Dr Oliver Mathematics
Mathematics
Integration Part 2
Past Examination Questions

This booklet consists of 26 questions across a variety of examination topics.
The total number of marks available is 197.

1. The line with equation $y = 3x + 20$ cuts the curve with equation $y = x^2 + 6x + 10$ and the points A and B, as shown in Figure 1.

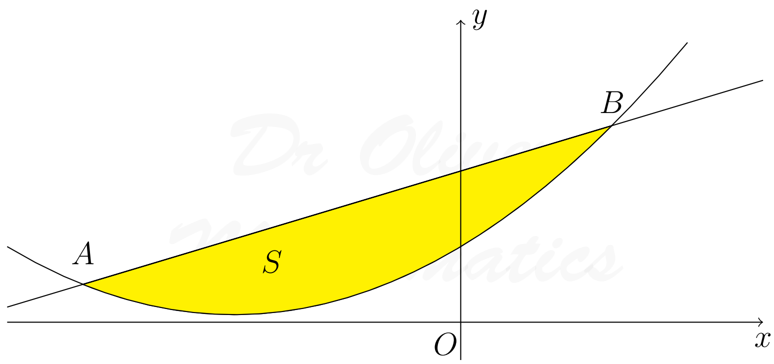

Figure 1: $y = 3x + 20$ and $y = x^2 + 6x + 10$

(a) Use algebra to find the coordinates of A and find the coordinates of B. (5)

Solution

$$\begin{aligned} x^2 + 6x + 10 = 3x + 20 &\Rightarrow x^2 + 3x - 10 = 0 \\ &\Rightarrow (x+5)(x-2) = 0 \\ &\Rightarrow \underline{\underline{x = -5 \text{ or } x = 2}}. \end{aligned}$$

The shaded region S is bounded by the line and the curve, as shown in Figure 1.

(b) Use calculus to find exact area of S. (7)

Solution

The area may be found by

$$\text{area} = \text{area of the line} - \text{area of the curve}.$$

Well, $A(-5, 5)$ and $B(2, 26)$.

$$\begin{aligned}\text{Area of the line} &= \tfrac{1}{2} \times (2 - (-5)) \times (5 + 26) \\ &= \tfrac{1}{2} \times 7 \times 31 \\ &= 108\tfrac{1}{2}\end{aligned}$$

and

$$\begin{aligned}\text{area of the curve} &= \int_{-5}^{2} (x^2 + 6x + 10)\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3 + 3x^2 + 10x\right]_{x=-5}^{2} \\ &= (\tfrac{8}{3} + 12 + 20) - (-\tfrac{125}{3} + 75 - 50) \\ &= 51\tfrac{1}{3}.\end{aligned}$$

Thus

$$\text{area} = 108\tfrac{1}{2} - 51\tfrac{1}{3} = \underline{\underline{57\tfrac{1}{6}\text{ units}^2}}.$$

2. Figure 2 shows part of a curve C with equation $y = 2x + \dfrac{8}{x^2} - 5$, $x > 0$. (8)

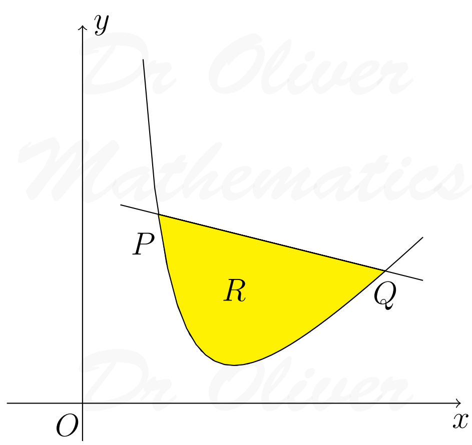

Figure 2: $y = 2x + \dfrac{8}{x^2} - 5$

The points P and Q lie on C and have x-coordinates 1 and 4 respectively. The region R, shaded in the figure, is bounded by C and the straight line joining P and Q. Find the exact area of R.

Solution

The area may be found by

$$\text{area} = \text{area of the line} - \text{area of the curve}.$$

Now, $P(1, 5)$ and $Q(4, 3\frac{1}{2})$ and so we have

$$\begin{aligned}\text{area of the line} &= \tfrac{1}{2} \times 3 \times (5 + 3\tfrac{1}{2}) \\ &= 12\tfrac{3}{4}.\end{aligned}$$

Also,

$$\begin{aligned}\text{area of the curve} &= \int_1^4 \left(2x + 8x^{-2} - 5\right)\,\mathrm{d}x \\ &= \left[x^2 - 8x^{-1} - 5x\right]_{x=1}^4 \\ &= (16 - 2 - 20) - (1 - 8 - 5) \\ &= 6.\end{aligned}$$

Thus

$$\text{area} = 12\tfrac{3}{4} - 6 = \underline{\underline{6\tfrac{3}{4}\text{ units}^2}}.$$

3. Figure 3 shows the shaded region R which is bounded by the curve $y = -2x^2 + 4x$ and the line $y = \frac{3}{2}$.

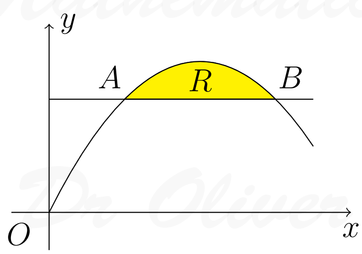

Figure 3: $y = -2x^2 + 4x$ and $y = \frac{3}{2}$

The points A and B are the points of intersection of the line and the curve. Find

(a) the x-coordinates of the points A and B, (4)

Solution

$$\begin{aligned}-2x^2+4x=\tfrac{3}{2} &\Rightarrow -4x^2+8x=3\\ &\Rightarrow 4x^2-8x+3=0\\ &\Rightarrow (2x-1)(2x-3)=0\\ &\Rightarrow \underline{\underline{x=\tfrac{1}{2} \text{ or } x=1\tfrac{1}{2}.}}\end{aligned}$$

(b) the exact area of R. (6)

Solution

We have

$$\text{area} = \text{area of the curve} - \text{area of the rectangle}.$$

Now,

$$\begin{aligned}\text{area of the curve} &= \int_{0.5}^{1.5} \left(-2x^2+4x\right)\,\mathrm{d}x\\ &= \left[-\tfrac{2}{3}x^3+2x^2\right]_{x=0.5}^{1.5}\\ &= \left(-\tfrac{9}{4}+\tfrac{9}{2}\right)-\left(-\tfrac{1}{12}+\tfrac{1}{2}\right)\\ &= 1\tfrac{5}{6}\end{aligned}$$

and

$$\text{area of the rectangle} = 1\times 1\tfrac{1}{2} = 1\tfrac{1}{2}.$$

Hence,

$$\text{area} = 1\tfrac{5}{6}-1\tfrac{1}{2} = \underline{\underline{\tfrac{1}{3}\text{ units}^2}}.$$

