Dr Oliver Mathematics
Further Pure Mathematics
Hyperbolic Equations
Past Examination Questions

This booklet consists of 57 questions across a variety of examination topics.
The total number of marks available is 422.

1. Using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials,

   (a) prove that $\cosh^2 x - \sinh^2 x \equiv 1$, (3)

   Solution

   $$\begin{aligned}\cosh^2 x - \sinh^2 x &\equiv \left[\tfrac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)\right]^2 - \left[\tfrac{1}{2}\left(\mathrm{e}^x - \mathrm{e}^{-x}\right)\right]^2 \\ &\equiv \tfrac{1}{4}\left(\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x}\right) - \tfrac{1}{4}\left(\mathrm{e}^{2x} - 2 + \mathrm{e}^{-2x}\right) \\ &\equiv \tfrac{1}{2} + \tfrac{1}{2} \\ &\equiv \underline{\underline{1}}.\end{aligned}$$

   (b) solve (4)

   $$\operatorname{cosech} x - 2\coth x = 2,$$

   giving your answers in the form $k \ln a$, where k and a are integers.

   Solution

   $$\begin{aligned}\operatorname{cosech} x - 2\coth x = 2 &\Rightarrow \frac{1}{\sinh x} - \frac{2\cosh x}{\sinh x} = 2 \\ &\Rightarrow \frac{1 - 2\cosh x}{\sinh x} = 2 \\ &\Rightarrow 1 - 2\cosh x = 2\sinh x \\ &\Rightarrow 1 - \left(\mathrm{e}^x + \mathrm{e}^{-x}\right) = \mathrm{e}^x - \mathrm{e}^{-x} \\ &\Rightarrow 1 - \mathrm{e}^x - \mathrm{e}^{-x} = \mathrm{e}^x - \mathrm{e}^{-x} \\ &\Rightarrow \mathrm{e}^x = \tfrac{1}{2} \\ &\Rightarrow x = \underline{\underline{\ln \tfrac{1}{2}}}.\end{aligned}$$

2. Given that $y = \sinh^{n-1} x \cosh x$,

(a) show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (n-1)\sinh^{n-2} x + n\sinh^n x$. (3)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \sinh^{n-1} x \times \frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) + \frac{\mathrm{d}}{\mathrm{d}x}(\sinh^{n-1} x) \times \cosh x \\ &= \sinh^n x + (n-1)\sinh^{n-2} x \cosh^2 x \\ &= \sinh^n x + (n-1)\sinh^{n-2} x(\sinh^2 x + 1) \\ &= \sinh^n x + (n-1)\sinh^n x + (n-1)\sinh^{n-2} x \\ &= \underline{\underline{(n-1)\sinh^n x + n\sinh^n x}}.\end{aligned}$$

The integral I_n is defined by

$$I_n = \int_0^{\operatorname{arsinh} 1} \sinh^n x \,\mathrm{d}x,\ n \geqslant 0.$$

(b) Using the result in part (a), or otherwise, show that (2)

$$nI_n = \sqrt{2} - (n-1)I_{n-2}\, n \geqslant 2.$$

Solution

$u = \sinh^{n-1} x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = (n-1)\sinh^{n-2} x \cosh x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sinh x \Rightarrow v = \cosh x$

and

$$\begin{aligned}I_n &= \int_0^{\operatorname{arsinh} 1} \sinh^n x \,\mathrm{d}x \\ &= \int_0^{\operatorname{arsinh} 1} \sinh^{n-1} x \sinh x \,\mathrm{d}x \\ &= \left[\sinh^{n-1} x \cosh x\right]_{x=0}^{\operatorname{arsinh} 1} - \int_0^{\operatorname{arsinh} 1} (n-1)\sinh^{n-2} x \cosh^2 x \,\mathrm{d}x \\ &= \left[\sinh^{n-1} x \cosh x\right]_{x=0}^{\operatorname{arsinh} 1} - (n-1)\int_0^{\operatorname{arsinh} 1} \sinh^{n-2} x(\sinh^2 x + 1)\,\mathrm{d}x \\ &= \left[\sinh^{n-1} x\sqrt{\sinh^2 x + 1}\right]_{x=0}^{\operatorname{arsinh} 1} - (n-1)(I_n + I_{n-2}) \\ &= (\sqrt{2} - 0) - (n-1)I_n - (n-1)I_{n-2}\end{aligned}$$

and so

$$\underline{\underline{nI_n = \sqrt{2} - (n-1)I_{n-2}}}.$$

(c) Hence find the value of I_4. (4)

Solution

$$\begin{aligned} I_4 &= \tfrac{1}{4}\left(\sqrt{2} - 3I_2\right) \\ &= \tfrac{\sqrt{2}}{4} - \tfrac{3}{4}I_2 \\ &= \tfrac{\sqrt{2}}{4} - \tfrac{3}{8}\left(\sqrt{2} - I_0\right) \\ &= -\tfrac{\sqrt{2}}{8} + \tfrac{3}{8}I_0 \\ &= -\tfrac{\sqrt{2}}{8} + \tfrac{3}{8}\int_0^{\operatorname{arsinh} 1} 1\,\mathrm{d}x \\ &= \underline{\underline{-\tfrac{\sqrt{2}}{8} + \tfrac{3}{8}\operatorname{arsinh} 1}} \text{ or } \underline{\underline{-\tfrac{\sqrt{2}}{8} + \tfrac{3}{8}\ln(1+\sqrt{2})}}. \end{aligned}$$

3. (a) Show that, for $x = \ln k$, where k is a positive constant, (3)

$$\cosh 2x = \frac{k^4+1}{2k^2}.$$

Solution

$$\begin{aligned} x = \ln k &\Rightarrow 2x = 2\ln k \\ &\Rightarrow 2x = \ln k^2 \\ &\Rightarrow \cosh 2x = \cosh(\ln k^2) \\ &\Rightarrow \cosh 2x = \tfrac{1}{2}\left(\mathrm{e}^{\ln k^2} + \mathrm{e}^{-\ln k^2}\right) \\ &\Rightarrow \cosh 2x = \tfrac{1}{2}\left(\mathrm{e}^{\ln k^2} + \mathrm{e}^{\ln \frac{1}{k^2}}\right) \\ &\Rightarrow \cosh 2x = \tfrac{1}{2}\left(k^2 + \frac{1}{k^2}\right) \\ &\Rightarrow \underline{\underline{\cosh 2x = \frac{k^4+1}{2k^2}}}. \end{aligned}$$

Given that $f(x) = px - \tanh 2x$, where p is a constant,

(b) find the value of p for which $f(x)$ has a stationary value at $x = \ln 2$, giving your answer as an exact fraction. (4)

Solution

$$f(x) = px - \tanh 2x \Rightarrow f'(x) = p - 2\,\text{sech}^2\, 2x$$

and

$$x = \ln 2 \Rightarrow \cosh 2x = \frac{2^4 + 1}{2 \times 2^2} = \frac{17}{8}.$$

Finally,

$$p = \frac{2}{\cosh^2 2x} = 2 \times \left(\frac{8}{17}\right)^2 = \underline{\underline{\frac{128}{289}}}.$$

4. Figure 1 shows a sketch of the curve with equation

$$y = x \operatorname{arcosh} x, \; 1 \leqslant x \leqslant 2.$$

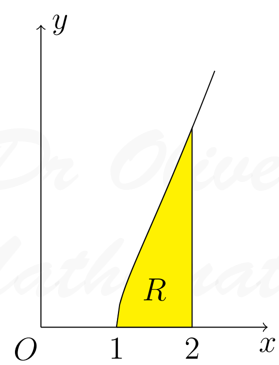


Figure 1: $y = x \operatorname{arcosh} x$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 2$. Show that the area of R is

$$\tfrac{7}{4} \ln(2 + \sqrt{3}) - \tfrac{\sqrt{3}}{2}.$$

Solution

$$u = \operatorname{arcosh} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{x^2-1}} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = x \Rightarrow v = \tfrac{1}{2}x^2,$$

$$\theta = \operatorname{arcosh} x \Rightarrow x = \cosh\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sinh\theta,$$

and

$$\theta = \operatorname{arcosh} 1 = 0 \text{ and } \theta = \operatorname{arcosh} 2.$$

$$\begin{aligned}
\int_1^2 x \operatorname{arcosh} x \,\mathrm{d}x &= \left[\tfrac{1}{2}x^2 \operatorname{arcosh} x\right]_{x=1}^2 - \int_1^2 \frac{x^2}{2\sqrt{x^2-1}}\,\mathrm{d}x \\
&= (2\operatorname{arcosh} 2 - \tfrac{1}{2}\operatorname{arcosh} 1) - \int_0^{\operatorname{arcosh} 2} \frac{\cosh^2\theta}{2\sqrt{\cosh^2\theta - 1}} \sinh\theta \,\mathrm{d}\theta \\
&= 2\operatorname{arcosh} 2 - \int_0^{\operatorname{arcosh} 2} \frac{\sinh\theta\cosh^2\theta}{2\sqrt{\sinh^2\theta}}\,\mathrm{d}\theta \\
&= 2\operatorname{arcosh} 2 - \int_0^{\operatorname{arcosh} 2} \frac{\sinh\theta\cosh^2\theta}{2\sinh\theta}\,\mathrm{d}\theta \\
&= 2\operatorname{arcosh} 2 - \tfrac{1}{2}\int_0^{\operatorname{arcosh} 2} \cosh^2\theta\,\mathrm{d}\theta \\
&= 2\operatorname{arcosh} 2 - \tfrac{1}{4}\int_0^{\operatorname{arcosh} 2} (1 + \cosh 2\theta)\,\mathrm{d}\theta \\
&= 2\operatorname{arcosh} 2 - \tfrac{1}{4}\left[\theta + \tfrac{1}{2}\sinh 2\theta\right]_{\theta=0}^{\operatorname{arcosh} 2} \\
&= 2\operatorname{arcosh} 2 - \tfrac{1}{4}\left\{(\operatorname{arcosh} 2 + \tfrac{1}{2}\sinh(2\operatorname{arcosh} 2)) - (0+0)\right\} \\
&= \tfrac{7}{4}\operatorname{arcosh} 2 - \tfrac{1}{8}\sinh(2\operatorname{arcosh} 2).
\end{aligned}$$

So, how do we do $\sinh(2\operatorname{arcosh} 2)$? Well,

$$\begin{aligned}
\sinh(2\operatorname{arcosh} 2) &= 2\sinh(\operatorname{arcosh} 2)\cosh(\operatorname{arcosh} 2) \\
&= 4\sinh(\operatorname{arcosh} 2) \\
&= 4\sinh(\ln(2+\sqrt{3})) \\
&= 4 \times \left[\tfrac{1}{2}\left(\mathrm{e}^{\ln(2+\sqrt{3})} - \mathrm{e}^{-\ln(2+\sqrt{3})}\right)\right] \\
&= 2\left(2+\sqrt{3} - \frac{1}{2+\sqrt{3}}\right) \\
&= 2 \times 2\sqrt{3} \\
&= 4\sqrt{3}.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_1^2 x \operatorname{arcosh} x \,\mathrm{d}x &= \tfrac{7}{4}\operatorname{arcosh} 2 - \tfrac{1}{8} \times 4\sqrt{3} \\
&= \tfrac{7}{4}\operatorname{arcosh} 2 - \tfrac{\sqrt{3}}{2} \\
&= \underline{\underline{\tfrac{7}{4}\ln(2+\sqrt{3}) - \tfrac{\sqrt{3}}{2}}}.
\end{aligned}$$

5. (a) Show that, for $0 < x \leqslant 1$, (3)

$$\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

Solution

$$\begin{aligned}\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) &= \ln\left(\frac{1-\sqrt{1-x^2}}{x} \times \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)\\ &= \ln\left(\frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})}\right)\\ &= \ln\left(\frac{x}{1+\sqrt{1-x^2}}\right)\\ &= \underline{\underline{-\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}}.\end{aligned}$$

(b) Using the definitions of $\cosh x$ or $\operatorname{sech} x$ in terms of exponentials, for $0 < x \leqslant 1$, (5)

$$\operatorname{arsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

Solution

$$\begin{aligned}y = \operatorname{arsech} x &\Rightarrow x = \operatorname{sech} y\\ &\Rightarrow x = \frac{2}{\mathrm{e}^y + \mathrm{e}^{-y}}\\ &\Rightarrow x\,\mathrm{e}^{2y} - 2\,\mathrm{e}^y + x = 0\\ &\Rightarrow \mathrm{e}^y = \frac{2+\sqrt{4-4x^2}}{2x} \text{ (since } 0 < x \leqslant 1)\\ &\Rightarrow \mathrm{e}^y = \frac{1+\sqrt{1-x^2}}{x}\\ &\Rightarrow \underline{\underline{y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}}.\end{aligned}$$

(c) Solve the equation (5)

$$3\tanh^2 x - 4\operatorname{sech} x + 1 = 0,$$

giving your answers in terms of natural logarithms.

Solution

$$\begin{aligned}3\tanh^2 x - 4\operatorname{sech} x + 1 = 0 &\Rightarrow 3(1-\operatorname{sech}^2 x) - 4\operatorname{sech} x + 1 = 0\\ &\Rightarrow 3 - 3\operatorname{sech}^2 x - 4\operatorname{sech} x + 1 = 0\\ &\Rightarrow 3\operatorname{sech}^2 x + 4\operatorname{sech} x - 4 = 0\\ &\Rightarrow (3\operatorname{sech} x - 2)(\operatorname{sech} x + 2) = 0\\ &\Rightarrow \operatorname{sech} x = \tfrac{2}{3}\\ &\Rightarrow \cosh x = \tfrac{3}{2}\\ &\Rightarrow x = \underline{\underline{\pm\ln(\tfrac{3+\sqrt{5}}{2})}}.\end{aligned}$$

6. Evaluate

$$\int_1^4 \frac{1}{\sqrt{x^2-2x+17}}\,\mathrm{d}x,$$

giving your answer as an exact logarthim.

