Dr Oliver Mathematics
AQA Further Maths Level 2
June 2021 Paper 1
1 hour 30 minutes

The total number of marks available is 80.
You must write down all the stages in your working.
You are **not** permitted to use a scientific or graphical calculator in this paper.

1. Work out the distance between the points $A(-3, 7)$ and $B(5, 1)$. (2)

Solution

$$\begin{aligned} AB &= \sqrt{[5 - (-3)]^2 + (7 - 1)^2} \\ &= \sqrt{8^2 + 6^2} \\ &= \sqrt{64 + 36} \\ &= \sqrt{100} \\ &= \underline{\underline{10 \text{ cm}}}. \end{aligned}$$

2. (3)

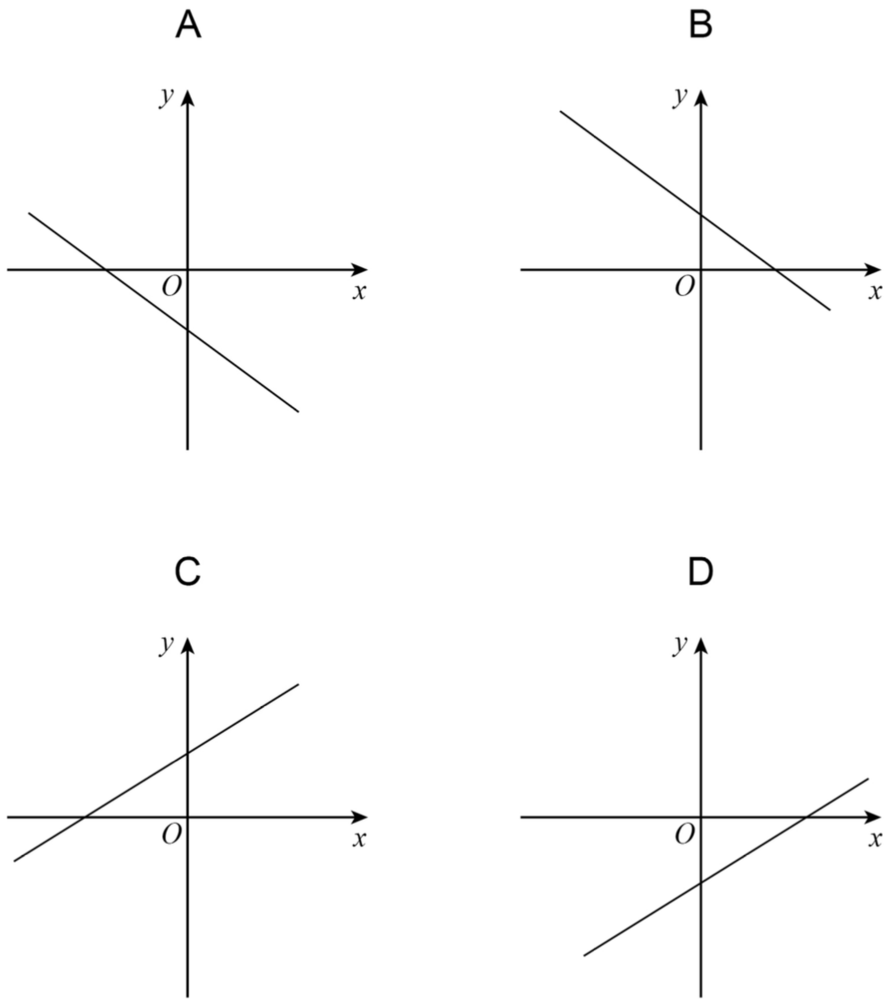
$$y = x(2x^4 - 7x^3).$$

Work out an expression for the rate of change of y with respect to x.

Solution

$$\begin{aligned} y = x(2x^4 - 7x^3) &\Rightarrow y = 2x^5 - 7x^4 \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 10x^4 - 28x^3}}. \end{aligned}$$

3. Here are four sketch graphs. (1)



Circle the letter of the sketch graph that represents

$$3x + 2y = 5.$$

Solution

$$\begin{aligned} 3x + 2y = 5 &\Rightarrow 2y = -3x + 5 \\ &\Rightarrow y = -\tfrac{3}{2}x + \tfrac{5}{2} \end{aligned}$$

so the answer is $\underline{\underline{\text{B}}}$.

