \Rightarrow \alpha = 0.588\ldots$$

Hence

$$3\sin x + 2\cos x \equiv \underline{\underline{\sqrt{13}\sin(x+0.588\ldots)}}.$$

Part (b)

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$$\begin{aligned}(3\sin x + 2\cos x)^4 &\equiv \left[\sqrt{13}\sin(x + 0.588\ldots)\right]^4 \\ &\equiv 169\sin^4(x + 0.588\ldots)\end{aligned}$$

and so the greatest value of $(3\sin x + 2\cos x)^4$ is $\underline{\underline{169}}$.

Part (c)

$$\begin{aligned}
& \sqrt{13}\sin(x + 0.588\ldots) = 1 \\
\Rightarrow\quad & \sin(x + 0.588\ldots) = \tfrac{1}{\sqrt{13}} \\
\Rightarrow\quad & x + 0.588\ldots = 0.281\ldots \text{ or } 2.8605\ldots \text{ or } 6.564\ldots \\
\Rightarrow\quad & x = 2.272\,555\,149 \text{ or } 5.976\,217\,605 \text{ (FCD)} \\
\Rightarrow\quad & x = \underline{\underline{2.273 \text{ or } 5.976 \text{ (3 dp)}}}
\end{aligned}$$

Find

$$\int \frac{1}{2\sin x - \cos x + 3}\,\mathrm{d}x.$$

This type of integral was part of the specification when P3 and P4 were Further Mathematics modules. And the substitution to use is

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An integration by substitution

Find
$$\int \frac{1}{2\sin x - \cos x + 3}\,\mathrm{d}x.$$
This type of integral was part of the specification when P3 and P4 were Further Mathematics modules. And the substitution to use is
$$t = \tan(\tfrac{1}{2}x).$$

The ‘Half-Angle Tangent’ Expression for $\sin x$

$$\sin x$$

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The 'Half-Angle Tangent' Expression for $\sin x$

$$\sin x \equiv \frac{\sin 2(\frac{1}{2}x)}{1}$$

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The 'Half-Angle Tangent' Expression for $\sin x$

$$\sin x \equiv \frac{\sin 2(\frac{1}{2}x)}{1}$$
$$\equiv \frac{2\sin(\frac{1}{2}x)\cos(\frac{1}{2}x)}{\cos^2(\frac{1}{2}x) + \sin^2(\frac{1}{2}x)}$$

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The 'Half-Angle Tangent' Expression for $\sin x$

$$\begin{aligned}\sin x &\equiv \frac{\sin 2(\frac{1}{2}x)}{1} \\ &\equiv \frac{2\sin(\frac{1}{2}x)\cos(\frac{1}{2}x)}{\cos^2(\frac{1}{2}x)+\sin^2(\frac{1}{2}x)} \\ &\equiv \frac{2\tan(\frac{1}{2}x)}{1+\tan^2(\frac{1}{2}x)}.\end{aligned}$$

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$$\begin{aligned}\sin x &\equiv \frac{\sin 2(\frac{1}{2}x)}{1} \\ &\equiv \frac{2\sin(\frac{1}{2}x)\cos(\frac{1}{2}x)}{\cos^2(\frac{1}{2}x)+\sin^2(\frac{1}{2}x)} \\ &\equiv \frac{2\tan(\frac{1}{2}x)}{1+\tan^2(\frac{1}{2}x)}.\end{aligned}$$

Note that the LHS is defined for all values of x but the RHS is not defined for $\frac{1}{2}x = \frac{1}{2}\pi + n\pi$, for any integer n, i.e., the RHS is not defined for $x = (2n+1)\pi$. This is because of the cancellation of $\cos^2(\frac{1}{2}x)$.