Solution

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x^2-2x+17}}\,\mathrm{d}x &= \int_1^4 \frac{1}{\sqrt{(x-1)^2+4^2}}\,\mathrm{d}x\\ &= \left[\operatorname{arsinh}\left(\tfrac{x-1}{4}\right)\right]_{x=1}^4\\ &= \operatorname{arsinh}\tfrac{3}{4} - 0\\ &= \underline{\underline{\ln 2}}.\end{aligned}$$

7. (a) Using the definitions of $\cosh x$ in terms of exponentials, prove that (3)

$$4\cosh^3 x - 3\cosh x \equiv \cosh 3x.$$

Solution

$$\begin{aligned}4\cosh^3 x - \cosh x &= 4\left[\tfrac{1}{2}\left(e^x + e^{-x}\right)\right]^3 - 3\left[\tfrac{1}{2}\left(e^x + e^{-x}\right)\right] \\ &\equiv \tfrac{1}{2}\left(e^{3x} + 3e^x + 3e^{-x} + e^{-3x}\right) - \tfrac{3}{2}\left(e^x + e^{-x}\right) \\ &\equiv \tfrac{1}{2}(e^{3x} + e^{-3x}) \\ &\equiv \underline{\underline{\cosh 3x}}.\end{aligned}$$

(b) Hence, or otherwise, solve the equation (4)

$$\cosh 3x = 5\cosh x,$$

giving your answer as natural logarithms.

Solution

$$\begin{aligned}\cosh 3x = 5\cosh x &\Rightarrow 4\cosh^3 x - 3\cosh x = 5\cosh x \\ &\Rightarrow 4\cosh^3 x - 8\cosh x = 0 \\ &\Rightarrow 4\cosh x(\cosh^2 x - 2) = 0 \\ &\Rightarrow \cosh x = \pm\sqrt{2} \\ &\Rightarrow x = \pm \operatorname{arcosh}\sqrt{2} \\ &\Rightarrow x = \underline{\underline{\pm\ln(\sqrt{2}+1)}}.\end{aligned}$$

8. (a) Show that $\operatorname{artanh}(\sin\frac{\pi}{4}) = \ln(1+\sqrt{2})$. (3)

Solution

$$\begin{aligned}\operatorname{artanh}\left(\sin\frac{\pi}{4}\right) &= \operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right) \\ &= \tfrac{1}{2}\ln(3+2\sqrt{2}) \\ &= \underline{\underline{\ln(1+\sqrt{2})}}.\end{aligned}$$

(b) Given that $y = \operatorname{artanh}(\sin x)$, show that $\dfrac{dy}{dx} = \sec x$. (2)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-\sin^2 x} \times \cos x = \frac{1}{\cos x} = \underline{\underline{\sec x}}.$$

(c) Find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \operatorname{artanh}(\sin x)\,\mathrm{d}x$. (5)

Solution

$$u = \operatorname{artanh}(\sin x) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \sec x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x$$

and

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sin x \operatorname{artanh}(\sin x)\,\mathrm{d}x &= [-\cos x \operatorname{artanh}(\sin x)]_{x=0}^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} 1\,\mathrm{d}x \\ &= (-\tfrac{1}{\sqrt{2}}\ln(1+\sqrt{2}) - 0) + [x]_{x=0}^{\frac{\pi}{4}} \\ &= -\tfrac{1}{\sqrt{2}}\ln(1+\sqrt{2}) + (\tfrac{\pi}{4} - 0) \\ &= \underline{\underline{\tfrac{\pi}{4} - \tfrac{1}{\sqrt{2}}\ln(1+\sqrt{2})}}.\end{aligned}$$

9. Find the values of x for which (6)

$$5\cosh x - 2\sinh x = 11,$$

giving your answer as natural logarithms.

Solution

$$\begin{aligned}5\cosh x - 2\sinh x = 11 &\Rightarrow \tfrac{5}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) - \left(\mathrm{e}^x - \mathrm{e}^{-x}\right) = 11 \\ &\Rightarrow \tfrac{3}{2}\mathrm{e}^x + \tfrac{7}{2}\mathrm{e}^{-x} - 11 = 0 \\ &\Rightarrow 3\mathrm{e}^{2x} - 22\mathrm{e}^x + 7 = 0 \text{ (as } \mathrm{e}^x > 0) \\ &\Rightarrow (3\mathrm{e}^x - 1)(\mathrm{e}^x - 7) = 0 \\ &\Rightarrow 3\mathrm{e}^x - 1 = 0 \text{ or } \mathrm{e}^x - 7 = 0 \\ &\Rightarrow \mathrm{e}^x = \tfrac{1}{3} \text{ or } \mathrm{e}^x = 7 \\ &\Rightarrow \underline{\underline{x = \ln\tfrac{1}{3}}} \text{ or } \underline{\underline{x = \ln 7}}.\end{aligned}$$

10. The curve with equation

$$y = -x + \tanh 4x,\ x \geqslant 0,$$

has a maximum turning point A.

(a) Find, in exact logarithmic form, the x-coordinate of A. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow -1 + 4\operatorname{sech}^2 4x = 0\\ &\Rightarrow \operatorname{sech}^2 4x = \tfrac{1}{4}\\ &\Rightarrow \operatorname{sech} 4x = \tfrac{1}{2}\\ &\Rightarrow \cosh 4x = 2\\ &\Rightarrow 4x = \pm\ln(2+\sqrt{3})\\ &\Rightarrow x = \pm\tfrac{1}{4}\ln(2+\sqrt{3}).\end{aligned}$$

But $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 16\operatorname{sech}^2 4x \tanh 4x < 0$ for $x = \underline{\underline{\tfrac{1}{4}\ln(2+\sqrt{3})}}$: this is the answer.

(b) Show that the y-coordinate of A is $\frac{1}{4}\left\{2\sqrt{3} - \ln(2+\sqrt{3})\right\}$. (3)

Solution

$$\begin{aligned}\operatorname{sech} 4x = \tfrac{1}{2} &\Rightarrow \operatorname{sech}^2 4x = \tfrac{1}{4}\\ &\Rightarrow \tanh^2 4x = \tfrac{3}{4}\\ &\Rightarrow \tanh 4x = \tfrac{\sqrt{3}}{2}\end{aligned}$$

and

$$\begin{aligned}x = \tfrac{1}{4}\ln(2+\sqrt{3}) &\Rightarrow y = -\tfrac{1}{4}\ln(2+\sqrt{3}) + \tanh 4x\\ &\Rightarrow y = -\tfrac{1}{4}\ln(2+\sqrt{3}) + \tfrac{\sqrt{3}}{2}\\ &\Rightarrow y = \underline{\underline{\tfrac{1}{4}\left\{2\sqrt{3} - \ln(2+\sqrt{3})\right\}}}.\end{aligned}$$

11. Figure 2 shows a sketch of the curve with equation (10)

$$y = x^2 \operatorname{arsinh} x.$$

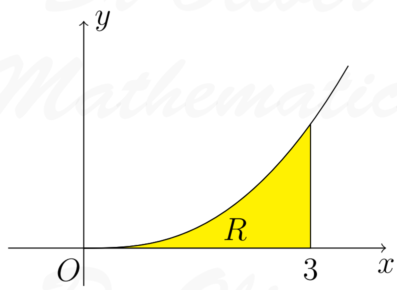


Figure 2: $y = x^2 \operatorname{arsinh} x$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 3$. Show that the area of R is

$$9 \ln(3 + \sqrt{10}) - \tfrac{1}{9}(2 + 7\sqrt{10}).$$

Solution

$$u = \operatorname{arsinh} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{x^2+1}} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = x^2 \Rightarrow v = \tfrac{1}{3}x^3,$$

$$u = x^2 + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \Rightarrow \mathrm{d}u = 2x\,\mathrm{d}x,$$

and

$$x = 0, u = 1 \text{ and } x = 3, u = 10.$$

Now,

$$\begin{aligned}
\int_0^3 x^2 \operatorname{arsinh} x \,\mathrm{d}x &= \left[\tfrac{1}{3}x^3 \operatorname{arsinh} x\right]_{x=0}^3 - \int_0^3 \frac{x^3}{3\sqrt{x^2+1}}\,\mathrm{d}x \\
&= (9 \operatorname{arsinh} 3 - 0) - \tfrac{1}{3}\int_1^{10} \frac{(u-1)^{\frac{3}{2}}}{u^{\frac{1}{2}} \times 2(u-1)^{\frac{1}{2}}}\,\mathrm{d}u \\
&= 9 \operatorname{arsinh} 3 - \tfrac{1}{6}\int_1^{10} \frac{u-1}{u^{\frac{1}{2}}}\,\mathrm{d}u \\
&= 9 \operatorname{arsinh} 3 - \tfrac{1}{6}\int_1^{10} (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\,\mathrm{d}u \\
&= 9 \operatorname{arsinh} 3 - \tfrac{1}{6}\left[\tfrac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_{u=1}^{10} \\
&= 9 \operatorname{arsinh} 3 - \tfrac{1}{6}\left\{(\tfrac{20}{3}\sqrt{10} - 2\sqrt{10}) - (\tfrac{2}{3} - 2)\right\} \\
&= 9 \operatorname{arsinh} 3 - \tfrac{1}{6}(\tfrac{14}{3}\sqrt{10} + \tfrac{4}{3}) \\
&= \underline{\underline{9 \ln(3 + \sqrt{10}) - \tfrac{1}{9}(7\sqrt{10} + 2)}}.
\end{aligned}$$

12.

$$I_n = \int x^n \cosh x \, \mathrm{d}x, \; n \geqslant 0.$$

(a) Show that, for $n \geqslant 2$, (4)

$$I_n = x^n \sinh x - nx^{n-1} \cosh x + n(n-1) I_{n-2}.$$

Solution

$$u = x^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \cosh x \Rightarrow v = \sinh x,$$

$$u = nx^{n-1} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = n(n-1)x^{n-2} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sinh x \Rightarrow v = \cosh x,$$

and we have

$$\begin{aligned} I_n &= \int x^n \cosh x \, \mathrm{d}x \\ &= x^n \cosh x - \int nx^{n-1} \sinh x \, \mathrm{d}x \\ &= x^n \cosh x - \left(nx^{n-1} \sinh x - \int n(n-1)x^{n-2} \cosh x \, \mathrm{d}x \right) \\ &= x^n \cosh x - nx^{n-1} \sinh x + \int n(n-1)x^{n-2} \cosh x \, \mathrm{d}x \\ &= \underline{\underline{x^n \sinh x - nx^{n-1} \cosh x + n(n-1) I_{n-2}}}. \end{aligned}$$

(b) Hence show that (5)

$$I_4 = \mathrm{f}(x) \sinh x + \mathrm{g}(x) \cosh x + c,$$

where $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are functions of x to be found, and c is an arbitrary constant.

Solution

$$\begin{aligned} I_4 &= \int x^4 \cosh x \, \mathrm{d}x \\ &= x^4 \sinh x - 4x^3 \cosh x + 12 I_2 \\ &= x^4 \sinh x - 4x^3 \cosh x + 12 \left[x^2 \sinh x - 2x \cosh x + 2 I_0 \right] \\ &= (x^4 + 12x^2) \sinh x - (4x^3 + 24x) \cosh x + 24 I_0 \\ &= (x^4 + 12x^2) \sinh x - (4x^3 + 24x) \cosh x + 24 \int \cosh x \, \mathrm{d}x \\ &= \underline{\underline{(x^4 + 12x^2 + 24) \sinh x - (4x^3 + 24x) \cosh x + c}}. \end{aligned}$$

(c) Find the exact value of $\displaystyle\int_0^1 x^4 \cosh x \, \mathrm{d}x$, given your answer in terms of e. (3)

Solution

$$J_n = \int_0^1 x^n \cosh x \, \mathrm{d}x.$$

$$\begin{aligned} J_n &= \int_0^1 x^n \cosh x \, \mathrm{d}x \\ &= \left[(x^4 + 12x^2 + 24) \sinh x - (4x^3 + 24x) \cosh x \right]_{x=0}^{1} \\ &= (37 \sinh 1 - 28 \cosh 1) - (0 - 0) \\ &= 37 \sinh 1 - 28 \cosh 1 \\ &= \tfrac{37}{2} \left(\mathrm{e} - \mathrm{e}^{-1} \right) - 14 \left(\mathrm{e} + \mathrm{e}^{-1} \right) \\ &= \underline{\underline{\tfrac{1}{2} \left(9\,\mathrm{e} - 65\,\mathrm{e}^{-1} \right)}}. \end{aligned}$$

13. Evaluate $\displaystyle\int_1^3 \frac{1}{\sqrt{x^2 + 4x - 5}} \, \mathrm{d}x$, giving your answer as a natural logarithm. (5)

Solution

$$\begin{aligned} \int_1^3 \frac{1}{\sqrt{x^2 + 4x - 5}} \, \mathrm{d}x &= \int_1^3 \frac{1}{\sqrt{(x+2)^2 - 3^2}} \, \mathrm{d}x \\ &= \left[\operatorname{arcosh} \left(\frac{x+2}{3} \right) \right]_{x=1}^{3} \\ &= \operatorname{arcosh} \tfrac{5}{3} - \operatorname{arcosh} 1 \\ &= \underline{\underline{\ln 3}}. \end{aligned}$$