4. The function f is given by

$$f(x) = 3x - 5.$$

The range is

$$13 < f(x) < 19.$$

(a) Work out the domain of the function. (1)

Solution
Well,

$$\begin{aligned} 3x - 5 = 13 &\Rightarrow 3x = 18 \\ &\Rightarrow x = 6 \end{aligned}$$

and

$$\begin{aligned} 3x - 5 = 19 &\Rightarrow 3x = 24 \\ &\Rightarrow x = 8; \end{aligned}$$

hence, the domain of the function is $\underline{\underline{6 < x < 8}}$.

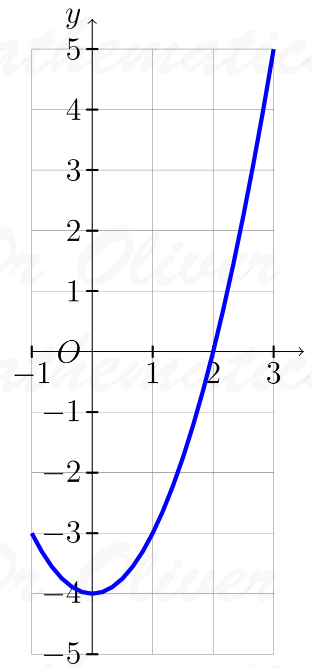
The function g is given by

$$g(x) = x^2 - 4$$

with domain $-1 < x < 3$.

(b) Work out the range of the function. (2)

Solution



Hence, the range of the function is $\underline{\underline{-4 \leqslant \mathrm{g}(x) < 5}}$.

The function h is given by

$$\mathrm{h}(x) = \frac{3+x}{2}.$$

(c) Work out $\mathrm{h}^{-1}(x)$. (2)

Solution

$$\begin{aligned} y = \frac{3+x}{2} &\Rightarrow 2y = 3 + x \\ &\Rightarrow 2y - 3 = x \end{aligned}$$

and

$$\mathrm{h}^{-1}(x) = \underline{\underline{2x-3}}.$$

5. The nth term of a sequence is

$$\frac{2n+47}{n+1}.$$

A term of the sequence has a value of 5.

(a) Work out the value of n. (2)

Solution

$$\frac{2n+47}{n+1} = 5 \Rightarrow 2n+47 = 5(n+1)$$
$$\Rightarrow 2n+47 = 5n+5$$
$$\Rightarrow 42 = 3n$$
$$\Rightarrow \underline{\underline{n = 14}}.$$

(b) Write down the limiting value of the sequence as $n \to \infty$. (1)

Solution

Divide top and bottom by n:

$$\frac{2n+47}{n+1} = \frac{2+\frac{47}{n}}{1+\frac{1}{n}}$$
$$\to \frac{2+0}{1+0}$$
$$= \underline{\underline{2}}$$

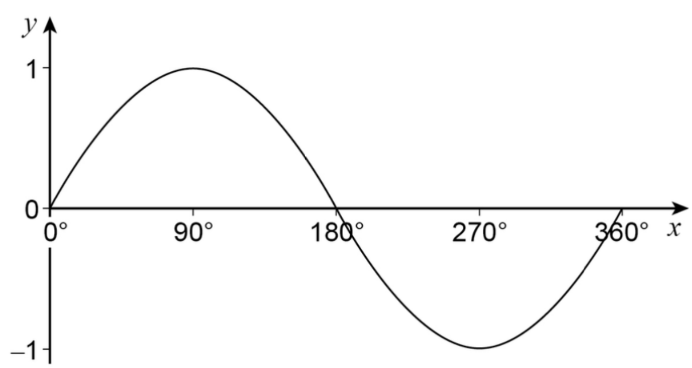
as $n \to \infty$.

6. Here is a sketch of (2)

$$y = \sin x$$

for

$$0^\circ \leqslant x \leqslant 360^\circ.$$



You are given that

$$\sin 220^\circ = -k.$$

Work out the two values of x for $0^\circ \leqslant x \leqslant 360^\circ$ for which $y = k$.

Solution

Well,

$$\sin 220^\circ = \sin(180 + 40)^\circ$$

so $\sin(180 - 40)^\circ$ has the opposite sign to $\sin(180 + 40)^\circ$.

Hence, the two values are $\underline{\underline{\sin 40^\circ}}$ and $\underline{\underline{\sin 140^\circ}}$.

7. Solve (3)

$$2x^2 + 4 > (2x - 3)(x + 1).$$

Solution

$\times$	$2x$	-3
x	$2x^2$	$-3x$
$+1$	$+2x$	-3

So,

$$\begin{aligned} 2x^2 + 4 > (2x - 3)(x + 1) &\Rightarrow 2x^2 + 4 > 2x^2 - x - 3 \\ &\Rightarrow \underline{\underline{x > -7}}. \end{aligned}$$

8. Simplify (2)

$$\sqrt{3}(\sqrt{75} + \sqrt{48}),$$

writing your answer as an integer.