4. Use calculus to find the exact value of $\displaystyle\int_1^2 \left(3x^2+5+\frac{4}{x^2}\right)\,\mathrm{d}x$. (5)

Solution

$$\begin{aligned}\int_1^2 \left(3x^2+5+\frac{4}{x^2}\right)\mathrm{d}x &= \int_1^2 \left(3x^2+5+4x^{-2}\right)\mathrm{d}x\\ &= \left[x^3+5x-4x^{-1}\right]_{x=1}^{2}\\ &= (8+10-2)-(1+5-4)\\ &= \underline{\underline{14\text{ units}^2}}.\end{aligned}$$

5. Figure 4 shows a sketch of part of the curve with equation $y = x^3 - 8x^2 + 20x$.

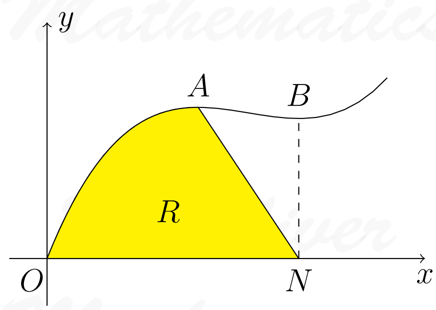

Figure 4: $y = x^3 - 8x^2 + 20x$

The curve has stationary points A and B.

(a) Use calculus to find the x-coordinates of A and B. (4)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 3x^2 - 16x + 20 = 0$$
$$\Rightarrow (3x - 10)(x - 2) = 0$$
$$\Rightarrow \underline{\underline{x = 2 \text{ or } x = 3\tfrac{1}{3}}}.$$

The line through B parallel to the y-axis meets the x-axis at the point N. The region R, shown shaded in the figure, is bounded by the curve, the x-axis, and the line from A and N.

(b) Find $\int \left(x^3 - 8x^2 + 20x\right)\,\mathrm{d}x$. (3)

Solution

$$\int \left(x^3 - 8x^2 + 20x\right)\,\mathrm{d}x = \underline{\underline{\tfrac{1}{4}x^4 - \tfrac{8}{3}x^3 + 10x^2 + c}}.$$

(c) Hence find the exact area of R. (5)

Solution

$A(2, 16)$, $B(3\frac{1}{3}, 14\frac{22}{27})$, and

area = area of the curve + area of the triangle.

Now,

$$\begin{aligned} \text{area of the curve} &= \left[\tfrac{1}{4}x^4 - \tfrac{8}{3}x^3 + 10x^2\right]_{x=0}^{2} \\ &= (4 - \tfrac{64}{3} + 40) - (0 - 0 + 0) \\ &= 22\tfrac{2}{3} \end{aligned}$$

and

$$\text{area of the triangle} = \tfrac{1}{2} \times 16 \times \tfrac{4}{3} = 10\tfrac{2}{3}.$$

Hence,

$$\text{area} = 22\tfrac{2}{3} + 10\tfrac{2}{3} = 33\tfrac{1}{3} \text{ units}^2.$$

6. Find $\displaystyle\int_1^2 \left(x^3 + 3x^2 + 5\right)\,\mathrm{d}x$. (4)

Solution

$$\begin{aligned} \int_1^2 \left(x^3 + 3x^2 + 5\right)\,\mathrm{d}x &= \left[\tfrac{1}{4}x^4 + x^3 + 5x\right]_{x=1}^{2} \\ &= (4 + 8 + 10) - (\tfrac{1}{4} + 1 + 5) \\ &= \underline{\underline{15\tfrac{3}{4}}}. \end{aligned}$$

7. Figure 5 shows a sketch of part of the curve C with equation (4)

$$y = x(x-1)(x-5).$$

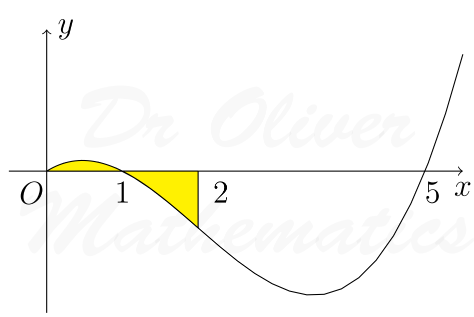

Figure 5: $y = x(x-1)(x-5)$

Use calculus to find the total area of the finite region, shown in the figure, that is, between $x = 0$ and $x = 2$ and is bounded by C, the x-axis, and the line $x = 2$.

Solution

We can find this by

$$\text{area} = \text{area of } [0,1] + \text{area of } [1,2].$$

Now,

$$\begin{aligned}\text{area of } [0,1] &= \int_0^1 x(x-1)(x-5)\,\mathrm{d}x \\ &= \int_0^1 x(x^2-6x+5)\,\mathrm{d}x \\ &= \int_0^1 \left(x^3-6x^2+5x\right)\,\mathrm{d}x \\ &= \left[\tfrac{1}{4}x^4 - 2x^3 + \tfrac{5}{2}x^2\right]_{x=0}^{1} \\ &= (\tfrac{1}{4} - 2 + \tfrac{5}{2}) - (0+0+0) \\ &= \tfrac{3}{4}.\end{aligned}$$

Also,

$$\begin{aligned}\int_1^2 \left(x^3-6x^2+5x\right)\,\mathrm{d}x &= \left[\tfrac{1}{4}x^4 - 2x^3 + \tfrac{5}{2}x^2\right]_{x=1}^{2} \\ &= (4-16+10) - (\tfrac{1}{4} - 2 + \tfrac{5}{2}) \\ &= -2\tfrac{3}{4}\end{aligned}$$

and

$$\text{area of } [1,2] = 2\tfrac{3}{4}.$$

Hence,

$$\text{area} = \tfrac{3}{4} + 2\tfrac{3}{4} = 3\tfrac{1}{2}\ \text{units}^2.$$

8. Evaluate $\displaystyle\int_1^8 \frac{1}{\sqrt{x}}\,\mathrm{d}x$, giving your answer in the form $a + b\sqrt{2}$, where a and b are integers. (4)

Solution

$$\begin{aligned}\int_1^8 \frac{1}{\sqrt{x}}\,\mathrm{d}x &= \int_1^8 x^{-\frac{1}{2}}\,\mathrm{d}x \\ &= \left[2x^{\frac{1}{2}}\right]_{x=1}^{8} \\ &= \underline{\underline{4\sqrt{2}-2}}.\end{aligned}$$