14. (a) Using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, prove that (3)

$$\cosh(A - B) \equiv \cosh A \cosh B - \sinh A \sinh B.$$

Solution

$$\begin{aligned}
&\cosh A \cosh B - \sinh A \sinh B \\
\equiv\ & \tfrac{1}{2}\left(e^{A} + e^{-A}\right) \times \tfrac{1}{2}\left(e^{B} + e^{-B}\right) - \tfrac{1}{2}\left(e^{A} - e^{-A}\right) \times \tfrac{1}{2}\left(e^{B} - e^{-B}\right) \\
\equiv\ & \tfrac{1}{4}\left(e^{A+B} + e^{A-B} + e^{-A+B} - e^{-A-B}\right) - \tfrac{1}{4}\left(e^{A+B} - e^{A-B} - e^{-A+B} - e^{-A-B}\right) \\
\equiv\ & \tfrac{1}{2}\left(e^{A-B} + e^{-A+B}\right) \\
\equiv\ & \underline{\underline{\cosh(A-B)}}.
\end{aligned}$$

(b) Hence, or otherwise, given that $\cosh(x-1) = \sinh x$, show that (4)

$$\tanh x = \frac{e^2 + 1}{e^2 + 2e - 1}.$$

Solution

$$\begin{aligned}
& \cosh(x-1) = \sinh x \\
\Rightarrow\ & \cosh x \cosh 1 - \sinh x \sinh 1 = \sinh x \\
\Rightarrow\ & \cosh x \cosh 1 = \sinh x (1 + \sinh 1) \\
\Rightarrow\ & \frac{\sinh x}{\cosh x} = \frac{\cosh 1}{1 + \sinh 1} \\
\Rightarrow\ & \tanh x = \frac{\frac{1}{2}\left(e + e^{-1}\right)}{1 + \frac{1}{2}\left(e - e^{-1}\right)} \\
\Rightarrow\ & \underline{\underline{\tanh x = \frac{e^2 + 1}{e^2 + 2e - 1}}}.
\end{aligned}$$

15. Figure 3 shows a sketch of the curve with equation (10)

$$y = \operatorname{arsinh} \sqrt{x},\ x \geqslant 0.$$

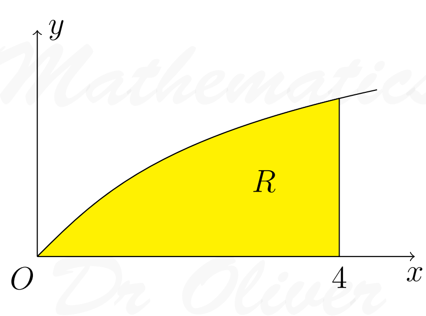


Figure 3: $y = \operatorname{arsinh} \sqrt{x}$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 4$. Using the substitution $x = \sinh^2\theta$, or otherwise, show that the area of R is

$$k\ln(2+\sqrt{5}) - \sqrt{5},$$

where k is a constant to be found.

Solution

$$\begin{aligned} \theta = \operatorname{arsinh}\sqrt{x} \Rightarrow x &= \sinh^2\theta \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} &= 2\sinh\theta\cosh\theta \\ \Rightarrow \mathrm{d}x &= 2\sinh\theta\cosh\theta\,\mathrm{d}\theta, \end{aligned}$$

$$x = 0 \Rightarrow \theta = 0 \text{ and } x = 2 \Rightarrow \theta = \operatorname{arsinh} 2,$$

$$u = \theta \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sinh 2\theta \Rightarrow v = \tfrac{1}{2}\cosh 2\theta,$$

and so we have

$$\begin{aligned} \int_0^4 \operatorname{arsinh}\sqrt{x}\,\mathrm{d}x &= \int_0^{\operatorname{arsinh} 2} \theta(2\sinh\theta\cosh\theta)\,\mathrm{d}\theta \\ &= \int_0^{\operatorname{arsinh} 2} \theta\sinh 2\theta\,\mathrm{d}\theta \\ &= \left[\tfrac{1}{2}\theta\cosh 2\theta\right]_{\theta=0}^{\operatorname{arsinh} 2} - \tfrac{1}{2}\int_0^{\operatorname{arsinh} 2} \cosh 2\theta\,\mathrm{d}\theta \\ &= \left[\tfrac{1}{2}\theta\cosh 2\theta - \tfrac{1}{4}\sinh 2\theta\right]_{\theta=0}^{\operatorname{arsinh} 2} \\ &= \left[\tfrac{1}{2}\theta(1+2\sinh^2\theta) - \tfrac{1}{2}\sinh\theta\cosh\theta\right]_{\theta=0}^{\operatorname{arsinh} 2} \\ &= \left[\tfrac{1}{2}\theta(1+2\sinh^2\theta) - \tfrac{1}{2}\sinh\theta\sqrt{1+\sinh^2\theta}\right]_{\theta=0}^{\operatorname{arsinh} 2} \\ &= (\tfrac{9}{2}\operatorname{arsinh} 2 - \tfrac{1}{2}\times 2\times\sqrt{5}) - (0-0) \\ &= \underline{\underline{\tfrac{9}{2}\ln(2+\sqrt{5}) - \sqrt{5}}}. \end{aligned}$$

16. Show that (4)

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(\tanh x)] = 2\operatorname{cosech} 2x,\ x > 0.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[\ln(\tanh x)] &= \frac{\operatorname{sech}^2 x}{\tanh x}\\ &= \frac{1}{\sinh x \cosh x}\\ &= \frac{2}{2\sinh x \cosh x}\\ &= \frac{2}{\sinh 2x}\\ &= \underline{\underline{2 \operatorname{cosech} 2x}}.\end{aligned}$$

17. Find the values of x for which (6)

$$8\cosh x - 4\sinh x = 13,$$

giving your answers as a natural logarithms.

Solution

$$\begin{aligned}8\cosh x - 4\sinh x = 13 &\Rightarrow 4\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right) - 2\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right) = 13\\ &\Rightarrow 2\,\mathrm{e}^{x} + 6\,\mathrm{e}^{-x} - 13 = 0\\ &\Rightarrow 2\,\mathrm{e}^{2x} - 13\,\mathrm{e}^{x} + 6 = 0\\ &\Rightarrow (2\,\mathrm{e}^{x} - 1)(\mathrm{e}^{x} - 6) = 0\\ &\Rightarrow \mathrm{e}^{x} = \tfrac{1}{2} \text{ or } \mathrm{e}^{x} = 6\\ &\Rightarrow \underline{\underline{x = \ln \tfrac{1}{2}}} \text{ or } \underline{\underline{x = \ln 6}}.\end{aligned}$$

18. Show that (7)

$$\int_5^6 \frac{3 + x}{\sqrt{x^2 - 9}}\,\mathrm{d}x = 3\ln\left(\frac{2 + \sqrt{3}}{3}\right) + 3\sqrt{3} - 4.$$

Solution

$$
\begin{aligned}
\int_5^6 \frac{3+x}{\sqrt{x^2-9}}\,\mathrm{d}x &= \int_5^6 \frac{3}{\sqrt{x^2-9}}\,\mathrm{d}x + \int_5^6 \frac{x}{\sqrt{x^2-9}}\,\mathrm{d}x \\
&= \int_5^6 \frac{3}{\sqrt{x^2-3^2}}\,\mathrm{d}x + \left[\sqrt{x^2-9}\right]_{x=5}^{6} \\
&= \left[3\operatorname{arcosh}\tfrac{x}{3} + \sqrt{x^2-9}\right]_{x=5}^{6} \\
&= \left[3\ln\left(\frac{x+\sqrt{x^2-9}}{3}\right) + \sqrt{x^2-9}\right]_{x=5}^{6} \\
&= (3\ln(6+3\sqrt{3}) + 3\sqrt{3}) - (3\ln 9 + 4) \\
&= \underline{\underline{3\ln(2+\sqrt{3}) + 3\sqrt{3} - 4}}.
\end{aligned}
$$

19. The curve C has equation $y = \operatorname{arsinh}(x^3)$, $x \geqslant 0$. The point P on C has x-coordinate $\sqrt{2}$. Show that an equation of the tangent to C at P is (5)

$$y = 2x - 2\sqrt{2} + \ln(3 + 2\sqrt{2}).$$

Solution

Well, $x = \sqrt{2} \Rightarrow y = \operatorname{arsinh}(2\sqrt{2}) = \ln(3 + 2\sqrt{2})$. Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{(x^3)^2+1} = \frac{3x^2}{x^6+1} \text{ and } x = \sqrt{2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2$$

and we have

$$y - \ln(3+2\sqrt{2}) = 2(x - \sqrt{2}) \Rightarrow \underline{\underline{y = 2x - 2\sqrt{2} + \ln(3+2\sqrt{2})}}.$$

20. Figure 4 shows a sketch of the curve with equation

$$y = \tfrac{1}{10}\cosh x \arctan(\sinh x),\ x \geqslant 0.$$

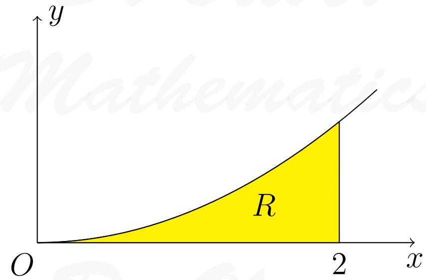


Figure 4: $y = \frac{1}{10}\cosh x \arctan(\sinh x)$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 2$.

(a) Find $\int \cosh x \arctan(\sinh x)\,\mathrm{d}x$. (5)

Solution

$$u = \arctan(\sinh x) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\cosh x}{\sinh^2 x + 1} = \frac{1}{\cosh x},$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cosh x \Rightarrow v = \sinh x,$$

and

$$\begin{aligned}\int \cosh x \arctan(\sinh x)\,\mathrm{d}x &= \sinh x \arctan(\sinh x) - \int \frac{\sinh x}{\cosh x}\,\mathrm{d}x \\ &= \underline{\underline{\sinh x \arctan(\sinh x) - \ln(\cosh x) + c}}.\end{aligned}$$

(b) Hence show that, to 2 significant figures, the area of R is 0.34. (2)

Solution

$$\begin{aligned}\int_0^2 \tfrac{1}{10}\cosh x \arctan(\sinh x)\,\mathrm{d}x &= \tfrac{1}{10}\left[\sinh x \arctan(\sinh x) - \ln(\cosh x)\right]_{x=0}^{2} \\ &= (\sinh 2 \arctan(\sinh 2) - \ln(\cosh 2)) - (0 - 0) \\ &= 0.339\,630\,027\,6 \text{ (FCD)} \\ &= \underline{\underline{0.34 \text{ (2 sf)}}}.\end{aligned}$$

21. (a) Starting from the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, prove that (2)

$$\sinh 2x \equiv 2 \sinh x \cosh x.$$

Solution

$$\begin{aligned} 2\sinh x\cosh x &\equiv 2 \times \tfrac{1}{2}\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right) \times \tfrac{1}{2}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right) \\ &\equiv \tfrac{1}{2}\left(\mathrm{e}^{2x} - \mathrm{e}^{-2x}\right) \\ &\equiv \underline{\underline{\sinh 2x}}. \end{aligned}$$

(b) Hence find the exact values of x for which (7)

$$\sinh 2x = 6\sinh^2 x + 7\sinh x.$$

Solution

$$\begin{aligned} \sinh 2x = 6\sinh^2 x + 7\sinh x &\Rightarrow 2\sinh x\cosh x = 6\sinh^2 x + 7\sinh x \\ &\Rightarrow 2\sinh x\cosh x - 6\sinh^2 x - 7\sinh x = 0 \\ &\Rightarrow \sinh x(2\cosh x - 6\sinh x - 7) = 0 \\ &\Rightarrow \sinh x = 0 \text{ or } 2\cosh x - 6\sinh x - 7 = 0. \end{aligned}$$

$\underline{\sinh x = 0}$:

$$\sinh x = 0 \Rightarrow \underline{\underline{x = 0}}.$$

$\underline{2\cosh x - 6\sinh x - 7 = 0}$:

$$\begin{aligned} 2\cosh x - 6\sinh x - 7 = 0 &\Rightarrow \left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right) - 3\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right) - 7 = 0 \\ &\Rightarrow -2\,\mathrm{e}^{x} + 4\,\mathrm{e}^{-x} - 7 = 0 \\ &\Rightarrow 2\,\mathrm{e}^{x} - 4\,\mathrm{e}^{-x} + 7 = 0 \\ &\Rightarrow 2\,\mathrm{e}^{2x} + 7\,\mathrm{e}^{x} - 4 = 0 \\ &\Rightarrow (2\,\mathrm{e}^{x} - 1)(\mathrm{e}^{x} + 4) = 0 \\ &\Rightarrow \mathrm{e}^{x} = \tfrac{1}{2} \\ &\Rightarrow \underline{\underline{x = \ln \tfrac{1}{2}}}. \end{aligned}$$

22. The curve C, with equation $y = \cosh 3x - 4x$, has a minimum point, as shown in Figure 5.

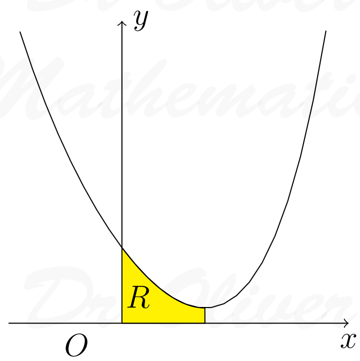


Figure 5: $y = \cosh 3x - 4x$

(a) Use calculus to find the x-coordinate of A. Give your answer in terms of natural logarithm. (5)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3\sinh 3x - 4 = 0\\ &\Rightarrow \tfrac{3}{2}\left(\mathrm{e}^{3x} - \mathrm{e}^{-3x}\right) - 4 = 0\\ &\Rightarrow 3\,\mathrm{e}^{3x} - 3\,\mathrm{e}^{-3x} - 8 = 0\\ &\Rightarrow 3\,\mathrm{e}^{6x} - 8\,\mathrm{e}^{3x} - 3 = 0\\ &\Rightarrow (3\,\mathrm{e}^{3x} + 1)(\mathrm{e}^{3x} - 3) = 0\\ &\Rightarrow \mathrm{e}^{3x} = 3\\ &\Rightarrow 3x = \ln 3\\ &\Rightarrow x = \underline{\underline{\tfrac{1}{3}\ln 3}}.\end{aligned}$$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, the y-axis, and the line through A parallel to the y-axis.