Solution

Now,

$$\begin{aligned}\sqrt{3} \times \sqrt{75} &= \sqrt{3 \times 75} \\ &= \sqrt{225} \\ &15\end{aligned}$$

and

$$\begin{aligned}\sqrt{3} \times \sqrt{48} &= \sqrt{3 \times 48} \\ &= \sqrt{144} \\ &12.\end{aligned}$$

Hence,

$$\begin{aligned}\sqrt{3}(\sqrt{75} + \sqrt{48}) &= (\sqrt{3} \times \sqrt{75}) + (\sqrt{3} \times \sqrt{48}) \\ &= 15 + 12 \\ &= \underline{\underline{27}}.\end{aligned}$$

9. Expand and simplify fully (3)

$$(2x - 5)(3x - 4)(x + 2).$$

Solution

Well,

$\times$	$2x$	-5
$3x$	$6x^2$	$-15x$
-4	$-8x$	$+20$

and

$$(2x - 5)(3x - 4) = 6x^2 - 23x + 20.$$

$$\begin{array}{c|ccc}
\hline
\times & 6x^2 & -23x & +20 \\
\hline
x & 6x^3 & -23x^2 & +20x \\
+2 & +12x^2 & -46x & +40 \\
\hline
\end{array}$$

and

$$(2x-5)(3x-4)(x+2) = \underline{\underline{6x^3 - 11x^2 - 26x + 40}}.$$

10. The first four terms of a quadratic sequence are (3)

$$0 \quad 1 \quad 0 \quad -3.$$

Work out an expression for the nth term.

Solution

Let the

$$n\text{th term} = an^2 + bn + c.$$

Then

$$\begin{array}{ccccccc}
{\color{red}0} & & 1 & & 0 & & -3 \\
& {\color{blue}1} & & -1 & & -3 & \\
& & {\color{magenta}-2} & & -2 & &
\end{array}$$

and

$$\begin{array}{ccccccc}
{\color{red}a+b+c} & & 4a+2b+c & & 9a+3b+c & & 16a+4b+c \\
& {\color{blue}3a+b} & & 5a+b & & 7a+b & \\
& & {\color{magenta}2a} & & 2a & &
\end{array}$$

We compare terms:

$${\color{magenta}2a = -2 \Rightarrow a = -1,}$$

$$\begin{aligned}
{\color{blue}3a + b = 6} &\Rightarrow 3 \times (-1) + b = 1 \\
&\Rightarrow b = 4,
\end{aligned}$$

and

$$a + b + c = 0 \Rightarrow -1 + 4 + c = 0$$
$$\Rightarrow c = -3;$$

hence,

$$n\text{th term} = \underline{\underline{-n^2 + 4n - 3}}.$$

11. (4)

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0.4 \end{pmatrix} = k\mathbf{I},$$

where k is a constant and $\mathbf{I}$ is the identity matrix.

Work out the values of a and b.

Solution

Well,

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0.4 \end{pmatrix} = \begin{pmatrix} 2a & 2b + 0.4 \\ 0 & 1.2 \end{pmatrix}.$$