9. In Figure 6 the curve C has equation $y = 6x - x^2$ and the line L has equation $y = 2x$.

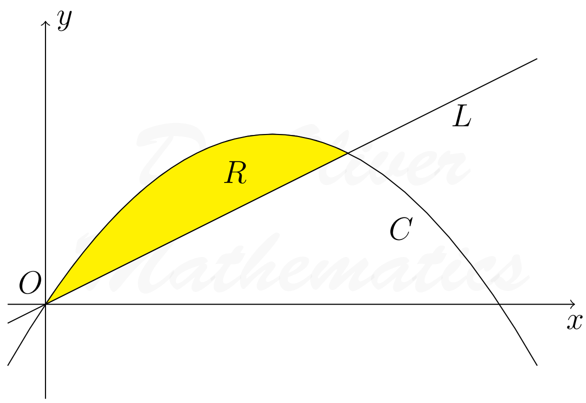

Figure 6: $y = 6x - x^2$ and $y = 2x$

(a) Show that the curve C intersects the x-axis at $x = 0$ and $x = 6$. (1)

Solution

$$\begin{aligned}6x - x^2 = 0 &\Rightarrow x(6-x) = 0 \\ &\Rightarrow \underline{\underline{x = 0 \text{ or } x = 6}}.\end{aligned}$$

(b) Show that the line L intersects the curve C at the points $(0, 0)$ and $(4, 8)$. (3)

Solution

$$\begin{aligned}6x - x^2 = 2x &\Rightarrow 4x - x^2 = 0 \\ &\Rightarrow x(4-x) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 4.\end{aligned}$$

Now, $x = 0 \Rightarrow y = 0$ and $x = 4 \Rightarrow y = 8$ and we have

$$\underline{\underline{(0,0) \text{ and } (4,8)}}.$$

The region R, bounded by the curve C and the line L, is shown shaded in the figure.

(c) Use calculus to find the exact area of R. (6)

Solution

We can find this by

$$\text{area} = \text{area of the curve} - \text{area of the triangle}.$$

Now,

$$\begin{aligned}\text{area of the curve} &= \int_0^4 (6x - x^2)\,\mathrm{d}x \\ &= \left[3x^2 - \tfrac{1}{3}x^3\right]_{x=0}^4 \\ &= (48 - \tfrac{64}{3}) - (0 - 0) \\ &= 26\tfrac{2}{3}\end{aligned}$$

and

$$\text{area of the triangle} = \tfrac{1}{2} \times 8 \times 4 = 16.$$

Hence

$$\text{area} = 26\tfrac{2}{3} - 16 = \underline{\underline{10\tfrac{2}{3} \text{ units}^2}}.$$

10. Figure 7 shows a sketch of part of the curve with equation $y = 10 + 8x + x^2 - x^3$.

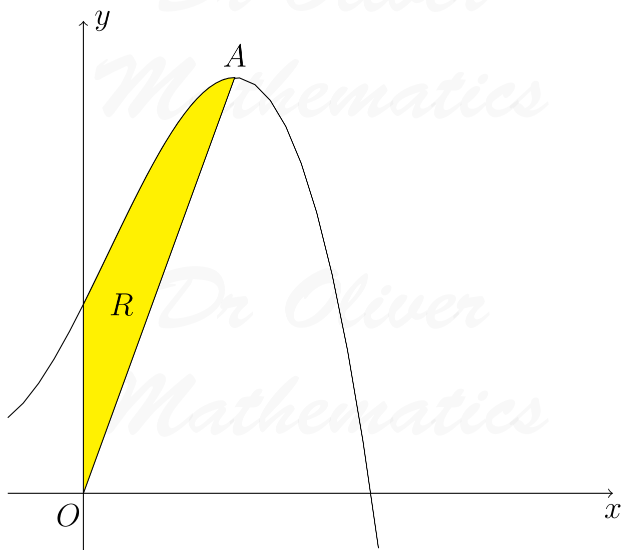

Figure 7: $y = 10 + 8x + x^2 - x^3$

The curve has a maximum turning point A.

(a) Using calculus, show that the x-coordinate of A is 2. (3)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 8 + 2x - 3x^2 = 0\\ &\Rightarrow (2 - x)(4 + 3x) = 0\\ &\Rightarrow \underline{\underline{x = 2}} \text{ or } x = -1\tfrac{1}{3}.\end{aligned}$$

The region R, shown shaded in the figure, is bounded by the curve, the y-axis, and the line from O to A, where O is the origin.

(b) Using calculus, find the exact area of R. (8)

Solution

We can find this by

$$\text{area} = \text{area of the curve} - \text{area of the triangle}.$$

Now, $A(2, 22)$,

$$\begin{aligned}\text{area of the curve} &= \int_0^2 \left(10 + 8x + x^2 - x^3\right)\,\mathrm{d}x \\ &= \left[10x + 4x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_{x=0}^2 \\ &= (20 + 16 + \tfrac{8}{3} - 4) - (0 + 0 + 0 - 0) \\ &= 34\tfrac{2}{3},\end{aligned}$$

and

$$\text{area of the triangle} = \tfrac{1}{2} \times 2 \times 22 = 22.$$

Hence,

$$\text{area} = 34\tfrac{2}{3} - 22 = \underline{\underline{12\tfrac{2}{3}\text{ units}^2}}.$$

11. Figure 8 shows part of the curve C with equation $y = (1 + x)(4 - x)$. (5)

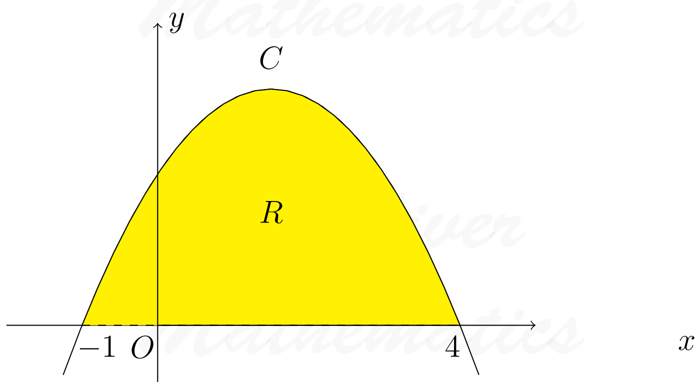

Figure 8: $y = (1 + x)(4 - x)$

The curve intersects the x-axis at $x = -1$ and $x = 4$. The region R, shown shaded in the figure, is bounded by C and the x-axis. Using calculus to find the exact area of R.