(b) Show that the area of R is $\frac{2}{9}[2 - (\ln 3)^2]$. (6)

Solution

$$\begin{aligned}
\text{Area} &= \int_0^{\frac{1}{3}\ln 3} (\cosh 3x - 4x)\,\mathrm{d}x \\
&= \left[\tfrac{1}{3}\sinh 3x - 2x^2\right]_{x=0}^{\frac{1}{3}\ln 3} \\
&= (\tfrac{1}{3}\sinh(\ln 3) - 2(\tfrac{1}{3}\ln 3)^2) - (0-0) \\
&= \tfrac{4}{3} - \tfrac{2}{3}(\ln 3)^2 \\
&= \underline{\underline{\tfrac{2}{9}[2 - (\ln 3)^2]}}.
\end{aligned}$$

23. (a) Using the substitution $x = \dfrac{a}{u}$, or otherwise, find (6)

$$\int \frac{1}{x\sqrt{a^2 - x^2}}\,\mathrm{d}x.$$

Solution

$$u = \frac{a}{x} \Rightarrow x = \frac{a}{u} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{a}{u^2} \Rightarrow \mathrm{d}x = -\frac{a}{u^2}\,\mathrm{d}u$$

and

$$\begin{aligned}
\int \frac{1}{x\sqrt{a^2 - x^2}}\,\mathrm{d}x &= \int \frac{1}{\frac{a}{u}\sqrt{a^2 - \left(\frac{a}{u}\right)^2}}(-\frac{a}{u^2})\,\mathrm{d}u \\
&= \int \frac{1}{\frac{a^2}{u}\sqrt{1 - \left(\frac{1}{u}\right)^2}}(-\frac{a}{u^2})\,\mathrm{d}u \\
&= \int \frac{1}{\frac{a^2}{u^2}\sqrt{u^2 - 1}}(-\frac{a}{u^2})\,\mathrm{d}u \\
&= \int \frac{u^2}{a^2\sqrt{u^2 - 1}}(-\frac{a}{u^2})\,\mathrm{d}u \\
&= -\frac{1}{a}\int \frac{1}{\sqrt{u^2 - 1}}\,\mathrm{d}u \\
&= -\frac{1}{a}\operatorname{arcosh} u + c \\
&= \underline{\underline{-\frac{1}{a}\operatorname{arcosh}\left(\frac{a}{x}\right) + c}}.
\end{aligned}$$

(b) Hence find (5)

$$\int_3^4 \frac{1}{x\sqrt{25-x^2}}\,\mathrm{d}x,$$

giving your answer in the form $a \ln b$, where a and b are rational numbers.

Solution

$$\begin{aligned}\int_3^4 \frac{1}{x\sqrt{25-x^2}}\,\mathrm{d}x &= \int_3^4 \frac{1}{x\sqrt{5^2-x^2}}\,\mathrm{d}x \\ &= -\frac{1}{5}\left[\operatorname{arcosh}\left(\frac{5}{x}\right)\right]_{x=3}^{4} \\ &= -\tfrac{1}{5}(\operatorname{arcosh}\tfrac{5}{4} - \operatorname{arcosh}\tfrac{5}{3}) \\ &= -\tfrac{1}{5}(\ln 2 - \ln 3) \\ &= -\tfrac{1}{5}\ln\tfrac{2}{3} \\ &= \underline{\underline{\tfrac{1}{5}\ln\tfrac{3}{2}}}.\end{aligned}$$

24. Solve the equation (5)

$$7\operatorname{sech} x - \tanh x = 5.$$

Give your answers in the form $\ln a$ where a is a rational number.

Solution

$$\begin{aligned}7\operatorname{sech} x - \tanh x = 5 &\Rightarrow \frac{14}{\mathrm{e}^x+\mathrm{e}^{-x}} - \frac{\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}} = 5 \\ &\Rightarrow 14 - (\mathrm{e}^x - \mathrm{e}^{-x}) = 5(\mathrm{e}^x + \mathrm{e}^{-x}) \\ &\Rightarrow 14 - \mathrm{e}^x + \mathrm{e}^{-x} = 5\,\mathrm{e}^x + 5\,\mathrm{e}^{-x} \\ &\Rightarrow 6\,\mathrm{e}^x + 4\,\mathrm{e}^{-x} - 14 = 0 \\ &\Rightarrow 3\,\mathrm{e}^x + 2\,\mathrm{e}^{-x} - 7 = 0 \\ &\Rightarrow 3\,\mathrm{e}^{2x} - 7\,\mathrm{e}^x + 2 = 0 \\ &\Rightarrow (3\,\mathrm{e}^x - 1)(\mathrm{e}^x - 2) = 0 \\ &\Rightarrow \mathrm{e}^x = \tfrac{1}{3} \text{ or } \mathrm{e}^x = 2 \\ &\Rightarrow \underline{\underline{x = \ln\tfrac{1}{3}}} \text{ or } \underline{\underline{x = \ln 2}}.\end{aligned}$$

25. Given that $y = \operatorname{arsinh}(\sqrt{x})$, $x > 0$,

(a) find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer as a simplified fraction. (3)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x+1}} \times \tfrac{1}{2}x^{-\frac{1}{2}} = \underline{\underline{\frac{1}{2\sqrt{x(x+1)}}}}.$$

(b) Hence, or otherwise, find (6)

$$\int_{\frac{1}{4}}^{4} \frac{1}{\sqrt{x(x+1)}}\,\mathrm{d}x,$$

giving your answer in the form $\ln\left(\frac{a+b\sqrt{5}}{2}\right)$, where a and b are integers.

Solution

$$\begin{aligned}\int_{\frac{1}{4}}^{4} \frac{1}{\sqrt{x(x+1)}}\,\mathrm{d}x &= \left[2\,\mathrm{arsinh}(\sqrt{x})\right]_{x=\frac{1}{4}}^{4} \\ &= 2\,\mathrm{arsinh}\,2 - 2\,\mathrm{arsinh}\,\tfrac{1}{2} \\ &= 2\ln(2+\sqrt{5}) - 2\ln(\tfrac{1+\sqrt{5}}{2}) \\ &= 2\ln(\tfrac{3+\sqrt{5}}{2}) \\ &= \ln(\tfrac{3+\sqrt{5}}{2})^2 \\ &= \underline{\underline{\ln(\tfrac{7+3\sqrt{5}}{2})}}.\end{aligned}$$

26.

$$R = 10\pi \int_0^1 \sqrt{16\,\mathrm{c}^2\,\theta + 9}\,\mathrm{d}c, \quad \text{where } c = \cos\theta.$$

Using the substitution $\cos\theta = \frac{3}{4}\sinh u$, or otherwise, evaluate the integral, showing all of your working and giving the final answer to 3 significant figures.

Solution

$$\mathrm{c} = \tfrac{3}{4}\sinh u \Rightarrow \frac{\mathrm{d}c}{\mathrm{d}u} = \tfrac{3}{4}\cosh u \Rightarrow \mathrm{d}c = \tfrac{3}{4}\cosh u\,\mathrm{d}u$$

and

$$\cos 0 \Rightarrow u = 0 \Rightarrow \cos\tfrac{\pi}{2} \Rightarrow u = \mathrm{arsinh}(\tfrac{4}{3}) = \ln 3.$$

Then

$$\begin{aligned}
R &= 10\pi \int_0^1 \sqrt{16\,\mathrm{c}^2\,\theta + 9}\,\mathrm{d}c \\
&= 10\pi \int_0^{\ln 3} \sqrt{9\sinh^2 u + 9}\,(\tfrac{3}{4}\cosh u)\,\mathrm{d}u \\
&= 10\pi \int_0^{\ln 3} \tfrac{9}{4}\cosh^2 u\,\mathrm{d}u \\
&= 10\pi \int_0^{\ln 3} \tfrac{9}{8}(1 + \cosh 2u)\,\mathrm{d}u \\
&= 10\pi \left[\tfrac{9}{8}u + \tfrac{9}{16}\sinh 2u\right]_{u=0}^{\ln 3} \\
&= 10\pi \left\{\tfrac{9}{8}(\tfrac{20}{9} + \ln 3) - (0 + 0)\right\} \\
&= 117.367\,979\,7 \text{ (FCD)} \\
&= \underline{\underline{117 \text{ (3 sf)}}}.
\end{aligned}$$

27. (a) Starting from the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, prove that (3)

$$\cosh 2x \equiv 1 + 2\sinh^2 x.$$

Solution

$$\begin{aligned}
1 + 2\sinh^2 x &\equiv 1 + \tfrac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})^2 \\
&\equiv 1 + \tfrac{1}{2}(\mathrm{e}^{2x} - 2 + \mathrm{e}^{-2x}) \\
&\equiv \tfrac{1}{2}(\mathrm{e}^{2x} + \mathrm{e}^{-2x}) \\
&\equiv \underline{\underline{\cosh 2x}}.
\end{aligned}$$

(b) Solve the equation (5)

$$\cosh 2x - 3\sinh x = 15,$$

giving your answers as exact logarithms.

Solution

$$\begin{aligned}\cosh 2x - 3\sinh x = 15 &\Rightarrow (1 + 2\sinh^2 x) - 3\sinh x = 15\\ &\Rightarrow 2\sinh^2 x - 3\sinh x - 14 = 0\\ &\Rightarrow (2\sinh x - 7)(\sinh x + 2) = 0\\ &\Rightarrow \sinh x = \tfrac{7}{2} \text{ or } \sinh x = -2\\ &\Rightarrow \underline{\underline{x = \ln(\tfrac{7+\sqrt{53}}{2})}} \text{ or } \underline{\underline{x = \ln(-2+\sqrt{5})}}.\end{aligned}$$

28. Given that $y = (\operatorname{arcosh} 3x)^2$, where $3x > 1$, show that

(a) $(9x^2 - 1)\left(\dfrac{dy}{dx}\right)^2 = 36y$, (5)

Solution

$$\frac{dy}{dx} = 2\operatorname{arcosh} 3x \times \frac{3}{\sqrt{(3x)^2 - 1}} = \frac{6\operatorname{arcosh} 3x}{\sqrt{9x^2 - 1}}$$

and

$$(9x^2 - 1)\left(\frac{dy}{dx}\right)^2 = (6\operatorname{arcosh} 3x)^2 = \underline{\underline{36y}}.$$

(b) $(9x^2 - 1)\dfrac{d^2y}{dx^2} + 9x\dfrac{dy}{dx} = 18$. (4)

Solution

Here there is no need to find $\dfrac{d^2y}{dx^2}$ explictly:

$$\begin{aligned} &(9x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 36y\\ \Rightarrow\quad &(9x^2 - 1) \times 2\frac{dy}{dx} \times \frac{d^2y}{dx^2} + 18x\left(\frac{dy}{dx}\right)^2 = 36\frac{dy}{dx}\\ \Rightarrow\quad &\underline{\underline{(9x^2 - 1)\frac{d^2y}{dx^2} + 9x\frac{dy}{dx} = 18.}}\end{aligned}$$

29. Given that $y = \arctan(3\,e^{2x})$, show that (5)

$$\frac{dy}{dx} = \frac{3}{5\cosh 2x + 4\sinh 2x}.$$

Solution

$$
\begin{aligned}
y = \arctan(3\,\mathrm{e}^{2x}) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6\,\mathrm{e}^{2x}}{(3\,\mathrm{e}^{2x})^2 + 1} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6\,\mathrm{e}^{2x}}{9\,\mathrm{e}^{4x} + 1} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3}{\frac{9}{2}\,\mathrm{e}^{2x} + \frac{1}{2}\,\mathrm{e}^{-2x}} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3}{\frac{5}{2}(\mathrm{e}^{2x} + \mathrm{e}^{2x}) + 2(\mathrm{e}^{2x} - \mathrm{e}^{-2x})} \\
&\Rightarrow \underline{\underline{\frac{3}{5\cosh 2x + 4\sinh 2x}}}.
\end{aligned}
$$

30. Show that (4)

$$\int_5^8 \frac{1}{\sqrt{x^2 - 10x + 34}}\,\mathrm{d}x = \ln(A + \sqrt{n}),$$

giving the values of the integers A and n.

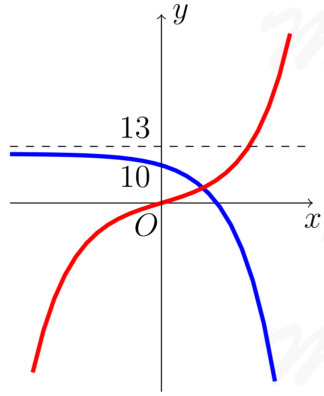
Solution

$$
\begin{aligned}
\int_5^8 \frac{1}{\sqrt{x^2 - 10x + 34}}\,\mathrm{d}x &= \int_5^8 \frac{1}{\sqrt{(x-5)^2 + 3^2}}\,\mathrm{d}x \\
&= \left[\operatorname{arsinh}\left(\frac{x-5}{3}\right)\right]_{x=5}^{8} \\
&= \operatorname{arsinh} 1 - 0 \\
&= \underline{\underline{\ln(1 + \sqrt{2})}}.
\end{aligned}
$$

31. The curve C_1 has equation $y = 3\sinh 2x$ and the curve C_2 has equation $y = 13 - 3\,\mathrm{e}^{2x}$.

(a) Sketch the graphs of the curves C_1 and C_2 on one set of axes, giving the equation of any asymptote and the coordinates of points where the curves cross the axes. (4)

Solution



(b) Solve the equation (5)

$$3\sinh 2x = 13 - 3\,\mathrm{e}^{2x},$$

giving your answer in the form $\frac{1}{2}\ln k$, where k is an integer.