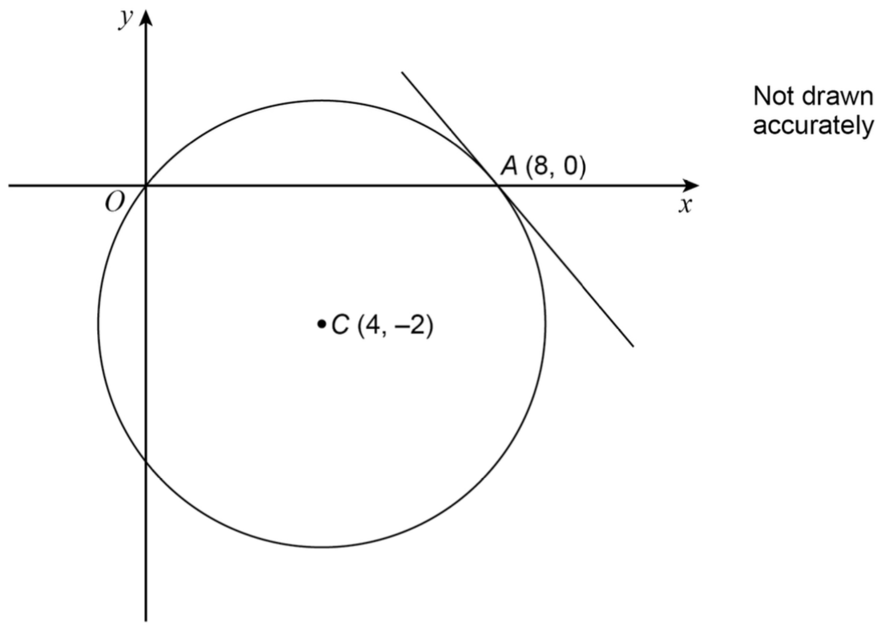
Now,

$$(1,2)\text{th is zero} \Rightarrow 2b + 0.4 = 0$$
$$\Rightarrow 2b = -0.4$$
$$\Rightarrow \underline{\underline{b = -0.2}}$$

and

$$(1,1)\text{th is } 1.2 \Rightarrow 2a = 1.2$$
$$\Rightarrow \underline{\underline{a = 0.6}}.$$

12. A circle, centre $C(4, -2)$, passes through the origin and point $A(8, 0)$ on the x-axis. The tangent at A is shown.



(a) Work out the equation of the circle. (2)

Solution

$$\begin{aligned} AC^2 &= (8-4)^2 + [(0-(-2)]^2 \\ &= 4^2 + 2^2 \\ &= 16 + 4 \\ &= 20 \end{aligned}$$

and, hence, the equation is

$$\underline{\underline{(x-4)^2 + (y+2)^2 = 20}}.$$

(b) Work out the equation of the tangent to the circle at A. (3)

Solution
Well,

$$\begin{aligned} m_{AC} &= \frac{0-(-2)}{8-4} \\ &= \tfrac{1}{2} \end{aligned}$$

and

$$m_{\text{tangent}} = -\frac{1}{\frac{1}{2}} = -2.$$

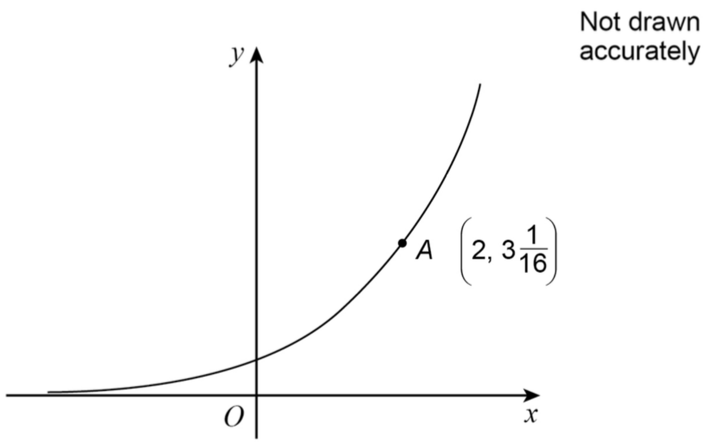
Hence, the equation of the tangent to the circle is

$$y - 0 = -2(x - 8) \Rightarrow \underline{\underline{y = -2x + 16}}.$$

13. Here is a sketch of

$$y = k^x,$$

where $k > 0$.
$A(2, 3\frac{1}{16})$ is a point on the curve.



(a) Work out the value of k. (2)

Solution
Well,

$$\begin{aligned} x = 2, y = 3\tfrac{1}{16} &\Rightarrow 3\tfrac{1}{16} = k^2 \\ &\Rightarrow \tfrac{49}{16} = k^2 \\ &\Rightarrow k = \sqrt{\frac{49}{16}} \\ &\Rightarrow k = \frac{\sqrt{49}}{\sqrt{16}} \\ &\Rightarrow \underline{\underline{k = \tfrac{7}{4}}}. \end{aligned}$$

B is a point on the curve with x-coordinate -1.

(b) Work out the y-coordinate of B. (1)

Solution

$$x = -1 \Rightarrow y = \left(\tfrac{7}{4}\right)^{-1}$$
$$\Rightarrow \underline{\underline{y = \tfrac{4}{7}}}.$$

14. Solve the simultaneous equations: (5)

$$4a - b + 3c = 27$$
$$3a + 2b - c = 5$$
$$2a - 5c = -7.$$

Do **not** use trial and improvement.
You **must** show your working.