Solution

$$\begin{aligned}\text{Area of the curve} &= \int_{-1}^{4} (1+x)(4-x)\,\mathrm{d}x \\ &= \int_{-1}^{4} (4+3x-x^2)\,\mathrm{d}x \\ &= \left[4x+\tfrac{3}{2}x^2-\tfrac{1}{3}x^3\right]_{x=-1}^{4} \\ &= (16+24-\tfrac{64}{3})-(-4+\tfrac{3}{2}+\tfrac{1}{3}) \\ &= \underline{\underline{20\tfrac{5}{6}\text{ units}^2}}.\end{aligned}$$

12. Use calculus to find the value of (5)

$$\int_{1}^{4} \left(2x+3\sqrt{x}\right)\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_{1}^{4} \left(2x+3\sqrt{x}\right)\,\mathrm{d}x &= \int_{1}^{4} \left(2x+3x^{\frac{1}{2}}\right)\,\mathrm{d}x \\ &= \left[x^2+2x^{\frac{3}{2}}\right]_{x=1}^{4} \\ &= (16+16)-(1+2) \\ &= \underline{\underline{29\text{ units}^2}}.\end{aligned}$$

13. The curve C has equation $y = x^2 - 5x + 4$.

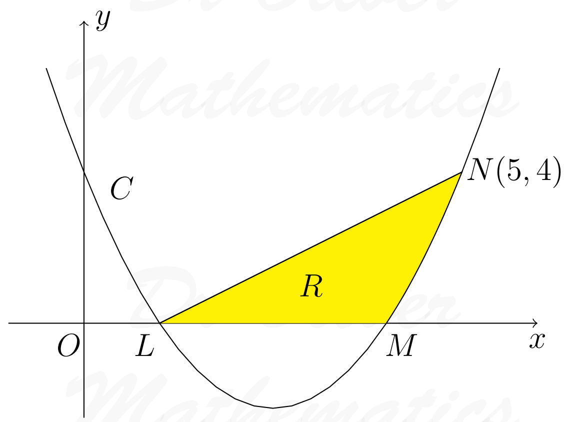

Figure 9: $y = x^2 - 5x + 4$

It cuts the x-axis at the points L and M, as shown the figure.

(a) Find the coordinates of L and find the coordinates of M. (2)

Solution

$$\begin{aligned} x^2 - 5x + 4 = 0 &\Rightarrow (x-1)(x-4) = 0 \\ &\Rightarrow x = 1 \text{ or } x = 4 \end{aligned}$$

and so $\underline{\underline{L(1, 0)}}$ and $\underline{\underline{M(4, 0)}}$.

(b) Show that the point $N(5, 4)$ lies on C. (1)

Solution

$$x = 5 : y = 5^2 - 5 \times 5 + 4 = 25 - 25 + 4 = \underline{\underline{4}}.$$

(c) Find $\int (x^2 - 5x + 4)\,\mathrm{d}x$. (2)

Solution

$$\int (x^2 - 5x + 4)\,\mathrm{d}x = \underline{\underline{\tfrac{1}{3}x^3 - \tfrac{5}{2}x^2 + 4x + c}}.$$

The finite region R is bounded by LM, LN, and the curve C, as shown the figure.

(d) Use your answer to part (c) to find the exact area of R. (5)

Solution

We can find this by

$$\text{area} = \text{area of the triangle} - \text{area of the curve}.$$

Now,

$$\text{area of the triangle} = \tfrac{1}{2} \times 4 \times 4 = 8$$

and

$$\begin{aligned}\int_4^5 (x^2 - 5x + 4)\,\mathrm{d}x &= \left[\tfrac{1}{3}x^3 - \tfrac{5}{2}x^2 + 4x\right]_{x=4}^{5} \\ &= \left(\tfrac{125}{3} - \tfrac{125}{2} + 20\right) - \left(\tfrac{64}{3} - 40 + 16\right) \\ &= 1\tfrac{5}{6}.\end{aligned}$$

Hence,

$$\text{area} = 8 - 1\tfrac{5}{6} = \underline{\underline{6\tfrac{1}{6}\text{ units}^2}}.$$

14. Figure 10 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + kx,$$

where k is a constant.

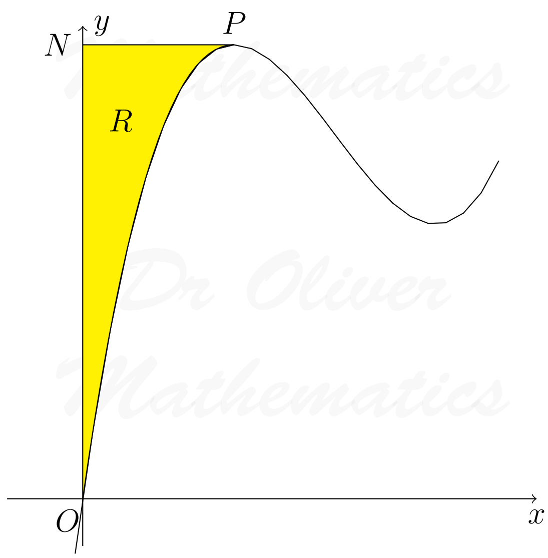

Figure 10: $y = x^3 - 10x^2 + kx$

The point P on C is the local maximum turning point. Given that the x-coordinate of P is 2,

(a) show that $k = 28$. (3)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 20x + k.$$

At $x = 2$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 12 - 40 + k = 0 \Rightarrow \underline{\underline{k = 28}}.$$

The line through P parallel to the x-axis cuts the y-axis at the point N. The region R is bounded by C, the y-axis, and PN, as shown in the figure.

(b) Use calculus to find the exact area of R. (6)

Solution

We can find this by

$$\text{area} = \text{area of the rectangle} - \text{area of the curve}.$$

Now, $P(2, 24)$,

$$\text{area of the rectangle} = 2 \times 24 = 48,$$

and

$$\begin{aligned}\text{area of the curve} &= \int_0^2 (x^3 - 10x^2 + 28x)\,\mathrm{d}x \\ &= \left[\tfrac{1}{4}x^4 - \tfrac{10}{3}x^3 + 14x^2\right]_{x=0}^{2} \\ &= (4 - \tfrac{80}{3} + 56) - (0 - 0 + 0) \\ &= 33\tfrac{1}{3}.\end{aligned}$$

Hence,

$$\text{area} = 48 - 33\tfrac{1}{3} = \underline{\underline{14\tfrac{2}{3}\text{ units}^2}}.$$

15. Figure 11 shows a sketch of part of the curve C with equation

$$y = (x + 1)(x - 5).$$

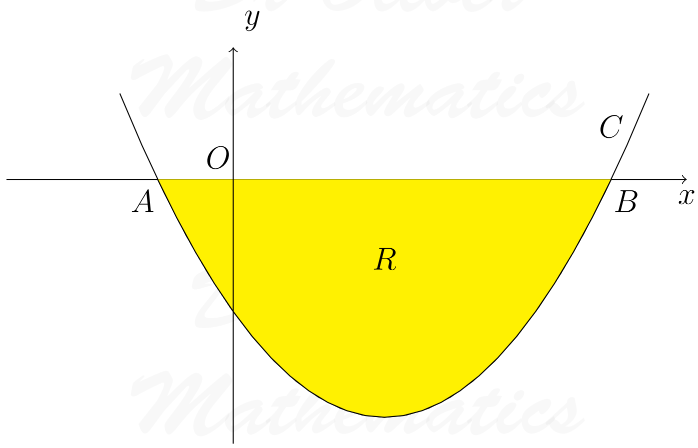

Figure 11: $y = (x + 1)(x - 5)$

The curve crosses the x-axis at the points A and B.