Solution

$$\begin{aligned} 3\sinh 2x = 13 - 3\,\mathrm{e}^{2x} &\Rightarrow \tfrac{3}{2}(\mathrm{e}^{2x} - \mathrm{e}^{-2x}) = 13 - 3\,\mathrm{e}^{2x} \\ &\Rightarrow 3\,\mathrm{e}^{4x} - 3 = 26\,\mathrm{e}^{2x} - 6\,\mathrm{e}^{4x} \\ &\Rightarrow 9\,\mathrm{e}^{4x} - 26\,\mathrm{e}^{2x} - 3 = 0 \\ &\Rightarrow (9\,\mathrm{e}^{2x} + 3)(\mathrm{e}^{2x} - 3) = 0 \\ &\Rightarrow \mathrm{e}^{2x} = 3 \\ &\Rightarrow 2x = \ln 3 \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}\ln 3}}. \end{aligned}$$

32. The hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

(a) Use calculus to show that the equation of the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$ may be written in the form (4)

$$xb\cosh\theta - ya\sinh\theta = ab.$$

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\sinh\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\theta} = b\cosh\theta$$

and we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b\cosh\theta}{a\sinh\theta}.$$

Finally,

$$\begin{aligned}
& y - b\sinh\theta = \frac{b\cosh\theta}{a\sinh\theta}(x - a\cosh\theta) \\
\Rightarrow\quad & a\sinh\theta(y - b\sinh\theta) = b\cosh\theta(x - a\cosh\theta) \\
\Rightarrow\quad & ya\sinh\theta - ab\sinh^2\theta = xb\cosh\theta - ab\cosh^2\theta \\
\Rightarrow\quad & xb\cosh\theta - ya\sinh\theta = ab(\cosh^2\theta - \sinh^2\theta) \\
\Rightarrow\quad & \underline{\underline{xb\cosh\theta - ya\sinh\theta = ab}}.
\end{aligned}$$

The line l_1 is the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$, $\theta \neq 0$. Given that $l_1$meets the x-axis at the point P,

(b) find, in terms of a and θ, the coordinates of P. (2)

Solution

$$y = 0 \Rightarrow xb\cosh\theta = ab \Rightarrow \underline{\underline{x = \frac{a}{\cosh\theta}}} \text{ or } \underline{\underline{x = a\,\mathrm{sech}\,\theta}}.$$

The line l_2 is the tangent to H at the point $(a, 0)$. Given that l_1 and l_2 meets the at the point Q,

(c) find, in terms of a, b, and θ, the coordinates of Q. (2)

Solution

$$ab\cosh\theta - ya\sinh\theta = ab \Rightarrow y\sinh\theta = b\cosh\theta - b \Rightarrow y = \frac{b(\cosh\theta - 1)}{\sinh\theta}$$

and so we have $\underline{\underline{Q\left(a, \frac{b(\cosh\theta - 1)}{\sinh\theta}\right)}}$.

(d) Show that, as θ varies, the locus of the midpoint of PQ has equation (6)

$$x(4y^2 + b^2) = ab^2.$$

Solution

The midpoint of PQ has $\left(\dfrac{a(\cosh\theta+1)}{2\cosh\theta}, \dfrac{b(\cosh\theta-1)}{2\sinh\theta}\right)$. Now,

$$\begin{aligned}
4y^2+b^2 &= 4\left(\frac{b(\cosh\theta-1)}{2\sinh\theta}\right)^2+b^2 \\
&= \frac{4b^2(\cosh^2\theta-2\cosh\theta+1)}{4\sinh^2\theta}+b^2 \\
&= \frac{4b^2(\cosh^2\theta-2\cosh\theta+1)+4b^2\sinh^2\theta}{4\sinh^2\theta} \\
&= \frac{b^2(\cosh^2\theta-2\cosh\theta+1+\sinh^2\theta)}{\sinh^2\theta} \\
&= \frac{b^2(\cosh^2\theta-2\cosh\theta+1+(\cosh^2\theta-1))}{\sinh^2\theta} \\
&= \frac{b^2(2\cosh^2\theta-2\cosh\theta)}{\sinh^2\theta} \\
&= \frac{2b^2\cosh\theta(\cosh\theta-1)}{\sinh^2\theta}
\end{aligned}$$

and

$$\begin{aligned}
x(4y^2+b^2) &= \frac{a(\cosh\theta+1)}{2\cosh\theta}\times\frac{2b^2\cosh\theta(\cosh\theta-1)}{\sinh^2\theta} \\
&= \frac{ab^2(\cosh^2\theta-1)}{\sinh^2\theta} \\
&= \frac{ab^2\sinh^2\theta}{\sinh^2\theta} \\
&= \underline{\underline{ab^2}}.
\end{aligned}$$

33. The curve C, as shown in Figure 6, has equation (6)

$$y=\tfrac{1}{3}\cosh 3x,\ 0\leqslant x\leqslant \ln a,$$

where a is a constant and $a>1$.

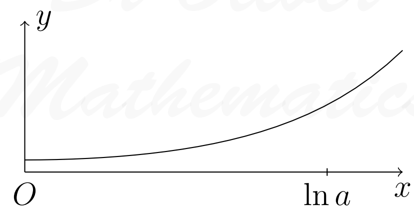


Figure 6: $y = \frac{1}{3}\cosh 3x$

Using calculus, show that the length of curve C is

$$k\left(a^3 - \frac{1}{a^3}\right)$$

and state the value of the constant k.

Solution

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + \sinh^2 3x} = \cosh 3x$$

and

$$\begin{aligned}
\text{Length of curve} &= \int_0^{\ln a} \cosh 3x \,\mathrm{d}x \\
&= \left[\tfrac{1}{3}\sinh 3x\right]_{x=0}^{\ln a} \\
&= \tfrac{1}{3}(\sinh(3\ln a) - 0) \\
&= \tfrac{1}{3}\sinh(\ln a^3) \\
&= \tfrac{1}{6}\left(\mathrm{e}^{\ln a^3} - \mathrm{e}^{-\ln a^3}\right) \\
&= \underline{\underline{\frac{1}{6}\left(a^3 - \frac{1}{a^3}\right)}}.
\end{aligned}$$

34. (a) Differentiate $x \operatorname{arsinh} 2x$ with respect to x. (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \operatorname{arsinh} 2x) = \operatorname{arsinh} 2x + \frac{2x}{\sqrt{(2x)^2 + 1}} = \underline{\underline{\operatorname{arsinh} 2x + \frac{2x}{\sqrt{4x^2 + 1}}}}.$$

(b) Hence, or otherwise, find the exact value of (7)

$$\int_0^{\sqrt{2}} x \operatorname{arsinh} 2x \, \mathrm{d}x,$$

giving your answer in the form $A \ln B + C$, where A, B, C are real numbers.

Solution

$$\begin{aligned}\int_0^{\sqrt{2}} x \operatorname{arsinh} 2x \, \mathrm{d}x &= [x \operatorname{arsinh} 2x]_{x=0}^{\sqrt{2}} - \int_0^{\sqrt{2}} \frac{2x}{\sqrt{4x^2+1}} \, \mathrm{d}x \\ &= (\sqrt{2} \operatorname{arsinh} 2\sqrt{2} - 0) - \left[\tfrac{1}{2}(4x^2+1)^{\frac{1}{2}}\right]_{x=0}^{\sqrt{2}} \\ &= \sqrt{2} \operatorname{arsinh} 2\sqrt{2} - (\tfrac{3}{2} - \tfrac{1}{2}) \\ &= \sqrt{2} \operatorname{arsinh} 2\sqrt{2} - 1 \\ &= \underline{\underline{\sqrt{2} \ln(3 + 2\sqrt{2}) - 1}}.\end{aligned}$$

35.

$$\mathrm{f}(x) = 5 \cosh x - 4 \sinh x.$$

(a) Show that $\mathrm{f}(x) = \frac{1}{2}(\mathrm{e}^x + 9\,\mathrm{e}^{-x})$. (2)

Solution

$$\begin{aligned}\mathrm{f}(x) &= 5 \cosh x - 4 \sinh x \\ &= \tfrac{5}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) - 2\left(\mathrm{e}^x - \mathrm{e}^{-x}\right) \\ &= \underline{\underline{\tfrac{1}{2}\left(\mathrm{e}^x + 9\,\mathrm{e}^{-x}\right)}}.\end{aligned}$$

Hence

(b) solve $\mathrm{f}(x) = 5$, (4)

Solution

$$\begin{aligned} f(x) = 5 &\Rightarrow \tfrac{1}{2}\left(e^x + 9e^{-x}\right) = 5 \\ &\Rightarrow e^x + 9e^{-x} = 10 \\ &\Rightarrow e^{2x} - 10e^x + 9 = 0 \\ &\Rightarrow (e^x - 1)(e^x - 9) = 0 \\ &\Rightarrow e^x = 1 \text{ or } e^x = 9 \\ &\Rightarrow \underline{\underline{x = 0}} \text{ or } \underline{\underline{x = \ln 9}} \end{aligned}$$

(c) show that (5)

$$\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\cosh x - 4\sinh x}\,dx = \tfrac{\pi}{18}.$$

Solution

$$\begin{aligned} \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\cosh x - 4\sinh x}\,dx &= \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2}{e^x + 9e^{-x}}\,dx \\ &= \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2e^x}{e^{2x} + 9}\,dx \\ &= \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2e^x}{(e^x)^2 + 3^2}\,dx \\ &= \left[\frac{2}{3}\arctan\left(\frac{e^x}{3}\right)\right]_{x=\frac{1}{2}\ln 3}^{\ln 3} \\ &= \tfrac{2}{3}(\arctan 1 - \arctan \tfrac{1}{\sqrt{3}}) \\ &= \tfrac{2}{3}(\tfrac{\pi}{4} - \tfrac{\pi}{6}) \\ &= \underline{\underline{\tfrac{\pi}{18}}}. \end{aligned}$$

36. (a) Find (2)

$$\int \frac{1}{\sqrt{4x^2 + 9}}\,dx.$$

Solution

$$\int \frac{1}{\sqrt{4x^2+9}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(2x)^2+3^3}}\,\mathrm{d}x$$
$$= \underline{\underline{\tfrac{1}{2}\operatorname{arsinh}\tfrac{2x}{3} + c}}.$$

(b) Use your answer to part (a) to find the exact value of (3)

$$\int_{-3}^{3} \frac{1}{\sqrt{4x^2+9}}\,\mathrm{d}x,$$

giving your answer in the form $k\ln(a+b\sqrt{5}$, where a and b are integers and k is a constant.

Solution

$$\begin{aligned}\int_{-3}^{3} \frac{1}{\sqrt{4x^2+9}}\,\mathrm{d}x &= \left[\tfrac{1}{2}\operatorname{arsinh}\tfrac{2x}{3}\right]_{x=-3}^{3} \\ &= \tfrac{1}{2}(\operatorname{arsinh}2 - \operatorname{arsinh}(-2)) \\ &= \tfrac{1}{2}(\ln(2+\sqrt{5}) - \ln(-2+\sqrt{5})) \\ &= \underline{\underline{\tfrac{1}{2}\ln(9+4\sqrt{5})}} \\ &= \ln(9+4\sqrt{5})^{\frac{1}{2}} \\ &= \underline{\underline{\ln(2+\sqrt{5})}}.\end{aligned}$$

37. The curve with parametric equations (7)

$$x = \cosh 2\theta, y = 4\sinh\theta, 0 \leqslant \theta \leqslant 1,$$

is rotated through 2π radians about the x-axis. Show that the area of of the surface generated is $\lambda(\cosh^3 1 - 1)$, where λ is a constant to be found.

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sinh 2\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\theta} = 4\cosh\theta$$

and

$$\begin{aligned}
\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} &= \sqrt{(2\sinh 2\theta)^2+(4\cosh\theta)^2}\\
&= \sqrt{4\sinh^2 2\theta+16\cosh^2\theta}\\
&= \sqrt{4(\cosh^2 2\theta-1)+8(1+\cosh 2\theta)}\\
&= \sqrt{4\cosh^2 2\theta+8\cosh 2\theta+4}\\
&= \sqrt{2(\cosh 2\theta+1)^2}\\
&= 2(\cosh 2\theta+1)\\
&= 2(2\cosh^2\theta)\\
&= 4\cosh^2\theta.
\end{aligned}$$

Now,

$$\begin{aligned}
\text{surface generated} &= 2\pi\int_0^1 (4\sinh\theta)(4\cosh^2\theta)\,\mathrm{d}\theta\\
&= 32\pi\int_0^1 \sinh\theta\cosh^2\theta\,\mathrm{d}\theta\\
&= 32\pi\left[\tfrac{1}{3}\cosh^3\theta\right]_{\theta=0}^{1}\\
&= 32\pi(\tfrac{1}{3}\cosh^3\theta-\tfrac{1}{3})\\
&= \underline{\underline{\tfrac{32\pi}{3}(\cosh^3\theta-1)}}.
\end{aligned}$$

38. Figure 7 shows a sketch of the curve with equation (7)

$$y = 40\,\operatorname{arcosh} x - 9x,\ x \geqslant 1.$$

Figure 7: $y = 40 \operatorname{arcosh} x - 9x$

Use calculus to find the exact coordinates of the turning point of the curve, giving your answer in the form $\left(\frac{p}{q}, r \ln 3 + s\right)$, where p, q, r, and s are integers.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \frac{40}{\sqrt{x^2-1}} - 9 = 0\\ &\Rightarrow 40 = 9\sqrt{x^2-1}\\ &\Rightarrow 1600 = 81(x^2-1)\\ &\Rightarrow x^2 - 1 = \tfrac{1600}{81}\\ &\Rightarrow x^2 = \tfrac{1681}{81}\\ &\Rightarrow x = \tfrac{41}{9}\end{aligned}$$

and $y = 40 \operatorname{arcosh} \frac{41}{9} - 41 = 40 \ln 9 - 41 = 80 \ln 3 - 41$. The answer is $\underline{\underline{\left(\frac{41}{9}, 80 \ln 3 - 41\right)}}$.