Solution

$$4a - b + 3c = 27 \quad (1)$$
$$3a + 2b - c = 5 \quad (2)$$
$$2a - 5c = -7 \quad (3)$$

Do $3 \times (1)$ and $4 \times (2)$:

$$12a - 3b + 9c = 81 \quad (4)$$
$$12a + 8b - 4c = 20 \quad (5)$$

and do $(4) - (5)$:

$$\boxed{-11b + 13c = 61 \quad (6)}$$

Now, do $2 \times (3)$:

$$4a - 10c = -14 \quad (7)$$

and do $(1) - (7)$:

$$\boxed{-b + 13c = 41 \quad (8)}$$

Next, do $(6) = (8)$:

$$-10b = 20 \Rightarrow b = -2;$$

from (6):

$$\begin{aligned} &\Rightarrow -11(-2) + 13c = 61 \\ &\Rightarrow 22 + 13c = 61 \\ &\Rightarrow 13c = 39 \\ &\Rightarrow c = 3; \end{aligned}$$

from (3):

$$\begin{aligned} &\Rightarrow 2a - 5(3) = -7 \\ &\Rightarrow 2a - 15 = -7 \\ &\Rightarrow 2a = 8 \\ &\Rightarrow a = 4. \end{aligned}$$

Hence,

$$\underline{\underline{a = 4,\ b = -2,\ \text{and } c = 3}}.$$

15. Work out the value of x where $0° \leqslant x \leqslant 90°$ for which (2)

$$3 \tan^2 x = 1.$$

Solution

Well,

$$3 \tan^2 x = 1 \Rightarrow \tan^2 x = \tfrac{1}{3}$$

we know $0° \leqslant x \leqslant 90°$:

$$\begin{aligned} &\Rightarrow \tan x = \tfrac{1}{\sqrt{3}} \\ &\Rightarrow \underline{\underline{x = 30}}. \end{aligned}$$

16.

$$\mathrm{f}(x) = 200x^3 + 100x^2 - 18x - 9.$$

(a) Use the factor theorem to show that $(2x + 1)$ is a factor of $\mathrm{f}(x)$. (2)

Solution

Well,

$$\begin{aligned} \mathrm{f}(-\tfrac{1}{2}) &= 200[(-\tfrac{1}{2})]^3 + 100[(-\tfrac{1}{2})]^2 - 18(-\tfrac{1}{2}) - 9 \\ &= -25 + 25 + 9 - 9 \\ &= 0; \end{aligned}$$

there is no remainder and, hence, $(2x + 1)$ is a factor of $\mathrm{f}(x)$.

(b) Hence solve $\mathrm{f}(x) = 0$. (3)

Solution

Well, we can spot the the first and third terms are precisely double the second and fourth terms respectively:

$$\begin{aligned} 200x^3 + 100x^2 - 18x - 9 &= 100x^2(2x + 1) - 9(2x + 1) \\ &= (100x^2 - 9)(2x + 1) \\ &= (10x - 3)(10x + 3)(2x + 1), \end{aligned}$$

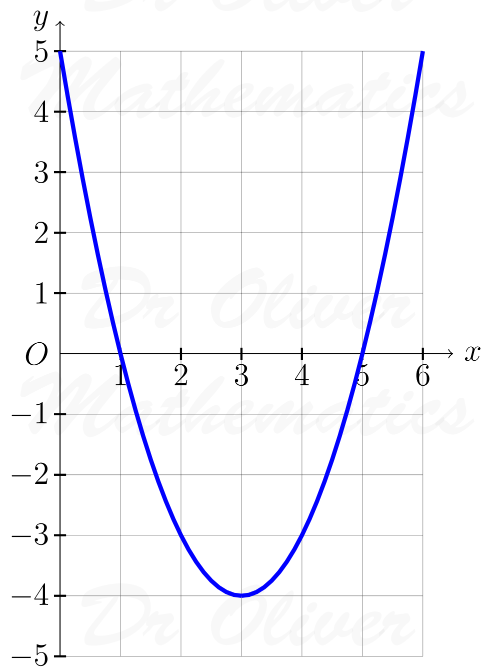
using the difference of two squares. Finally,

$$\begin{aligned} 200x^3 + 100x^2 - 18x - 9 = 0 &\Rightarrow (10x - 3)(10x + 3)(2x + 1) = 0 \\ &\Rightarrow 10x - 3 = 0,\ 10x + 3 = 0,\ \text{or } 2x + 1 = 0 \\ &\Rightarrow x = \tfrac{3}{10},\ x = -\tfrac{3}{10},\ \text{or } x = -\tfrac{1}{2}. \end{aligned}$$

17. Here is the graph of (3)

$$y = x^2 - 6x + 5,$$

for values of x between 0 and 6.