(a) Write down the x-coordinates of A and B. (1)

Solution
$\underline{\underline{A(-1, 0)}}$ and $\underline{\underline{B(5, 0)}}$.

The region R is bounded by C and the x-axis, as shown in the figure.

(b) Use integration to find the exact area of R. (6)

Solution

$$\begin{aligned}\int_{-1}^{5} (x+1)(x-5)\,\mathrm{d}x &= \int_{-1}^{5} (x^2 - 4x - 5)\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3 - 2x^2 - 5x\right]_{x=-1}^{5} \\ &= (\tfrac{125}{3} - 50 - 25) - (-\tfrac{1}{3} - 2 + 5) \\ &= -36\end{aligned}$$

and hence

$$\text{area} = \underline{\underline{36 \text{ units}^2}}.$$

16. The straight line with equation $y = x + 4$ cuts the curve with equation $y = -x^2 + 2x + 24$ at the points A and B, as shown in Figure 12.

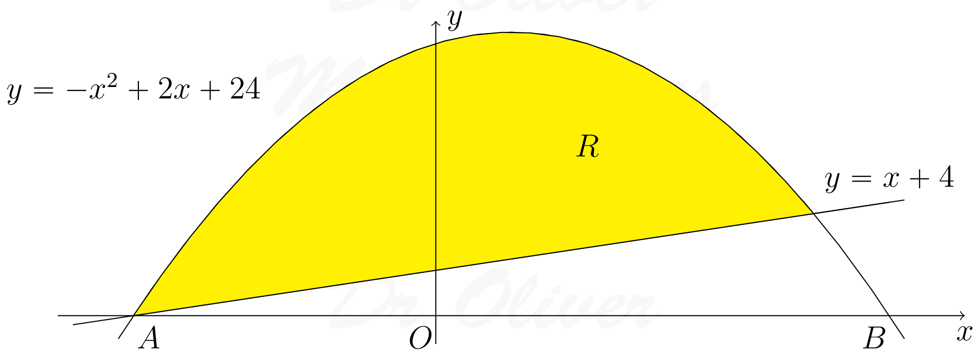

Figure 12: $y = -x^2 + 2x + 24$ and $y = x + 4$

(a) Use algebra to find the coordinates of the points A and B. (4)

Solution

$$\begin{aligned} x + 4 = -x^2 + 2x + 24 &\Rightarrow x^2 - x - 20 = 0 \\ &\Rightarrow (x+4)(x-5) = 0 \\ &\Rightarrow x = -4 \text{ or } x = 5, \end{aligned}$$

and so we have $\underline{\underline{A(-4, 0)}}$ and $\underline{\underline{B(5, 0)}}$.

(b) Use calculus to find the exact area of R. (7)

Solution

We can find this by

area = area of the curve − area of the triangle.

Now,

$$\begin{aligned} \text{area of the curve} &= \int_{-4}^{5} (-x^2 + 2x + 24)\,\mathrm{d}x \\ &= \left[-\tfrac{1}{3}x^3 + x^2 + 24x\right]_{x=-4}^{5} \\ &= (-\tfrac{125}{3} + 25 + 120) - (\tfrac{64}{3} + 16 - 96) \\ &= 162 \end{aligned}$$

and

$$\text{area of the triangle} = \tfrac{1}{2} \times 9 \times 9 = 40\tfrac{1}{2}.$$

Hence,

$$\text{area} = 162 - 40\tfrac{1}{2} = \underline{\underline{121\tfrac{1}{2}\text{ units}^2}}.$$

17. Figure 13 shows the graph of the curve with equation (5)

$$y = \frac{16}{x^2} - \frac{x}{2} + 1,\ x > 0.$$

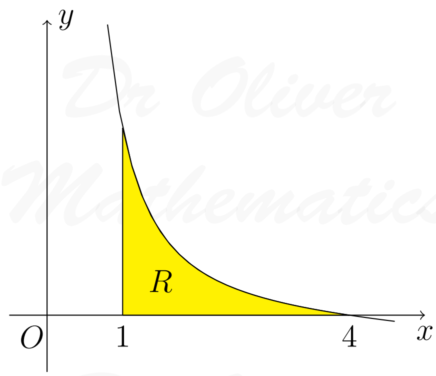

Figure 13: $y = \frac{16}{x^2} - \frac{x}{2} + 1$

The finite region R, bounded by the lines $x = 1$, the x-axis, and the curve, is shown shaded in the figure. The curve crosses the x-axis at the point $(4, 0)$. Use integration to find the exact value for the area of R.

Solution

$$\begin{aligned}\text{Area of the curve} &= \int_1^4 (16x^{-2} - \tfrac{1}{2}x + 1)\,\mathrm{d}x \\ &= \left[-16x^{-1} - \tfrac{1}{4}x^2 + x\right]_{x=1}^{4} \\ &= (-4 - 4 + 4) - (-16 - \tfrac{1}{4} + 1) \\ &= \underline{\underline{11\tfrac{1}{4}\text{ units}^2}}.\end{aligned}$$

18. Figure 14 shows the straight line with equation $y = 10 - x$ and the curve with equation $y = 10x - x^2 - 8$.

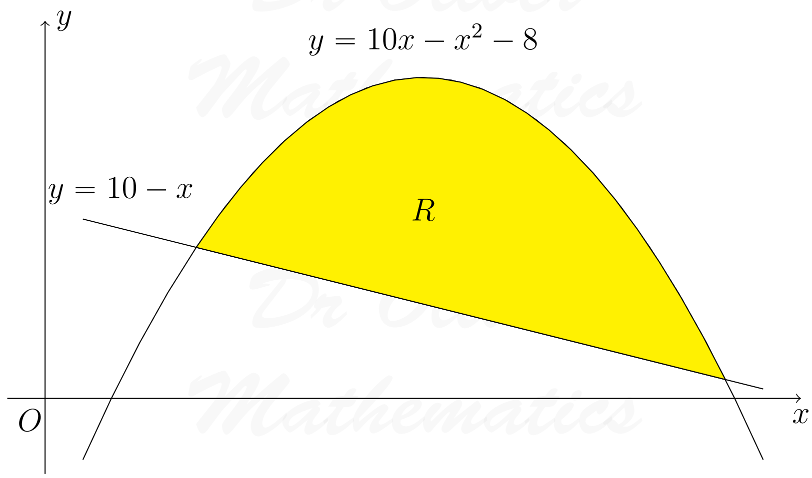

Figure 14: $y = 10 - x$ and $y = 10x - x^2 - 8$

The line and the curve intersect at the points A and B, and O is the origin.

