Dr Oliver Mathematics
Pi Function

1 Introduction

We want

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

for *all* $n \in \mathbb{R}$. So, we want a function, $\mathrm{f}(x)$, so that

(i) $\mathrm{f}(1) = 1$, and
(ii) $\mathrm{f}(n) = n \cdot \mathrm{f}(n-1)$.

And that is the *Pi function*:

$$\Pi(x) = \int_0^\infty t^x \mathrm{e}^{-t} \, \mathrm{d}t.$$

2 $\Pi(1) = 1$

We use the tabular method.

	D	**I**
$+$	t	e^{-t}
$-$	1	$-\mathrm{e}^{-t}$
$+$	0	e^{-t}

$$\begin{aligned}
\Pi(1) &= \int_0^\infty t \cdot \mathrm{e}^{-t} \, \mathrm{d}t \\
&= \left[-t\mathrm{e}^{-t} - \mathrm{e}^{-t}\right]_{t=0}^{\infty} \\
&= \left[\frac{-t}{\mathrm{e}^t} - \mathrm{e}^{-t}\right]_{t=0}^{\infty} \\
&= \lim_{u \to \infty} \left\{ \left[\frac{-t}{\mathrm{e}^t} - \mathrm{e}^{-t}\right]_{t=0}^{u} \right\}.
\end{aligned}$$

Now,

$$\lim_{t\to\infty}\left(\frac{-t}{\mathrm{e}^t}\right)=\frac{-\infty}{\infty}$$

and so we use L'Hôpital's Rule:

$$\lim_{t\to\infty}\left(\frac{\frac{\mathrm{d}}{\mathrm{d}t}(-t)}{\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^t)}\right)=\lim_{t\to\infty}\left(\frac{-1}{-\mathrm{e}^t}\right)=0.$$

Finally,

$$\begin{aligned}\Pi(1)&=\lim_{t\to\infty}\left[(0-\mathrm{e}^{-t})-(-0\cdot\mathrm{e}^{-0}-\mathrm{e}^{-0})\right]\\&=0-(-1)\\&=1,\end{aligned}$$

as required.

3 $\Pi(n)=n\cdot\Pi(n-1)$

	D	**I**
$+$	t^n	e^{-t}
$-$	$n\cdot t^{n-1}$	$-\mathrm{e}^{-t}$

$$\begin{aligned}\Pi(n)&=\int_0^\infty t^n\mathrm{e}^{-t}\,\mathrm{d}t\\&=\left[-t^n\mathrm{e}^{-t}\right]_{t=0}^{\infty}+\int_0^\infty n\cdot t^n\mathrm{e}^{-t}\,\mathrm{d}t\\&=\left[-t^n\mathrm{e}^{-t}\right]_{t=0}^{\infty}+n\int_0^\infty t^n\mathrm{e}^{-t}\,\mathrm{d}t\\&=\left[-t^n\mathrm{e}^{-t}\right]_{t=0}^{\infty}+n\cdot\Pi(n-1);\end{aligned}$$

and we just want to prove

$$\left[\frac{-t^n}{\mathrm{e}^t}\right]_{t=0}^{\infty}=0.$$

Now,

$$\lim_{t\to\infty}\left(\frac{-t^n}{e^t}\right) = \frac{-\infty}{\infty}$$
$$\lim_{t\to\infty}\left(\frac{\frac{d}{dt}(t^n)}{\frac{d}{dt}(e^t)}\right) = \lim_{t\to\infty}\left(\frac{-nt^{n-1}}{e^t}\right) = \frac{-\infty}{\infty}$$
$$\lim_{t\to\infty}\left(\frac{\frac{d}{dt}(-nt^{n-1})}{\frac{d}{dt}(e^t)}\right) = \lim_{t\to\infty}\left(\frac{-n(n-1)t^{n-2}}{e^t}\right) = \frac{-\infty}{\infty}$$

and so on down to

$$\lim_{t\to\infty}\left(\frac{-n!}{e^t}\right) = 0.$$

4 0!

$$\begin{aligned} 0! &= \Pi(0) \\ &= \int_0^\infty e^{-t}\,dt \\ &= \left[-\frac{1}{e^t}\right]_{t=0}^\infty \\ &= 1. \end{aligned}$$

5 Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

In particular,

$$\Gamma(n) = (n-1)!$$

6 An integral

Maybe you have see this before:

$$\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}.$$

We will use that in the next section.

7 $(\frac{1}{2})!$

What is $(\frac{1}{2})!$?

$$u = t^{\frac{1}{2}} \Rightarrow u^2 = t \Rightarrow 2u\,\frac{\mathrm{d}u}{\mathrm{d}t} = 1 \Rightarrow 2u\,\mathrm{d}u = \mathrm{d}t.$$

Now,

$$\begin{aligned}(\tfrac{1}{2})! &= \int_0^\infty t^{\frac{1}{2}}\mathrm{e}^{-t}\,\mathrm{d}t \\ &= \int_0^\infty u\mathrm{e}^{-u^2}\,2u\,\mathrm{d}u \\ &= 2\int_0^\infty u \cdot u\mathrm{e}^{-u^2}\,\mathrm{d}u.\end{aligned}$$

	D	**I**
$+$	u	$u\mathrm{e}^{-u^2}$
$-$	1	$-\frac{1}{2}\mathrm{e}^{-u^2}$

$$\begin{aligned}&= 2\left[-\tfrac{1}{2}u\mathrm{e}^{-u^2}\right]_{u=0}^{\infty} + 2\int_0^\infty \tfrac{1}{2}\mathrm{e}^{-u^2}\,\mathrm{d}u \\ &= \left[\frac{-u}{\mathrm{e}^{u^2}}\right]_{u=0}^{\infty} + \int_0^\infty \mathrm{e}^{-u^2}\,\mathrm{d}u.\end{aligned}$$

Next,

$$\lim_{u\to\infty}\left(\frac{-u}{\mathrm{e}^{u^2}}\right) = \frac{-\infty}{\infty}$$

and so we apply L'Hôpital's Rule:

$$\lim_{u\to\infty}\left(\frac{\frac{\mathrm{d}}{\mathrm{d}u}(-u)}{\frac{\mathrm{d}}{\mathrm{d}u}(\mathrm{e}^{u^2})}\right) = \lim_{t\to\infty}\left(\frac{-1}{2u\,\mathrm{e}^{u^2}}\right) = 0.$$

Finally,

$$\begin{aligned}(\tfrac{1}{2})! &= (0-0) + \int_0^\infty \mathrm{e}^{-u^2}\,\mathrm{d}u \\ &= \int_0^\infty \mathrm{e}^{-u^2}\,\mathrm{d}u \\ &= \frac{\sqrt{\pi}}{2}.\end{aligned}$$

8 $(-\frac{1}{2})!$

$$
\begin{aligned}
(\tfrac{1}{2})! = (\tfrac{1}{2}) \cdot (\tfrac{1}{2} - 1)! &\Rightarrow (\tfrac{1}{2})! = (\tfrac{1}{2}) \cdot (-\tfrac{1}{2})! \\
&\Rightarrow (-\tfrac{1}{2})! = 2 \cdot (\tfrac{1}{2})! \\
&\Rightarrow (-\tfrac{1}{2})! = 2 \cdot \frac{\sqrt{\pi}}{2} \\
&\Rightarrow (-\tfrac{1}{2})! = \pi.
\end{aligned}
$$