By drawing a suitable **linear** graph on the grid, work out approximate solutions to

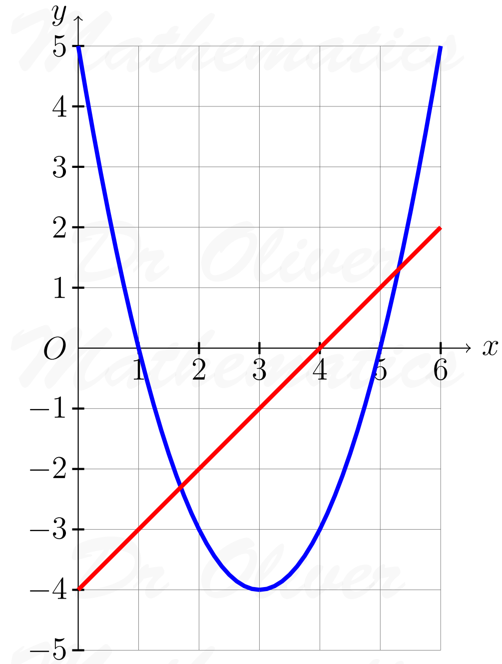
$$x^2 - 7x + 9 = 0.$$

Solution

Well,

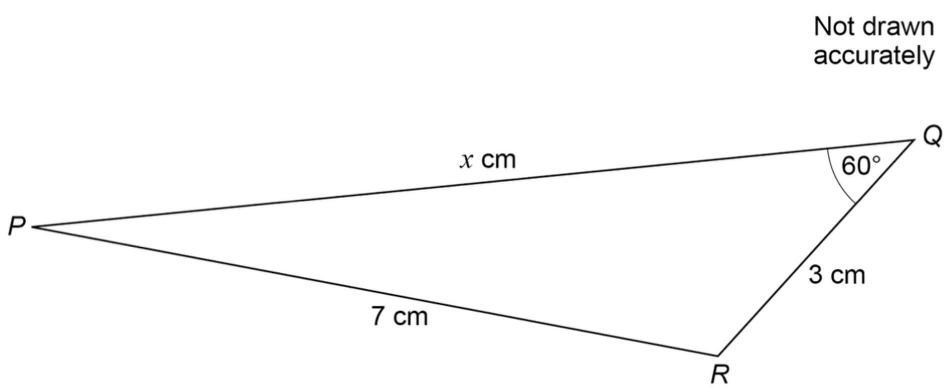
$$x^2 - 7x + 9 = 0 \Leftrightarrow x^2 - 6x + 5 = x - 4$$

and so we need to draw on $y = x - 4$:



Correct read-off: approximately $\underline{\underline{x = 1.8}}$ and $\underline{\underline{x = 5.2}}$.

18. Here is a triangle. (4)



Use the cosine rule to work out the value of x.

Solution

Now,

$$
\begin{aligned}
& PR^2 = PQ^2 + QR^2 - 2 \times PQ \times QR \times \cos PQR \\
\Rightarrow \quad & 7^2 = x^2 + 3^2 - 2 \times x \times 3 \times \cos 60^\circ \\
\Rightarrow \quad & 49 = x^2 + 9 - 3x \\
\Rightarrow \quad & x^2 - 3x - 40 = 0
\end{aligned}
$$

$$
\left.\begin{array}{ll} \text{add to:} & -3 \\ \text{multiply to:} & -40 \end{array}\right\} -8, +5
$$

$$
\begin{aligned}
\Rightarrow \quad & (x-8)(x+5) = 0 \\
\Rightarrow \quad & x - 8 = 0 \text{ or } x + 5 = 0 \\
\Rightarrow \quad & x = 8 \text{ or } x = -5.
\end{aligned}
$$

But $x \neq -5$ — it is a length! — so $\underline{\underline{x = 8}}$.

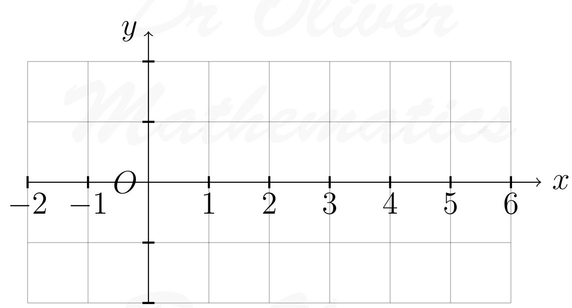
19. $y = \mathrm{f}(x)$ is the graph of a cubic function. (4)

- $y < 0$ for $x < 5$.
- $y \geqslant 0$ for $x \geqslant 5$.