(a) Calculate the coordinates of A and find the coordinates of B. (5)

Solution

$$\begin{aligned} 10 - x = 10x - x^2 - 8 &\Rightarrow x^2 - 11x + 18 = 0 \\ &\Rightarrow (x-2)(x-9) = 0 \\ &\Rightarrow x = 2 \text{ or } x = 9, \end{aligned}$$

and so we have $\underline{\underline{A(2, 8)}}$ and $\underline{\underline{B(9, 1)}}$.

The shaded area R is bounded by the line and the curve, is shown shaded in the figure.

(b) Calculate the exact area of R. (7)

Solution

We can find this by

$$\text{area} = \text{area of the curve} - \text{area of the trapezium.}$$

Now,

$$\begin{aligned} \text{area of the curve} &= \int_2^9 (10x - x^2 - 8)\,\mathrm{d}x \\ &= \left[5x^2 - \tfrac{1}{3}x^3 - 8x\right]_{x=2}^{9} \\ &= (405 - 243 - 72) - (20 - \tfrac{8}{3} - 16) \\ &= 88\tfrac{2}{3} \end{aligned}$$

and

$$\text{area of the trapezium} = \tfrac{1}{2} \times (8+1) \times 7 = 31\tfrac{1}{2}.$$

Hence,

$$\text{area} = 88\tfrac{2}{3} - 31\tfrac{1}{2} = \underline{\underline{57\tfrac{1}{6} \text{ units}^2}}.$$

19. The finite region R, as shown in Figure 15, is bounded by the x-axis and the curve with equation (6)

$$y = 27 - 2x - 9\sqrt{x} - \frac{16}{x^2},\ x > 0.$$

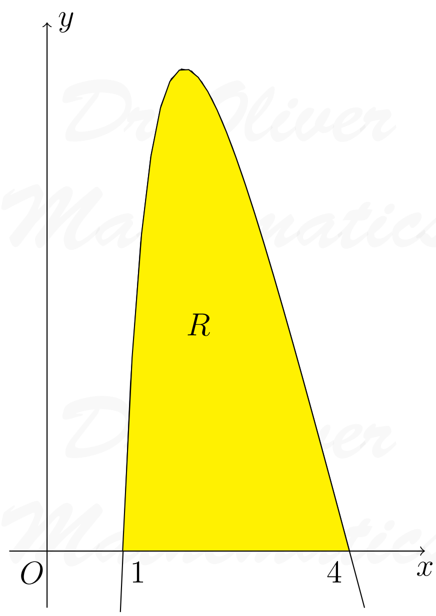

Figure 15: $y = 27 - 2x - 9\sqrt{x} - \dfrac{16}{x^2}$

The curve crosses the x-axis at the point $(1, 0)$ and $(4, 0)$. Use integration to find the exact value for the area of R.

Solution

$$\begin{aligned}\text{Area of the curve} &= \int_1^4 (27 - 2x - 9x^{\frac{1}{2}} - 16x^{-2})\,\mathrm{d}x \\ &= \left[27x - x^2 - 6x^{\frac{3}{2}} + 16x^{-1}\right]_{x=1}^{4} \\ &= (108 - 16 - 48 + 4) - (27 - 1 - 6 + 16) \\ &= \underline{\underline{12 \text{ units}^2}}.\end{aligned}$$

20. Figure 16 shows a sketch of part of the curve C with equation

$$y = x(x+4)(x-2).$$

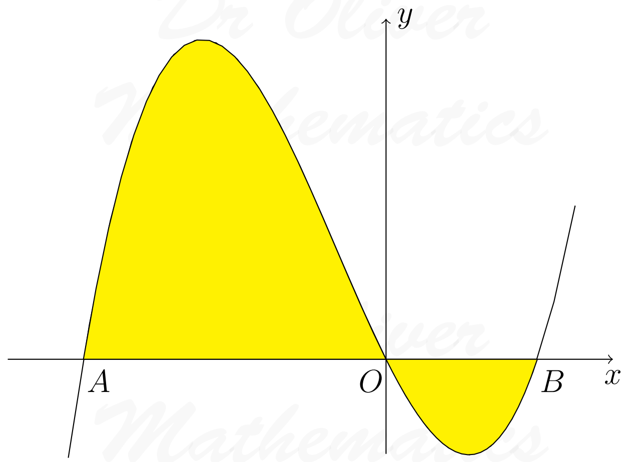

Figure 16: $y = x(x+4)(x-2)$

The curve C crosses the x-axis at the origin O and at the points A and B.

(a) Find the x-coordinates of the points A and B. (1)

Solution
$\underline{\underline{A(-4, 0)}}$ and $\underline{\underline{B(2, 0)}}$.

The finite region, shown shaded in the figure, is bounded by the curve C and the x-axis.

(b) Use integration to find the total area of the finite region. (7)

Solution

We can find this by

$$\text{area} = \text{area of [-4,0]} + \text{area of [0,2]}.$$

$$\begin{aligned}
\text{Area of [-4,0]} &= \int_{-4}^{0} x(x+4)(x-2)\,\mathrm{d}x \\
&= \int_{-4}^{0} x(x^2+2x-8)\,\mathrm{d}x \\
&= \int_{-4}^{0} (x^3+2x^2-8x)\,\mathrm{d}x \\
&= \left[\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3 - 4x^2\right]_{x=-4}^{0} \\
&= (0+0-0) - (64 - \tfrac{128}{3} - 64) \\
&= 42\tfrac{2}{3}.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_{0}^{2} x(x+4)(x-2)\,\mathrm{d}x &= \left[\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3 - 4x^2\right]_{x=0}^{2} \\
&= (4 + \tfrac{16}{3} - 16) - (0+0-0) \\
&= -6\tfrac{2}{3}
\end{aligned}$$

and

$$\text{area of [0,2]} = 6\tfrac{2}{3}.$$

Hence,

$$\text{area} = 42\tfrac{2}{3} + 6\tfrac{2}{3} = \underline{\underline{49\tfrac{1}{3}\text{ units}^2}}.$$

21. The line with equation $y = 10$ cuts the curve with equation $y = x^2 + 2x + 2$ at the points A and B, as shown in Figure 17.