The 'Half-Angle Tangent' Expression for $\cos x$

$$\cos x$$

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The 'Half-Angle Tangent' Expression for $\cos x$

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$$\cos x \equiv \frac{\cos 2(\frac{1}{2}x)}{1}$$

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The 'Half-Angle Tangent' Expression for $\cos x$

$$\cos x \equiv \frac{\cos 2(\frac{1}{2}x)}{1}$$
$$\equiv \frac{\cos^2(\frac{1}{2}x) - \sin^2(\frac{1}{2}x)}{\cos^2(\frac{1}{2}x) + \sin^2(\frac{1}{2}x)}$$

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The 'Half-Angle Tangent' Expression for $\cos x$

$$\begin{aligned}\cos x &\equiv \frac{\cos 2(\frac{1}{2}x)}{1} \\ &\equiv \frac{\cos^2(\frac{1}{2}x) - \sin^2(\frac{1}{2}x)}{\cos^2(\frac{1}{2}x) + \sin^2(\frac{1}{2}x)} \\ &\equiv \frac{1 - \tan^2(\frac{1}{2}x)}{1 + \tan^2(\frac{1}{2}x)}.\end{aligned}$$

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The 'Half-Angle Tangent' Expression for $\cos x$

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Again, the LHS is defined for all values of x but the RHS is not defined for $\frac{1}{2}x = \frac{1}{2}\pi + n\pi$, for any integer n, i.e., the RHS is not defined for $x = (2n+1)\pi$.

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$\tan x$

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The 'Half-Angle Tangent' Expression for $\tan x$

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$$\tan x \equiv \tan 2(\tfrac{1}{2}x)$$

The 'Half-Angle Tangent' Expression for $\tan x$

$$\begin{aligned}\tan x &\equiv \tan 2(\tfrac{1}{2}x) \\ &\equiv \frac{2\tan(\tfrac{1}{2}x)}{1-\tan^2(\tfrac{1}{2}x)}.\end{aligned}$$

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The 'Half-Angle Tangent' Expression for $\tan x$

$$\begin{aligned}\tan x &\equiv \tan 2(\tfrac{1}{2}x) \\ &\equiv \frac{2\tan(\frac{1}{2}x)}{1-\tan^2(\frac{1}{2}x)}.\end{aligned}$$

The issue of when the two sides are defined is left as an exercise for the interested reader ...

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The 'Half-Angle Tangent' Expressions in Action

Let $t = \tan(\frac{1}{2}x)$. Then

$$3 \sin x + 2 \cos x = 1$$

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Let $t = \tan(\frac{1}{2}x)$. Then

$$\begin{aligned} & 3\sin x + 2\cos x = 1 \\ \Rightarrow \quad & 3\left(\frac{2t}{1+t^2}\right) + 2\left(\frac{1-t^2}{1+t^2}\right) = 1 \end{aligned}$$

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Let $t = \tan(\frac{1}{2}x)$. Then

$$\begin{aligned} & 3\sin x + 2\cos x = 1 \\ \Rightarrow \quad & 3\left(\frac{2t}{1+t^2}\right) + 2\left(\frac{1-t^2}{1+t^2}\right) = 1 \\ \Rightarrow \quad & 6t + 2 - 2t^2 = 1 + t^2 \end{aligned}$$

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Let $t = \tan(\frac{1}{2}x)$. Then

$$\begin{aligned} & 3\sin x + 2\cos x = 1 \\ \Rightarrow \quad & 3\left(\frac{2t}{1+t^2}\right) + 2\left(\frac{1-t^2}{1+t^2}\right) = 1 \\ \Rightarrow \quad & 6t + 2 - 2t^2 = 1 + t^2 \\ \Rightarrow \quad & 0 = 3t^2 - 6t - 1 \\ \Rightarrow \quad & t = \frac{3 \pm 2\sqrt{3}}{3} \end{aligned}$$

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The 'Half-Angle Tangent' Expressions in Action

$$\tan(\tfrac{1}{2}x) = \frac{3 + 2\sqrt{3}}{3}$$

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$$\tan(\tfrac{1}{2}x) = \frac{3 + 2\sqrt{3}}{3}$$
$$\Rightarrow \quad \tfrac{1}{2}x = 1.136\,277\,574 \text{ (FCD)}$$