39. Figure 8 shows a sketch of the curve with equations

$$y = 6 \cosh x \text{ and } y = 9 - 2 \sinh x.$$

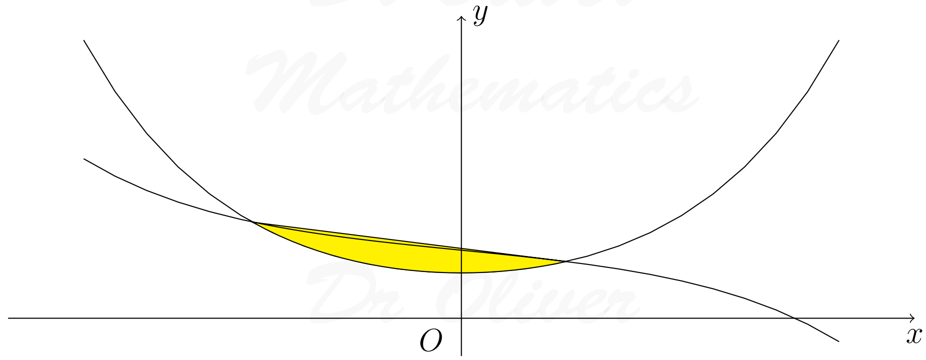


Figure 8: $y = 6\cosh x$ and $y = 9 - 2\sinh x$

(a) Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x, find exact values for the x-coordinates of the two points where the curves intersect. (6)

Solution

$$\begin{aligned}
6\cosh x = 9 - 2\sinh x &\Rightarrow 3\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) = 9 - \left(\mathrm{e}^x - \mathrm{e}^{-x}\right) \\
&\Rightarrow 4\,\mathrm{e}^x - 9 + 2\,\mathrm{e}^{-x} = 0 \\
&\Rightarrow 4\,\mathrm{e}^{2x} - 9\,\mathrm{e}^x + 2 = 0 \\
&\Rightarrow (4\,\mathrm{e}^x - 1)(\mathrm{e}^x - 2) = 0 \\
&\Rightarrow \mathrm{e}^x = \tfrac{1}{4} \text{ or } \mathrm{e}^x = 2 \\
&\Rightarrow \underline{\underline{x = \ln \tfrac{1}{4}}} \text{ or } \underline{\underline{x = \ln 2}}.
\end{aligned}$$

The finite region between the two curves is shown shaded in the figure.

(b) Using calculus, find the area of the shaded region, giving your answer in the form $a \ln b + c$, where a, b, and c are integers. (6)

Solution

$$\begin{aligned}
\text{Area} &= \int_{\ln \frac{1}{4}}^{\ln 2} (9 - 2\sinh x - 6\cosh x)\,\mathrm{d}x \\
&= \int_{\ln \frac{1}{4}}^{\ln 2} (9 - 4\,\mathrm{e}^x - 2\,\mathrm{e}^{-x})\,\mathrm{d}x \\
&= \left[9x - 4\,\mathrm{e}^x + 2\,\mathrm{e}^{-x}\right]_{x=\ln \frac{1}{4}}^{\ln 2} \\
&= (9\ln 2 - 8 + 1) - (9\ln \tfrac{1}{4} - 1 + 8) \\
&= \underline{\underline{9\ln 8 - 14}}.
\end{aligned}$$

40. The curve C, shown in Figure 9, has equation

$$y = 2x^{\frac{1}{2}},\ 1 \leqslant x \leqslant 8.$$

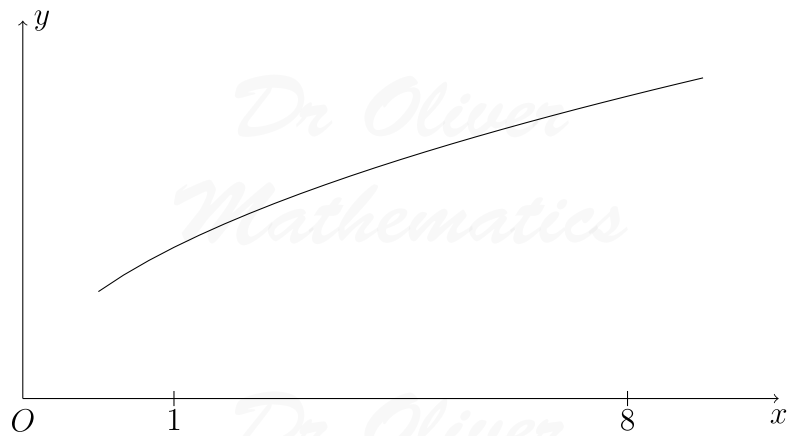


Figure 9: $y = 2x^{\frac{1}{2}}$

(a) Show that the length s of the curve C is given by the equation (2)

$$s = \int_1^8 \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^{\frac{1}{2}}}$$

and we have

$$s = \int_1^8 \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x.$$

(b) Using the substitution $x = \sinh^2 u$, or otherwise, find an exact value for s. Give your answer in the form $a\sqrt{2} + \ln(b + c\sqrt{2})$, where a, b, and c are integers. (9)

Solution

$$u = \operatorname{arsinh}\sqrt{x} \Rightarrow \sinh u = \sqrt{x} \Rightarrow x = \sinh^2 u,$$

$$x = \sinh^2 u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = 2\sinh u \cosh u \Rightarrow \mathrm{d}x = 2\sinh u \cosh u\,\mathrm{d}u,$$

and

$$x = 1 \Rightarrow u = \operatorname{arsinh} 1 \text{ and } x = 8 \Rightarrow u = \operatorname{arsinh} 2\sqrt{2}.$$

$$\begin{aligned}
s &= \int_1^8 \sqrt{1 + \frac{1}{x}}\, \mathrm{d}x \\
&= \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} \sqrt{1 + \frac{1}{\sinh^2 u}} 2 \sinh u \cosh u \, \mathrm{d}u \\
&= \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} \sqrt{\frac{\sinh^2 u + 1}{\sinh^2 u}} 2 \sinh u \cosh u \, \mathrm{d}u \\
&= \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} 2 \coth u \sinh u \cosh u \, \mathrm{d}u \\
&= \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} 2 \cosh^2 u \, \mathrm{d}u \\
&= \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} (1 + \cosh 2u) \, \mathrm{d}u \\
&= \left[u + \tfrac{1}{2} \sinh 2u\right]_{u=\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} \\
&= (\operatorname{arsinh} 2\sqrt{2} + \tfrac{1}{2} \sinh(2 \operatorname{arsinh} 2\sqrt{2})) - (\operatorname{arsinh} 1 + \tfrac{1}{2} \sinh(2 \operatorname{arsinh} 1)) \\
&= (\ln(3 + 2\sqrt{2}) + 6\sqrt{2}) - (\ln(1 + \sqrt{2}) + \sqrt{2}) \\
&= \underline{\underline{\ln(1 + \sqrt{2}) + 5\sqrt{2}}}
\end{aligned}$$

and we have, of course,

$$\begin{aligned}
\tfrac{1}{2} \sinh(2 \operatorname{arsinh} 2\sqrt{2}) &= \sinh(\operatorname{arsinh} 2\sqrt{2}) \cosh(\operatorname{arsinh} 2\sqrt{2}) \\
&= 2\sqrt{2} \cosh(\ln(3 + 2\sqrt{2})) \\
&= \sqrt{2} \left(\mathrm{e}^{\ln(3+2\sqrt{2})} + \mathrm{e}^{-\ln(3+2\sqrt{2})}\right) \\
&= \sqrt{2} \left(3 + 2\sqrt{2} + \frac{1}{3 + 2\sqrt{2}}\right) \\
&= 6\sqrt{2}
\end{aligned}$$

and

$$\begin{aligned}\tfrac{1}{2}\sinh(2\operatorname{arsinh}1) &= \sinh(\operatorname{arsinh}1)\cosh(\operatorname{arsinh}1)\\ &= \cosh(\ln(1+\sqrt{2}))\\ &= \tfrac{1}{2}\left(\mathrm{e}^{\ln(1+\sqrt{2})}+\mathrm{e}^{-\ln(1+\sqrt{2})}\right)\\ &= \tfrac{1}{2}\left(1+\sqrt{2}+\frac{1}{1+\sqrt{2}}\right)\\ &= \sqrt{2}.\end{aligned}$$

41. Using calculus, find the exact value of

(a) $\displaystyle\int_1^2 \frac{1}{\sqrt{x^2-2x+3}}\,\mathrm{d}x,$ (4)

Solution

$$\begin{aligned}\int_1^2 \frac{1}{\sqrt{x^2-2x+3}}\,\mathrm{d}x &= \int_1^2 \frac{1}{\sqrt{(x-1)^2+(\sqrt{2})^2}}\,\mathrm{d}x\\ &= \left[\operatorname{arsinh}\left(\frac{x-1}{\sqrt{2}}\right)\right]_{x=1}^2\\ &= \operatorname{arsinh}\frac{1}{\sqrt{2}}-0\\ &= \underline{\underline{\ln\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)}}.\end{aligned}$$

(b) $\displaystyle\int_0^1 \mathrm{e}^{2x}\sinh x\,\mathrm{d}x.$ (4)

Solution

$$
\begin{aligned}
\int_0^1 e^{2x} \sinh x \, dx &= \int_0^1 \tfrac{1}{2} e^{2x} \left(e^x - e^{-x} \right) \, dx \\
&= \int_0^1 \tfrac{1}{2} \left(e^{3x} - e^x \right) \, dx \\
&= \tfrac{1}{2} \left[\tfrac{1}{3} e^{3x} - e^x \right]_{x=0}^1 \\
&= \tfrac{1}{2} \left\{ \left(\tfrac{1}{3} e^3 - e \right) - \left(\tfrac{1}{3} - 1 \right) \right\} \\
&= \tfrac{1}{2} \left(\tfrac{1}{3} e^3 - e + \tfrac{2}{3} \right) \\
&= \underline{\underline{\tfrac{1}{6} (e^3 - 3e + 2)}}.
\end{aligned}
$$

42. Using the definitions of hyperbolic functions in terms of exponentials,

(a) show that (3)

$$\operatorname{sech}^2 x \equiv 1 - \tanh^2 x,$$

Solution

$$
\begin{aligned}
1 - \tanh^2 x &\equiv 1 - \frac{\sinh^2 x}{\cosh^2 x} \\
&\equiv \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\
&\equiv \frac{1}{\cosh^2 x} \\
&\equiv \underline{\underline{\operatorname{sech}^2 x}}.
\end{aligned}
$$

(b) solve the equation (4)

$$4 \sinh x - 3 \cosh x = 3.$$

Solution

$$
\begin{aligned}
4\sinh x - 3\cosh x = 3 &\Rightarrow 2\left(e^{x} - e^{-x}\right) - \tfrac{3}{2}\left(e^{x} + e^{-x}\right) = 3 \\
&\Rightarrow \tfrac{1}{2}e^{x} - \tfrac{7}{2}e^{-x} - 3 = 0 \\
&\Rightarrow e^{x} - 7e^{-x} - 6 = 0 \\
&\Rightarrow e^{2x} - 6e^{x} - 7 = 0 \\
&\Rightarrow (e^{x} - 7)(e^{x} + 1) = 0 \\
&\Rightarrow e^{x} = 7 \\
&\Rightarrow \underline{\underline{x = \ln 7}}.
\end{aligned}
$$

43. Given that $y = \operatorname{artanh} \dfrac{x}{\sqrt{1+x^2}}$, show that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}$.

Solution

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{1 - \frac{x^2}{1+x^2}} \times \frac{\sqrt{1+x^2} \times 1 - x \times \frac{x}{\sqrt{1+x^2}}}{1+x^2} \\
&= \frac{1}{\frac{(1+x^2)-x^2}{1+x^2}} \times \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} \\
&= \frac{1}{\frac{1}{1+x^2}} \times \frac{1+x^2-x^2}{(1+x^2)^{\frac{3}{2}}} \\
&= (1+x^2) \times \frac{1}{(1+x^2)^{\frac{3}{2}}} \\
&= \underline{\underline{\frac{1}{\sqrt{1+x^2}}}}.
\end{aligned}
$$

44. Solve the equation

$$5\tanh x + 7 = 5\operatorname{sech} x.$$

Give each answer in the form $\ln k$ where k is a rational number.