The function is

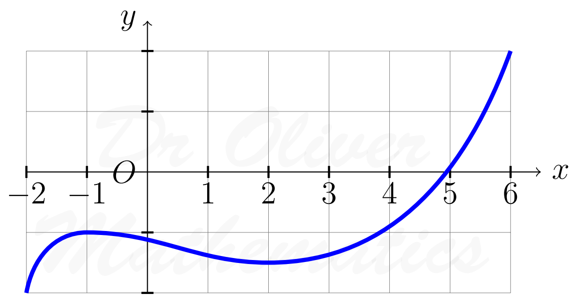
- increasing for $x < -1$,
- decreasing for $-1 < x < 2$, and
- increasing for $x > 2$.

Draw a possible sketch of $y = \mathrm{f}(x)$ for values of x from -2 to 6.



Solution

E.g.,



20. Miriam's date of birth is 14/09/2006. (3)

She makes a 4-digit number code using digits from her date of birth.

The 4-digit number she makes must

- not start with 0 and
- have all different digits.

How many codes can she make?

Solution

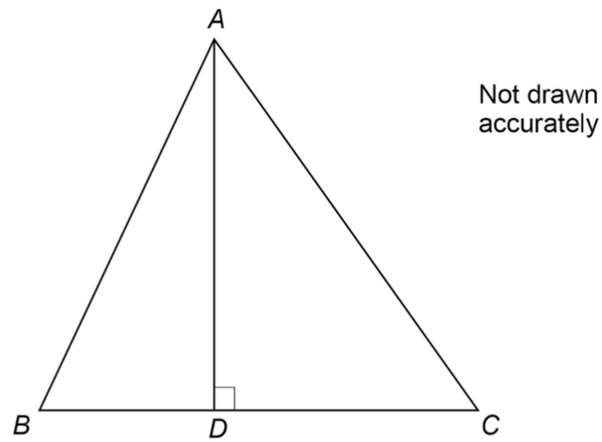
She has six digits (1, 4, 0, 9, 2, or 6) and she has

- 5 choices for the first one $(1 - 9)$,
- 5 choices for the second one (all different digits),
- 4 choices for the second one (all different digits), and
- 3 choices for the second one (all different digits).

So,

$$5 \times 5 \times 4 \times 3 = \underline{\underline{300 \text{ codes}}}.$$

21. ABC is a triangle. (5)
The perpendicular from A meets BC at D.
$BC = (6 + 2\sqrt{7})$ cm.



Area of triangle $ABC = (13 + 3\sqrt{7})$ cm^2.

Work out the length, in cm, of AD.
Give your answer in the form

$$a + b\sqrt{c},$$

where a, b, and c are integers.

Solution

Now,

$$A = \tfrac{1}{2}bh \Rightarrow 13 + 3\sqrt{7} = \tfrac{1}{2} \times (6 + 2\sqrt{7}) \times AD$$
$$\Rightarrow AD = \frac{26 + 6\sqrt{7}}{6 + 2\sqrt{7}}$$
$$\Rightarrow AD = \frac{26 + 6\sqrt{7}}{6 + 2\sqrt{7}} \times \frac{6 - 2\sqrt{7}}{6 - 2\sqrt{7}}.$$

Next,

$\times$	26	$+6\sqrt{7}$
6	156	$+36\sqrt{7}$
$-2\sqrt{7}$	$-52\sqrt{7}$	-84

and

$\times$	6	$+2\sqrt{7}$
6	36	$+12\sqrt{7}$
$-2\sqrt{7}$	$-12\sqrt{7}$	-28

So,

$$\begin{aligned} AD &= \frac{72 - 16\sqrt{7}}{8} \\ &= \underline{\underline{\left(9 - 2\sqrt{7}\right) \text{ cm}}}. \end{aligned}$$

22. Solve (4)

$$8^x = \frac{2^{56} - 4^{26}}{30}.$$

Solution

Well,

$$\begin{aligned} 2^{56} - 4^{26} &\Rightarrow 2^{56} - (2^2)^{26} \\ &= 2^{56} - 2^{2 \times 26} \\ &= 2^{56} - 2^{52} \\ &= 2^{52}(2^4 - 1) \\ &= 2^{52}(16 - 1) \\ &= 15 \times 2^{52} \end{aligned}$$