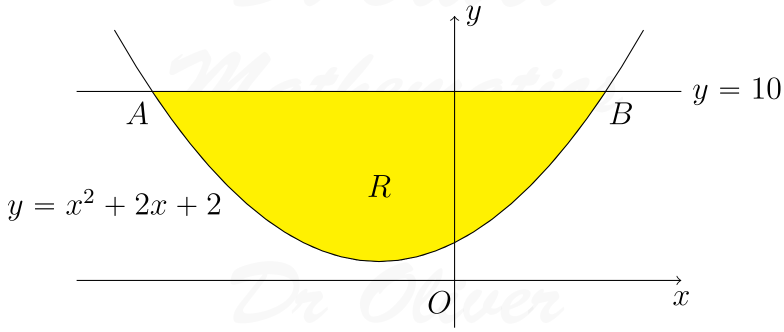

Figure 17: $y = x^2 + 2x + 2$ and $y = 10$

(a) Find the x-coordinate of A and the x-coordinate of B. (2)

Solution

$$\begin{aligned} x^2 + 2x + 2 = 10 &\Rightarrow x^2 + 2x - 8 = 0 \\ &\Rightarrow (x+4)(x-2) = 0 \\ &\Rightarrow x = -4 \text{ or } x = 2; \end{aligned}$$

the x-coordinate of A is $\underline{\underline{-4}}$ and x-coordinate of B is $\underline{\underline{2}}$.

The shaded region R is bounded by line with equation $y = 10$ and the curve.

(b) Use calculus find the exact area of R. (7)

Solution

We can find this by

$$\text{area} = \text{area of the rectangle} - \text{area of the curve}.$$

Now,

$$\text{area of the rectangle} = 6 \times 10 = 60$$

and

$$\begin{aligned} \text{area of the curve} &= \int_{-4}^{2} (x^2 + 2x + 2)\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3 + x^2 + 2x\right]_{x=-4}^{2} \\ &= \left(\tfrac{8}{3} + 4 + 4\right) - \left(-\tfrac{64}{3} + 16 - 8\right) \\ &= 24. \end{aligned}$$

Hence,

$$\text{area} = 60 - 24 = \underline{\underline{36 \text{ units}^2}}.$$

22. Use integration to find (5)

$$\int_1^{\sqrt{3}} \left(\frac{x^3}{6} + \frac{1}{3x^2}\right) \, \mathrm{d}x,$$

giving your answer in the form $a + b\sqrt{3}$, where a and b are constant to be determined.

Solution

$$\begin{aligned}\int_1^{\sqrt{3}} \left(\frac{x^3}{6} + \frac{1}{3x^2}\right) \, \mathrm{d}x &= \int_1^{\sqrt{3}} \left(\tfrac{1}{6}x^3 + \tfrac{1}{3}x^{-2}\right) \, \mathrm{d}x \\ &= \left[\tfrac{1}{24}x^4 - \tfrac{1}{3}x^{-1}\right]_{x=1}^{\sqrt{3}} \\ &= \left(\tfrac{3}{8} - \tfrac{\sqrt{3}}{9}\right) - \left(\tfrac{1}{24} - \tfrac{1}{3}\right) \\ &= \underline{\underline{\left(\tfrac{2}{3} - \tfrac{1}{9}\sqrt{3}\right) \text{ units}^2}}.\end{aligned}$$

23. Figure 18 shows a sketch of part of the curve C with equation (7)

$$y = \tfrac{1}{8}x^3 + \tfrac{3}{4}x^2, \; x \in \mathbb{R}.$$

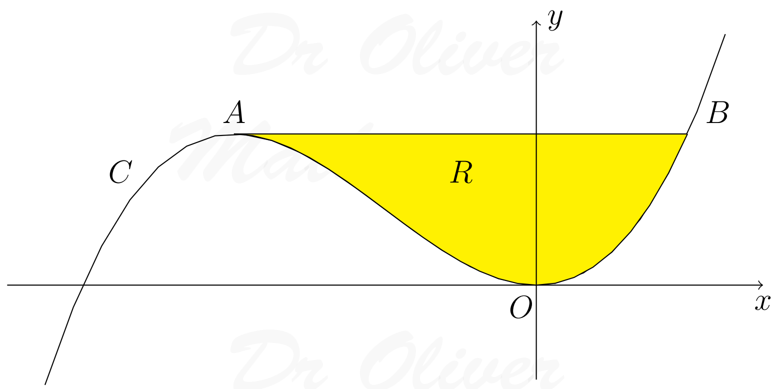

Figure 18: $y = \frac{1}{8}x^3 + \frac{3}{4}x^2$

The curve C has a maximum turning point at the point A and a minimum turning point at the origin O. The line l touches the curve C at the point A and cuts the curve C at the point B. The x-coordinate of A is -4 and the x-coordinate of B is 2. The finite region R, shown shaded in the figure, is bounded by the curve C and the line l. Use integration to find the area of the finite region R.

Solution

We can find this by

$$\text{area} = \text{area of the rectangle} - \text{area under the curve}.$$

Now, $A(-4,4)$, $B(2,4)$,

$$\text{area of the rectangle} = 6 \times 4 = 24,$$

and

$$\begin{aligned}\text{area under the curve} &= \int_{-4}^{2} \left(\tfrac{1}{8}x^3 + \tfrac{3}{4}x^2\right)\,\mathrm{d}x \\ &= \left[\tfrac{1}{32}x^4 + \tfrac{1}{4}x^3\right]_{x=-4}^{2} \\ &= \left(\tfrac{1}{2} + 2\right) - (8 - 16) \\ &= 10\tfrac{1}{2}.\end{aligned}$$

Hence,

$$\text{area} = 24 - 10\tfrac{1}{2} = \underline{\underline{13\tfrac{1}{2}\text{ units}^2}}.$$

24. (a) Find (4)

$$\int 10x(x^{\frac{1}{2}} - 2)\,\mathrm{d}x,$$

giving each term in it simplest form.

Solution

$$\begin{aligned}\int 10x(x^{\frac{1}{2}} - 2)\,\mathrm{d}x &= \int (10x^{\frac{3}{2}} - 20x)\,\mathrm{d}x \\ &= \underline{\underline{4x^{\frac{5}{2}} - 10x^2 + c}}.\end{aligned}$$

Figure 19 shows a sketch of part of the curve C with equation

$$y = 10x(x^{\frac{1}{2}} - 2),\ x \geqslant 0.$$

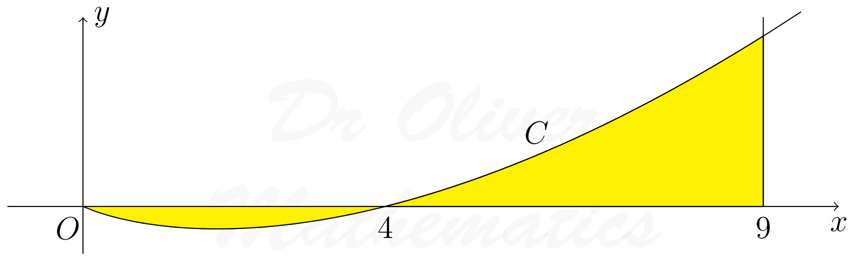

Figure 19: $y = 10x(x^{\frac{1}{2}} - 2)$

The curve C starts at the origin and crosses the x-axis at the point $(4, 0)$. The area, as shown in the figure, consists of two finite regions and is bounded by the curve C, the x-axis, and the line $x = 9$.