Solution

$$\begin{aligned}5\tanh x + 7 = 5\operatorname{sech} x &\Rightarrow \frac{5(\mathrm{e}^{x} - \mathrm{e}^{-x})}{\mathrm{e}^{x} + \mathrm{e}^{-x}} + 7 = \frac{10}{\mathrm{e}^{x} + \mathrm{e}^{-x}} \\ &\Rightarrow 5(\mathrm{e}^{x} - \mathrm{e}^{-x}) + 7(\mathrm{e}^{x} + \mathrm{e}^{-x}) = 10 \\ &\Rightarrow 12\,\mathrm{e}^{x} + 2\,\mathrm{e}^{-x} - 10 = 0 \\ &\Rightarrow 6\,\mathrm{e}^{x} + \mathrm{e}^{-x} - 5 = 0 \\ &\Rightarrow 6\,\mathrm{e}^{2x} - 5\,\mathrm{e}^{x} + 1 = 0 \\ &\Rightarrow (3\,\mathrm{e}^{x} - 1)(2\,\mathrm{e}^{x} - 1) = 0 \\ &\Rightarrow \mathrm{e}^{x} = \tfrac{1}{3} \text{ or } \mathrm{e}^{x} = \tfrac{1}{2} \\ &\Rightarrow \underline{\underline{x = \ln\tfrac{1}{3}}} \text{ or } \underline{\underline{x = \ln\tfrac{1}{2}}}.\end{aligned}$$

45.

$$9x^2 + 6x + 5 \equiv a(x + b)^2 + c.$$

(a) Find the values of the constants a, b, and c. (3)

Solution

$$\begin{aligned}9x^2 + 6x + 5 &\equiv 9(x^2 + \tfrac{2}{3}x) + 5 \\ &\equiv 9[(x + \tfrac{1}{3})^2 - \tfrac{1}{9}] + 5 \\ &\equiv \underline{\underline{9(x + \tfrac{1}{3})^2 + 4}}.\end{aligned}$$

Hence, or otherwise, find

(b) $\displaystyle\int \frac{1}{9x^2 + 6x + 5}\,\mathrm{d}x$, (2)

Solution

$$\begin{aligned}\int \frac{1}{9x^2 + 6x + 5}\,\mathrm{d}x &= \int \frac{1}{9(x + \frac{1}{3})^2 + 2^2}\,\mathrm{d}x \\ &= \underline{\underline{\frac{1}{6}\arctan\left(\frac{3x + 1}{2}\right) + c}}.\end{aligned}$$

(c) $\displaystyle\int \frac{1}{\sqrt{9x^2 + 6x + 5}}\,\mathrm{d}x$. (2)

Solution

$$\int \frac{1}{\sqrt{9x^2+6x+5}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(3(x+\frac{1}{3}))^2+2^2}}\,\mathrm{d}x$$
$$= \underline{\underline{\frac{1}{3}\operatorname{arsinh}\left(\frac{3x+1}{2}\right)+c.}}$$

46. The curve C has equation $y = \frac{1}{2}\ln(\coth x)$, $x > 0$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\operatorname{cosech} 2x$. (3)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\operatorname{cosech}^2}{2\coth x}\\ &= -\frac{\sinh x}{2\sinh^2 x\cosh x}\\ &= -\frac{1}{2\sinh x\cosh x}\\ &= -\frac{1}{\sinh 2x}\\ &= \underline{\underline{-\operatorname{cosech} 2x}}.\end{aligned}$$

The points A and B lie on C. The x-coordinates of A and B are $\ln 2$ and $\ln 3$ respectively.

(b) Find the length of the arc AB, giving your answer in the form $p\ln q$, where p and q are rational numbers. (6)

Solution

$$\begin{aligned}\text{Length of the arc} &= \int_{\ln 2}^{\ln 3}\sqrt{1+\operatorname{cosech}^2 2x}\,\mathrm{d}x\\ &= \int_{\ln 2}^{\ln 3}\coth 2x\,\mathrm{d}x\\ &= \left[\tfrac{1}{2}\ln(\sinh 2x)\right]_{x=\ln 2}^{\ln 3}\\ &= \tfrac{1}{2}[\ln(\sinh 2\ln 3)-\ln(\sinh 2\ln 2)].\end{aligned}$$

Now,

$$\begin{aligned}\sinh(2\ln 3) &= \tfrac{1}{2}(\mathrm{e}^{2\ln 3} - \mathrm{e}^{-2\ln 3})\\ &= \tfrac{1}{2}(\mathrm{e}^{\ln 3^2} - \mathrm{e}^{-\ln 3^2})\\ &= \tfrac{1}{2}(9 - \tfrac{1}{9})\\ &= \tfrac{40}{9}\end{aligned}$$

and

$$\begin{aligned}\sinh(2\ln 2) &= \tfrac{1}{2}(\mathrm{e}^{2\ln 2} - \mathrm{e}^{-2\ln 2})\\ &= \tfrac{1}{2}(\mathrm{e}^{\ln 2^2} - \mathrm{e}^{-\ln 2^2})\\ &= \tfrac{1}{2}(4 - \tfrac{1}{4})\\ &= \tfrac{15}{8}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{length of the arc} &= \tfrac{1}{2}(\ln \tfrac{40}{9} - \ln \tfrac{15}{8})\\ &= \underline{\underline{\tfrac{1}{2}\ln \tfrac{64}{27}}}\\ &= \tfrac{1}{2}\ln(\tfrac{4}{3})^3\\ &= \underline{\underline{\tfrac{3}{2}\ln \tfrac{4}{3}}}.\end{aligned}$$

47. The curve C has equation

$$y = \mathrm{e}^{-x},\ x \in \mathbb{R}.$$

The part of the curve C between $x = 0$ and $x = \ln 3$ is rotated through 2π radians about the x-axis.

(a) Show that the area S of the curved surface generated is given by (3)

$$S = 2\pi \int_0^{\ln 3} \mathrm{e}^{-x}\sqrt{1 + \mathrm{e}^{-2x}}\,\mathrm{d}x.$$

Solution

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + (-\mathrm{e}^{-x})^2} = \sqrt{1 + \mathrm{e}^{-2x}}$$

and so we have

$$\underline{\underline{S = 2\pi \int_0^{\ln 3} \mathrm{e}^{-x}\sqrt{1 + \mathrm{e}^{-2x}}\,\mathrm{d}x}}.$$

(b) Use the substitution $e^{-x} = \sinh u$ to show that (5)

$$S = 2\pi \int_{\operatorname{arsinh}\alpha}^{\operatorname{arsinh}\beta} \cosh^2 u \, du,$$

where α and β are constants to be determined.

Solution

$$u = \operatorname{arsinh} e^{-x} \Rightarrow e^{-x} = \sinh u \Rightarrow -x = \ln(\sinh u) \Rightarrow x = -\ln(\sinh u),$$

$$\frac{dx}{du} = -\coth u \Rightarrow dx = -\coth u \, du,$$

$$x = 0 \Rightarrow u = \operatorname{arsinh} 1 \text{ and } x = \ln 3 \Rightarrow u = \operatorname{arsinh} \tfrac{1}{3}.$$

$$\begin{aligned} S &= 2\pi \int_0^{\ln 3} e^{-x} \sqrt{1 + e^{-2x}} \, dx \\ &= 2\pi \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} \frac{1}{3}} \sinh u \sqrt{1 + \sinh^2 u} (-\coth u) \, du \\ &= -2\pi \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} \frac{1}{3}} \sinh u \cosh u \coth u \, du \\ &= -2\pi \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} \frac{1}{3}} \cosh^2 u \, du \\ &= \underline{\underline{2\pi \int_{\operatorname{arsinh} \frac{1}{3}}^{\operatorname{arsinh} 1} \cosh^2 u \, du.}} \end{aligned}$$

(c) Show that (2)

$$2 \int \cosh^2 u \, du = \tfrac{1}{2} \sinh 2u + u + k,$$

where k is an arbitrary constant.

Solution

$$\begin{aligned} 2 \int \cosh^2 u \, du &= \int (\cosh 2u + 1) \, du \\ &= \underline{\underline{\tfrac{1}{2} \sinh 2u + u + k.}} \end{aligned}$$

(d) Hence find the value of S, giving your answer to 3 decimal places. (2)

Solution

$$\begin{aligned}
S &= 2\pi \int_0^{\ln 3} \mathrm{e}^{-x}\sqrt{1+\mathrm{e}^{-2x}}\,\mathrm{d}x \\
&= 2\pi \int_{\operatorname{arsinh}\frac{1}{3}}^{\operatorname{arsinh} 1} \cosh^2 u\,\mathrm{d}u \\
&= \pi\left[\tfrac{1}{2}\sinh 2u + u\right]_{u=\operatorname{arsinh}\frac{1}{3}}^{\operatorname{arsinh} 1} \\
&= \pi[(\tfrac{1}{2}\sinh(2\operatorname{arsinh} 1) + \operatorname{arsinh} 1) - (\tfrac{1}{2}\sinh(2\operatorname{arsinh}\tfrac{1}{3}) + \operatorname{arsinh}\tfrac{1}{3})] \\
&= 5.079\,241\,597 \text{ (FCD)} \\
&= \underline{\underline{5.079 \text{ (3 dp)}}}.
\end{aligned}$$

48. Solve the equation (6)

$$2\cosh^2 x - 3\sinh x = 1,$$

giving your answers in terms of natural logarithms.

Solution

$$\begin{aligned}
2\cosh^2 x - 3\sinh x = 1 &\Rightarrow 2(1+\sinh^2 x) - 3\sinh x = 1 \\
&\Rightarrow 2\sinh^2 x - 3\sinh x + 1 = 0 \\
&\Rightarrow (2\sinh x - 1)(\sinh x - 1) = 0 \\
&\Rightarrow \sinh x = \tfrac{1}{2} \text{ or } \sinh x = 1 \\
&\Rightarrow x = \operatorname{arsinh}\tfrac{1}{2} \text{ or } x = \operatorname{arsinh} 1 \\
&\Rightarrow \underline{\underline{x = \ln(\tfrac{1+\sqrt{5}}{2})}} \text{ or } \underline{\underline{x = \ln(1+\sqrt{2})}}.
\end{aligned}$$

49. A curve has equation (5)

$$y = \cosh x,\ 1 \leqslant x \leqslant \ln 5.$$

Find the length of this curve. Give your answer in terms of e.

Solution

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1+\sinh^2 x} = \cosh x$$

and we have

$$\begin{aligned}\text{length of the curve} &= \int_1^{\ln 5} \cosh x \, \mathrm{d}x \\ &= [\sinh x]_{x=1}^{\ln 5} \\ &= \sinh(\ln 5) - \sinh 1 \\ &= \tfrac{12}{5} - \tfrac{1}{2}(\mathrm{e} - \mathrm{e}^{-1}) \\ &= \underline{\underline{\tfrac{12}{5} - \tfrac{1}{2}\,\mathrm{e} + \tfrac{1}{2}\,\mathrm{e}^{-1}}}.\end{aligned}$$

50. The curve C has equation

$$y = \frac{1}{\sqrt{x^2 + 2x - 3}},\ x > 1.$$

(a) Find $\int y \, \mathrm{d}x$. (3)

Solution

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 + 2x - 3}} \, \mathrm{d}x &= \int \frac{1}{\sqrt{(x+1)^2 - 2^2}} \, \mathrm{d}x \\ &= \underline{\underline{\operatorname{arcosh}\left(\frac{x+1}{2}\right) + c}}.\end{aligned}$$

The region R is bounded by the curve C, the x-axis, and the lines with equations $x = 2$ and $x = 3$. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid generated. Give your answer in the form $p\pi \ln q$, where p and q are rational numbers to be found. (4)

Solution

$$\begin{aligned}
\text{Volume} &= \pi \int_2^3 \frac{1}{(x+1)^2 - 2^2}\,\mathrm{d}x \\
&= \tfrac{1}{4}\pi \left[\ln \left| \frac{(x+1)-2}{(x+1)+2} \right| \right]_{x=2}^{3} \\
&= \tfrac{1}{4}\pi \left[\ln \left| \frac{x-1}{x+3} \right| \right]_{x=2}^{3} \\
&= \tfrac{1}{4}\pi (\ln \tfrac{1}{3} - \ln \tfrac{1}{5}) \\
&= \underline{\underline{\tfrac{1}{4}\pi \ln \tfrac{5}{3}}}.
\end{aligned}$$

51. The hyperbola H is given by the equation $x^2 - y^2 = 1$.

(a) Write down the equations of the two asymptotes of H. (1)

Solution

$\underline{\underline{y = x}}$ and $\underline{\underline{y = -x}}$

(b) Show that an equation of the tangent to H at the point $P(\cosh t, \sinh t)$ is (3)

$$y \sinh t = x \cosh t - 1.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cosh t}{\sinh t}$$

and we have

$$\begin{aligned}
y - \sinh t = \frac{\cosh t}{\sinh t}(x - \cosh t) &\Rightarrow \sinh t(y - \sinh t) = \cosh t(x - \cosh t) \\
&\Rightarrow y \sinh t - \sinh^2 t = x \cosh t - \cosh^2 t \\
&\Rightarrow y \sinh t = x \cosh t - \cosh^2 t + \sinh^2 t \\
&\Rightarrow \underline{\underline{y \sinh t = x \cosh t - 1}}.
\end{aligned}$$

The tangent at P meets the asymptotes of H at the points Q and R.