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The 'Half-Angle Tangent' Expressions in Action

$$
\begin{aligned}
\tan(\tfrac{1}{2}x) &= \frac{3 + 2\sqrt{3}}{3} \\
\Rightarrow \quad \tfrac{1}{2}x &= 1.136\,277\,574 \text{ (FCD)} \\
\Rightarrow \quad x &= \underline{\underline{2.273 \text{ (3 dp)}}}
\end{aligned}
$$

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The ‘Half-Angle Tangent’ Expressions in Action

$$\begin{aligned} \tan(\tfrac{1}{2}x) &= \frac{3 + 2\sqrt{3}}{3} \\ \Rightarrow \quad \tfrac{1}{2}x &= 1.136\,277\,574 \text{ (FCD)} \\ \Rightarrow \quad x &= \underline{\underline{2.273 \text{ (3 dp)}}} \end{aligned}$$

and

$$\tan(\tfrac{1}{2}x) = \frac{3 - 2\sqrt{3}}{3}$$

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$$\begin{aligned} \tan(\tfrac{1}{2}x) &= \frac{3 + 2\sqrt{3}}{3} \\ \Rightarrow \quad \tfrac{1}{2}x &= 1.136\,277\,574 \text{ (FCD)} \\ \Rightarrow \quad x &= \underline{\underline{2.273 \text{ (3 dp)}}} \end{aligned}$$

and

$$\begin{aligned} \tan(\tfrac{1}{2}x) &= \frac{3 - 2\sqrt{3}}{3} \\ \Rightarrow \quad \tfrac{1}{2}x &= -0.153\,483\,851 \end{aligned}$$

The 'Half-Angle Tangent' Expressions in Action

$$\begin{aligned} \tan(\tfrac{1}{2}x) &= \frac{3+2\sqrt{3}}{3} \\ \Rightarrow \quad \tfrac{1}{2}x &= 1.136\,277\,574 \text{ (FCD)} \\ \Rightarrow \quad x &= \underline{\underline{2.273 \text{ (3 dp)}}} \end{aligned}$$

and

$$\begin{aligned} \tan(\tfrac{1}{2}x) &= \frac{3-2\sqrt{3}}{3} \\ \Rightarrow \quad \tfrac{1}{2}x &= -0.153\,483\,851 \\ &\quad \text{or } 2.988\,108\,803 \text{ (FCD)} \end{aligned}$$

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$$\tan(\tfrac{1}{2}x) = \frac{3 + 2\sqrt{3}}{3}$$
$$\Rightarrow \quad \tfrac{1}{2}x = 1.136\,277\,574 \text{ (FCD)}$$
$$\Rightarrow \quad x = \underline{\underline{2.273 \text{ (3 dp)}}}$$

and

$$\tan(\tfrac{1}{2}x) = \frac{3 - 2\sqrt{3}}{3}$$
$$\Rightarrow \quad \tfrac{1}{2}x = -0.153\,483\,851$$
$$\text{or } 2.988\,108\,803 \text{ (FCD)}$$
$$\Rightarrow \quad x = \underline{\underline{5.976 \text{ (3 dp)}}}$$

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The General Case

Let $t = \tan(\frac{1}{2}x)$. Then

$$\begin{aligned}
& a\sin x + b\cos x = c \\
\Rightarrow \quad & a\left(\frac{2t}{1+t^2}\right) + b\left(\frac{1-t^2}{1+t^2}\right) = c \\
\Rightarrow \quad & 2at + b - bt^2 = c + ct^2 \\
\Rightarrow \quad & 0 = (b+c)t^2 - 2at + (c-b)
\end{aligned}$$

This has solutions provided the discriminant is non-negative:

$$\begin{aligned}
& (-2a)^2 - 4(b+c)(c-b) \geqslant 0 \\
\Rightarrow \quad & 4a^2 - 4(c^2 - b^2) \geqslant 0 \\
\Rightarrow \quad & 4(a^2 + b^2 - c^2) \geqslant 0 \\
\Rightarrow \quad & a^2 + b^2 \geqslant c^2 \\
\Rightarrow \quad & -\sqrt{a^2+b^2} \leqslant a\sin x + b\cos x \leqslant \sqrt{a^2+b^2}.
\end{aligned}$$

A small problem ...