(b) Use your answer from part (a) to find the total area of the shaded regions. (5)

Solution

We can find this by

$$\text{area} = \text{area of } [0,4] - \text{area of } [4,9].$$

Now,

$$\begin{aligned}\int_0^4 10x(x^{\frac{1}{2}} - 2)\,\mathrm{d}x &= \left[4x^{\frac{5}{2}} - 10x^2\right]_{x=0}^{4} \\ &= (128 - 160) - (0 - 0) \\ &= -32\end{aligned}$$

and

$$\text{area of } [0,4] = 32.$$

And

$$\begin{aligned}\text{area of } [4,9] &= \left[4x^{\frac{5}{2}} - 10x^2\right]_{x=4}^{9} \\ &= (972 - 810) - (128 - 160) \\ &= 194\end{aligned}$$

and hence

$$\text{area} = 32 + 194 = \underline{\underline{226 \text{ units}^2}}.$$

25. Figure 20 shows a sketch of part of the curve with equation

$$y = 3x - x^{\frac{3}{2}},\ x \geqslant 0.$$

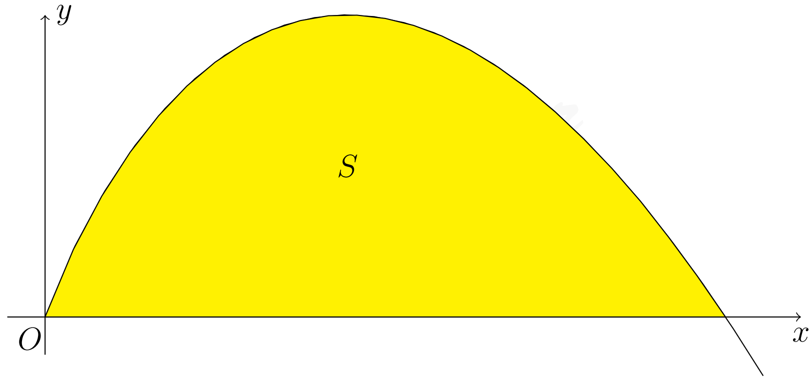

Figure 20: $y = 10x(x^{\frac{1}{2}} - 2)$

The finite region S, bounded by the x-axis and the curve, is shown shaded in the figure.

(a) Find (3)

$$\int \left(3x - x^{\frac{3}{2}}\right)\,\mathrm{d}x.$$

Solution

$$\int \left(3x - x^{\frac{3}{2}}\right)\,\mathrm{d}x = \underline{\underline{\tfrac{3}{2}x^2 - \tfrac{2}{5}x^{\frac{5}{2}} + c}}.$$

(b) Hence find the area of S. (3)

Solution

$$\begin{aligned}\text{Area under the curve} &= \left[\tfrac{3}{2}x^2 - \tfrac{2}{5}x^{\frac{5}{2}}\right]_{x=0}^{9} \\ &= \left(\tfrac{243}{2} - \tfrac{486}{5}\right) - (0 - 0) \\ &= \underline{\underline{24.3 \text{ units}^2}}.\end{aligned}$$

26. Figure 21 shows a sketch of part of the curve with equation (3)

$$y = 4x^3 + 9x^2 - 30x - 8,\ -0.5 \leqslant x \leqslant 2.2.$$

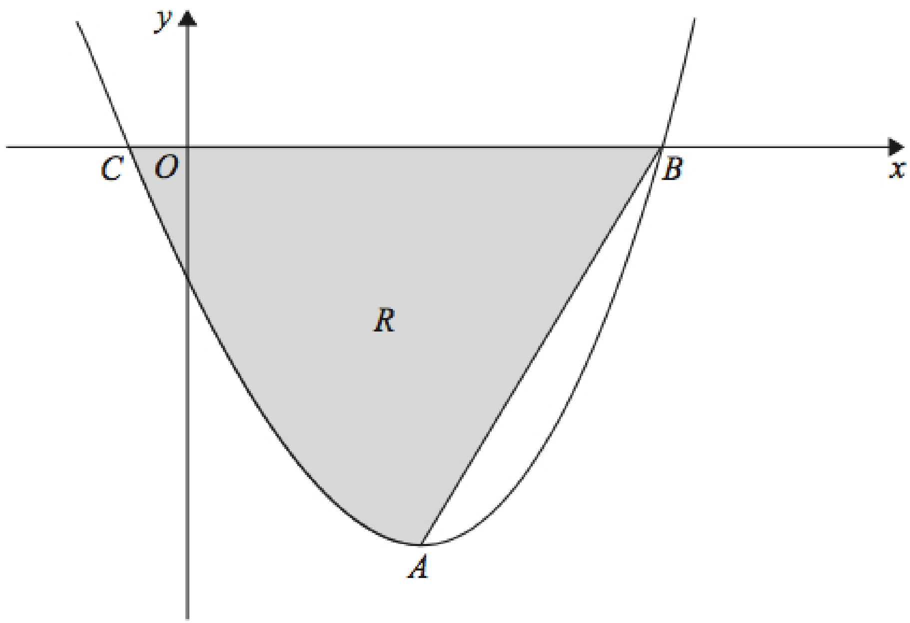

Figure 21: $y = 4x^3 + 9x^2 - 30x - 8$

The curve has a turning point at the point 1. The curve crosses the x-axis at the points $B(2, 0)$ and $C(-\frac{1}{4}, 0)$. The finite region R is bounded by the curve, the line AB, and the x-axis. Use integration to find the area of the finite region R, giving your answer to 2 decimal places.

Solution

$-0.5 \leqslant x \leqslant 1$:

$$\begin{aligned}\int_{-\frac{1}{4}}^{1} (4x^3 + 9x^2 - 30x - 8)\,\mathrm{d}x &= \left[x^4 + 3x^3 - 15x^2 - 8x\right]_{x=-\frac{1}{4}}^{1} \\ &= (1 + 3 - 15 - 8) - \left(\tfrac{1}{256} - \tfrac{3}{64} - \tfrac{15}{16} + 2\right) \\ &= -20\tfrac{5}{256}.\end{aligned}$$

$1 \leqslant x \leqslant 2$: Now,

$$x = 1 \Rightarrow y = 4 + 9 - 30 - 8 = -25$$

and

$$\int_{1}^{2} (4x^3 + 9x^2 - 30x - 8)\,\mathrm{d}x = \tfrac{1}{2} \times 1 \times (-25) = -12\tfrac{1}{2}.$$

Finally,

$$
\begin{aligned}
\text{area} &= \text{area of [-0.5,1] + area of [1,2]} \\
&= 20\tfrac{5}{256} + 12\tfrac{1}{2} \\
&= 32\tfrac{133}{256} \\
&= \underline{\underline{32.52 \text{ (2 dp)}}}.
\end{aligned}
$$