(c) Show that P is the midpoint of QR. (3)

Solution

$$\begin{aligned} y = x \Rightarrow x \sinh t &= x \cosh t - 1 \\ \Rightarrow x(\cosh t - \sinh t) &= 1 \\ \Rightarrow x = \frac{1}{\cosh t - \sinh t} &\text{ and } y = \frac{1}{\cosh t - \sinh t} \end{aligned}$$

and

$$\begin{aligned} y = -x \Rightarrow -x \sinh t &= x \cosh t - 1 \\ \Rightarrow x(\cosh t + \sinh t) &= 1 \\ \Rightarrow x = \frac{1}{\cosh t + \sinh t} &\text{ and } y = -\frac{1}{\cosh t + \sinh t}. \end{aligned}$$

Now,

$$\begin{aligned} & \frac{1}{2}\left(\frac{1}{\cosh t - \sinh t} + \frac{1}{\cosh t + \sinh t}\right) \\ = \quad & \frac{1}{2}\left(\frac{(\cosh t + \sinh t) + (\cosh t - \sinh t)}{(\cosh t - \sinh t)(\cosh t + \sinh t)}\right) \\ = \quad & \frac{1}{2}\left(\frac{2\cosh t}{\cosh^2 t - \sinh^2 t}\right) \\ = \quad & \cosh t \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2}\left(\frac{1}{\cosh t - \sinh t} - \frac{1}{\cosh t + \sinh t}\right) \\ = \quad & \frac{1}{2}\left(\frac{(\cosh t + \sinh t) - (\cosh t - \sinh t)}{(\cosh t - \sinh t)(\cosh t + \sinh t)}\right) \\ = \quad & \frac{1}{2}\left(\frac{2\sinh t}{\cosh^2 t - \sinh^2 t}\right) \\ = \quad & \sinh t. \end{aligned}$$

Thus, $\underline{\underline{P \text{ is the midpoint of } QR}}$.

(d) Show that the area of the triangle OQR, where O is the origin, is independent of t. (3)

Solution

The gradient of QR is

$$\frac{\frac{1}{\cosh t - \sinh t} + \frac{1}{\cosh t + \sinh t}}{\frac{1}{\cosh t - \sinh t} - \frac{1}{\cosh t + \sinh t}} = \frac{\frac{2\cosh t}{\cosh^2 t - \sinh^2 t}}{\frac{2\sinh t}{\cosh^2 t - \sinh^2 t}}$$
$$= \frac{\cosh t}{\sinh t}$$

and

$$y - \frac{1}{\cosh t - \sinh t} = \frac{\cosh t}{\sinh t}\left(x - \frac{1}{\cosh t - \sinh t}\right).$$

$y = 0$:

$$-\frac{1}{\cosh t - \sinh t} = \frac{\cosh t}{\sinh t}\left(x - \frac{1}{\cosh t - \sinh t}\right)$$
$$\Rightarrow \quad -\frac{\sinh t}{\cosh t(\cosh t - \sinh t)} = x - \frac{1}{\cosh t - \sinh t}$$
$$\Rightarrow \quad x = \frac{1}{\cosh t - \sinh t} - \frac{\sinh t}{\cosh t(\cosh t - \sinh t)}$$
$$\Rightarrow \quad x = \frac{1}{\cosh t - \sinh t}\left(1 - \frac{\sinh t}{\cosh t}\right)$$
$$\Rightarrow \quad x = \frac{1}{\cosh t - \sinh t}\left(\frac{\cosh t - \sinh t}{\cosh t}\right)$$
$$\Rightarrow \quad x = \frac{1}{\cosh t}$$

and so $A\left(\frac{1}{\cosh t}, 0\right)$.

$$\begin{aligned}
& \text{Area of the triangle} \\
= \; & \text{area of } OAQ + \text{area of } OAR \\
= \; & \frac{1}{2} \times \frac{1}{\cosh t} \times \frac{1}{\cosh t - \sinh t} + \frac{1}{2} \times \frac{1}{\cosh t} \times \frac{1}{\cosh t + \sinh t} \\
= \; & \frac{1}{2\cosh t}\left[\frac{1}{\cosh t - \sinh t} + \frac{1}{\cosh t + \sinh t}\right] \\
= \; & \frac{1}{2\cosh t} \times 2\cosh t \\
= \; & 1
\end{aligned}$$

and so the area is independent of t.

52. (a) Prove that (3)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arcoth} x) = \frac{1}{1-x^2}.$$

Solution

$$\begin{aligned} y = \operatorname{arcoth} x \Rightarrow x &= \coth y \\ \Rightarrow x &= \frac{\cosh y}{\sinh y} \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{\sinh^2 y - \cosh^2 y}{\sinh^2 y} \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= -\frac{1}{\sinh^2 y} \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= -\operatorname{cosech}^2 y \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= 1 - \coth^2 y \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1 - \coth^2 y} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1-x^2}. \end{aligned}$$

Given that $y = (\operatorname{arcoth} x)^2$,

(b) show that (5)

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{1-x^2},$$

where k is a constant to be determined.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\operatorname{arcoth} x}{1-x^2}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2(1-x^2) \times \frac{1}{1-x^2} - 2\operatorname{arcoth} x \times (-2x)}{(1-x^2)^2} = \frac{2 + 4x\operatorname{arcoth} x}{(1-x^2)^2}.$$

Now,

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2+4x\operatorname{arcoth}x}{1-x^2} - \frac{4x\operatorname{arcoth}x}{1-x^2}$$
$$= \underline{\underline{\frac{2}{1-x^2}}}.$$

53. (a) Show that (3)

$$5\cosh x - 4\sinh x = \frac{\mathrm{e}^{2x}+9}{2\,\mathrm{e}^x}.$$

Solution

$$5\cosh x - 4\sinh x = \tfrac{5}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) - 2\left(\mathrm{e}^x - \mathrm{e}^{-x}\right)$$
$$= \tfrac{1}{2}\,\mathrm{e}^x + \tfrac{9}{2}\,\mathrm{e}^{-x}$$
$$= \frac{\mathrm{e}^x + 9\,\mathrm{e}^{-x}}{2}$$
$$= \underline{\underline{\frac{\mathrm{e}^{2x}+9}{2\,\mathrm{e}^x}}}.$$

(b) Hence, using the substitution $u = \mathrm{e}^x$ or otherwise, find (4)

$$\int \frac{1}{5\cosh x - 4\sinh x}\,\mathrm{d}x.$$

Solution

$$u = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow \frac{1}{u}\,\mathrm{d}u = \mathrm{d}x$$

and

$$\int \frac{1}{5\cosh x - 4\sinh x}\,\mathrm{d}x = \int \frac{2\,\mathrm{e}^x}{\mathrm{e}^{2x}+9}\,\mathrm{d}x$$
$$= \int \frac{2u}{u(u^2+9)}\,\mathrm{d}u$$
$$= \int \frac{2}{u^2+9}\,\mathrm{d}u$$
$$= \frac{2}{3}\arctan(\frac{u}{3}) + c$$
$$= \underline{\underline{\frac{2}{3}\arctan\left(\frac{\mathrm{e}^x}{3}\right) + c.}}$$

54. Given that $y = \operatorname{arsinh}(\tanh x)$, show that (5)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\operatorname{sech}^2 x}{\sqrt{1+\tanh^2 x}}.$$

Solution

$$y = \operatorname{arsinh}(\tanh x) \Rightarrow \sinh y = \tanh x$$
$$\Rightarrow \cosh y \frac{\mathrm{d}y}{\mathrm{d}x} = \operatorname{sech}^2 x$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\operatorname{sech}^2 x}{\cosh y}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\operatorname{sech}^2 x}{\sqrt{\cosh^2 y}}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\operatorname{sech}^2 x}{\sqrt{1+\sinh^2 y}}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\operatorname{sech}^2 x}{\sqrt{1+\tanh^2 x}}.}}$$

55. (a) Using the definition for $\cosh x$ in terms of exponentials, show that (3)

$$\cosh 2x \equiv 2\cosh^2 x - 1.$$

Solution

$$\begin{aligned} 2\cosh^2 x - 1 &\equiv 2\left[\left(\frac{e^x + e^{-x}}{2}\right)\right]^2 - 1 \\ &=\equiv \frac{e^{2x} + 2 + e^{-2x}}{2} - 1 \\ &\equiv \frac{e^{2x} + e^{-2x}}{2} \\ &\equiv \underline{\underline{\cosh 2x}}. \end{aligned}$$

(b) Find the exact values of x for which (6)

$$29\cosh x - 3\cosh 2x = 38,$$

giving your answers in terms of natural logarithms.

Solution

$$\begin{aligned} 29\cosh x - 3\cosh 2x = 38 &\Rightarrow 29\cosh x - 3(2\cosh^2 x - 1) = 38 \\ &\Rightarrow 29\cosh x - 6\cosh^2 x + 3 = 38 \\ &\Rightarrow 6\cosh^2 x - 29\cosh x + 35 = 0 \\ &\Rightarrow (3\cosh x - 7)(2\cosh x - 5) = 0 \\ &\Rightarrow \cosh x = \tfrac{7}{3} \text{ or } \cosh x = \tfrac{5}{2} \\ &\Rightarrow \underline{\underline{x = \ln \tfrac{7 \pm 2\sqrt{10}}{3}}} \text{ or } \underline{\underline{x = \ln \tfrac{5 \pm \sqrt{21}}{2}}}. \end{aligned}$$

56.

$$I_n = \int_0^{\ln 2} \cosh^n x \, dx, \; n \geqslant 0.$$

(a) Show that, for $n \geqslant 2$, (6)

$$I_n = \frac{3a^{n-1}}{nb^n} + \frac{n-1}{n} I_{n-2},$$

where a and b are integers to be found.

Solution

$u = \cosh^{n-1} x \Rightarrow \dfrac{du}{dx} = -(n-1)\sinh x \cosh^{n-2} x$ and $\dfrac{dv}{dx} = \cosh x \Rightarrow v = \sinh x$

and

$$\int_0^{\ln 2} \cosh^n x \,\mathrm{d}x = \left[\sinh x \cosh^{n-1} x\right]_{x=0}^{\ln 2} + (n-1)\int_0^{\ln 2} \sinh^2 x \cosh^{n-2} x \,\mathrm{d}x$$
$$= \left(\tfrac{3}{4} \times (\tfrac{5}{4})^{n-1} - 0\right) + (n-1)\int_0^{\ln 2} (1 - \cosh^2 x)\cosh^{n-2} x \,\mathrm{d}x$$
$$= \tfrac{3}{4} \times (\tfrac{5}{4})^{n-1} + (n-1)I_{n-2} - (n-1)I_n$$

and

$$nI_n = \frac{3 \times 5^{n-1}}{4^n} + (n-1)I_{n-2} \Rightarrow \underline{\underline{I_n = \frac{3 \times 5^{n-1}}{n \times 4^n} + \frac{n-1}{n} I_{n-2}.}}$$

(b) Hence, or otherwise, find the exact value of (4)

$$\int_0^{\ln 2} \cosh^4 x \,\mathrm{d}x.$$

Solution

$$\int_0^{\ln 2} \cosh^4 x \,\mathrm{d}x = \frac{3 \times 5^3}{4 \times 4^4} + \tfrac{3}{4} I_2$$
$$= \tfrac{375}{1024} + \tfrac{3}{4}\left(\frac{3 \times 5}{2 \times 4^2} + \tfrac{1}{2} I_0\right)$$
$$= \tfrac{375}{1024} + \tfrac{45}{128} + \tfrac{3}{8}\int_0^{\ln 2} 1 \,\mathrm{d}x$$
$$= \underline{\underline{\tfrac{735}{1024} + \tfrac{3}{8}\ln 2.}}$$

57. The curve C has equation

$$y = \ln\left(\frac{\mathrm{e}^x + 1}{\mathrm{e}^x - 1}\right), \ln 2 \leqslant x \leqslant 3.$$

(a) Show that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2\,\mathrm{e}^x}{\mathrm{e}^{2x} - 1}.$$

Solution

$$y = \ln\left(\frac{\mathrm{e}^x+1}{\mathrm{e}^x-1}\right) \Rightarrow y = \ln(\mathrm{e}^x+1) - \ln(\mathrm{e}^x-1)$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x}{\mathrm{e}^x+1} - \frac{\mathrm{e}^x}{\mathrm{e}^x-1}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x(\mathrm{e}^x-1) - \mathrm{e}^x(\mathrm{e}^x+1)}{(\mathrm{e}^x+1)(\mathrm{e}^x-1)}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2\,\mathrm{e}^x}{\mathrm{e}^{2x}-1}}}.$$

(b) Find the length of the curve C, giving your answer in the form $\ln a$, where a is a rational number. (6)

Solution

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1+\left(-\frac{2\,\mathrm{e}^x}{\mathrm{e}^{2x}-1}\right)^2}$$
$$= \sqrt{1+\frac{4\,\mathrm{e}^{2x}}{(\mathrm{e}^{2x}-1)^2}}$$
$$= \sqrt{\frac{(\mathrm{e}^{2x}-1)^2+4\,\mathrm{e}^{2x}}{(\mathrm{e}^{2x}-1)^2}}$$
$$= \sqrt{\frac{(\mathrm{e}^{4x}-2\,\mathrm{e}^{2x}+1)+4\,\mathrm{e}^{2x}}{(\mathrm{e}^{2x}-1)^2}}$$
$$= \sqrt{\frac{\mathrm{e}^{4x}+2\,\mathrm{e}^{2x}+1}{(\mathrm{e}^{2x}-1)^2}}$$
$$= \sqrt{\frac{(\mathrm{e}^{2x}+1)^2}{(\mathrm{e}^{2x}-1)^2}}$$
$$= \frac{\mathrm{e}^{2x}+1}{\mathrm{e}^{2x}-1}$$

and

$$\begin{aligned}
\text{length of the curve} &= \int_{\ln 2}^{\ln 3} \frac{\mathrm{e}^{2x}+1}{\mathrm{e}^{2x}-1}\,\mathrm{d}x \\
&= \int_{\ln 2}^{\ln 3} \coth x\,\mathrm{d}x \\
&= [\ln\sinh x]_{x=\ln 2}^{\ln 3} \\
&= \ln\sinh(\ln 3) - \ln\sinh(\ln 2) \\
&= \ln\tfrac{4}{3} - \ln\tfrac{3}{4} \\
&= \underline{\underline{\ln\tfrac{16}{9}}}.
\end{aligned}$$