As we have seen the identities for both sine and cosine are not valid for the odd multiples of π. But, for any $n \in \mathbb{Z}$,

$$\sin(2n+1)\pi = 0 \text{ and } \cos(2n+1)\pi = -1$$

so this boils down to the case where the trigonometric equation has the form

$$a \sin x + b \cos x = -b.$$

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$$\begin{aligned} & a \sin x + b \cos x = -b \\ \Rightarrow \quad & a\left(\frac{2t}{1+t^2}\right) + b\left(\frac{1-t^2}{1+t^2}\right) = -b \\ \Rightarrow \quad & 2at + b - bt^2 = -b - bt^2 \\ \Rightarrow \quad & 2at = -2b \\ \Rightarrow \quad & t = -\tfrac{b}{a}. \end{aligned}$$

$$
\begin{aligned}
& a \sin x + b \cos x = a \\
\Rightarrow\quad & a\left(\frac{2t}{1+t^2}\right) + b\left(\frac{1-t^2}{1+t^2}\right) = a \\
\Rightarrow\quad & 2at + b - bt^2 = a + at^2 \\
\Rightarrow\quad & (a+b)t^2 - 2at + (a-b) = 0 \\
\Rightarrow\quad & \left[(a+b)t - (a-b)\right]\left[t-1\right] = 0 \\
\Rightarrow\quad & t = 1 \text{ or } t = \tfrac{a-b}{a+b}.
\end{aligned}
$$

Note that the case $t = 1$ generates the angles which have a sine of 1 and a cosine of 0.

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$$\begin{aligned}
& a\sin x + b\cos x = -a \\
\Rightarrow\quad & a\left(\frac{2t}{1+t^2}\right) + b\left(\frac{1-t^2}{1+t^2}\right) = -a \\
\Rightarrow\quad & 2at + b - bt^2 = -a - at^2 \\
\Rightarrow\quad & (a-b)t^2 - 2at + (a+b) = 0 \\
\Rightarrow\quad & \left[(a-b)t + (a+b)\right]\left[t+1\right] = 0 \\
\Rightarrow\quad & t = -1 \text{ or } t = \tfrac{-a-b}{a-b}.
\end{aligned}$$

Note that the case $t = -1$ generates the angles which have a sine of -1 and a cosine of 0.

$$\begin{aligned} & a \sin x + b \cos x = b \\ \Rightarrow \quad & a\left(\frac{2t}{1+t^2}\right) + b\left(\frac{1-t^2}{1+t^2}\right) = b \\ \Rightarrow \quad & 2at + b - bt^2 = b + bt^2 \\ \Rightarrow \quad & 2bt^2 - 2at = 0 \\ \Rightarrow \quad & 2t(bt - a) = 0 \\ \Rightarrow \quad & t = 0 \text{ or } t = \tfrac{a}{b}. \end{aligned}$$

Note that the case $t = 0$ generates the angles which have a sine of 0 and a cosine of 1.

And the answer to that integral is . . .

$$
\begin{aligned}
&\int \frac{1}{2\sin x - \cos x + 3}\,\mathrm{d}x \\
&= \int \frac{1}{2\left(\frac{2t}{1+t^2}\right) - \left(\frac{1-t^2}{1+t^2}\right) + 3} \times \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{2}{2(2t) - (1-t^2) + 3(1+t^2)}\,\mathrm{d}t \\
&= \int \frac{2}{4t^2 + 4t + 2}\,\mathrm{d}t \\
&= \int \frac{2}{(2t+1)^2 + 1}\,\mathrm{d}t \\
&= \arctan(2t+1) + c \\
&= \arctan(1 + 2\tan\tfrac{1}{2}x) + c.
\end{aligned}
$$

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