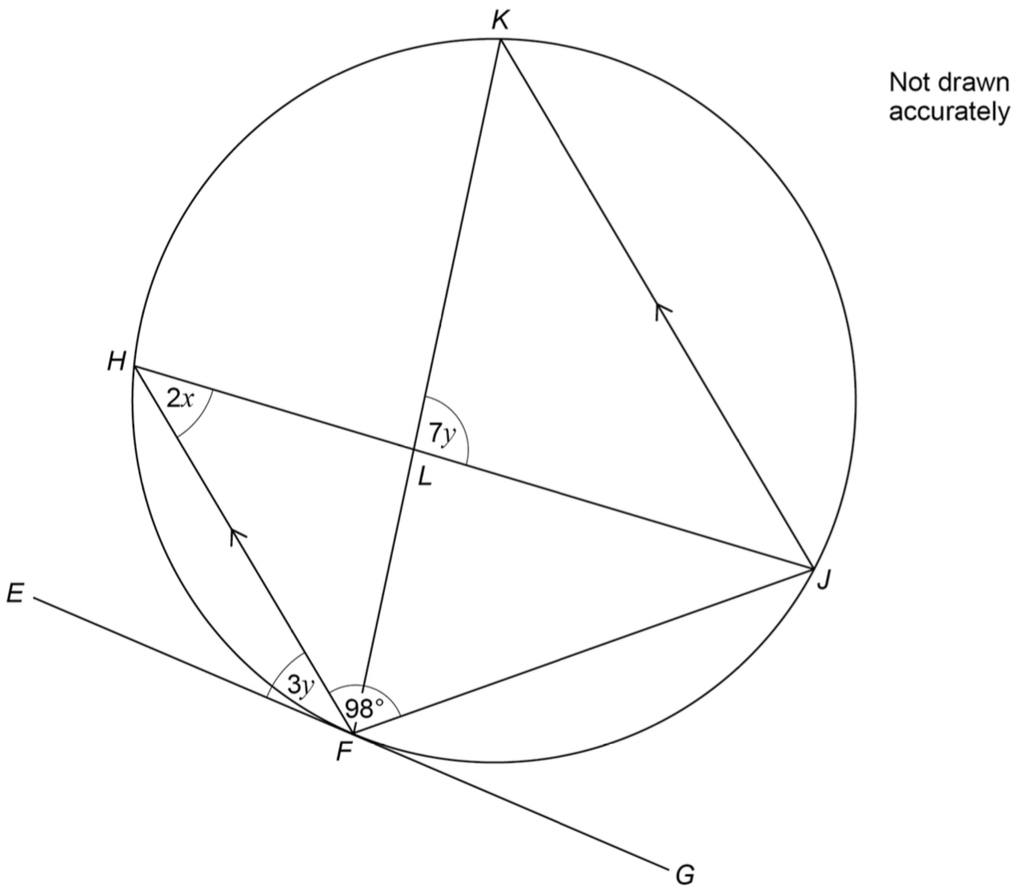
and

$$8^3 = (2^3)^x = 2^{3x}.$$

Finally,

$$\begin{aligned} 8^x = \frac{2^{56} - 4^{26}}{30} &\Rightarrow 2^{3x} = \frac{15 \times 2^{52}}{30} \\ &\Rightarrow 2^{3x} = \frac{2^{52}}{2} \\ &\Rightarrow 2 \times 2^{3x} = 2^{52} \\ &\Rightarrow 2^{3x+1} = 2^{52} \\ &\Rightarrow 3x + 1 = 52 \\ &\Rightarrow 3x = 51 \\ &\Rightarrow \underline{\underline{x = 17}}. \end{aligned}$$

23.
- F, H, K, and J are points on a circle.
- Chords HJ and KF intersect at L.
- EFG is a tangent to the circle.
- FH and JK are parallel.



Angle $FHJ = 2x$.

(a) Give reasons why angle FKJ and angle HJK are also equal to $2x$. (2)

Solution

$\angle FKJ = 2x$ — angles in the same segment.
$\angle HJK = 2x$ — alternate angles.

(b) Work out the values of x and y. (4)
You **must** show your working.
Do **not** use trial and improvement.

Solution

Alternate segment theorem:

$$\angle FJH = 3y$$

and angles in a triangle:

$$2x + 3y + 98 = 180 \Rightarrow \boxed{2x + 3y = 82 \quad (1)}$$

Angles in a triangle:

$$2x + 2x + 7y = 180 \Rightarrow \boxed{4x + 7y = 180 \quad (2)}$$

Do $2 \times (1)$:

$$4x + 6y = 164 \quad (3)$$

and do $(2) - (3)$:

$$\underline{\underline{y = 16}}.$$

Substitute y into (1):

$$\begin{aligned} 2x + 3(16) = 82 &\Rightarrow 2x + 48 = 82 \\ &\Rightarrow 2x = 34 \\ &\Rightarrow \underline{\underline{x = 17}}. \end{aligned}$$