Dr Oliver Mathematics
Mathematics
Expectation and Variance
Past Examination Questions

This booklet consists of 31 questions across a variety of examination topics.
The total number of marks available is 338.

1. The discrete random variable X has probability function

$$\mathrm{P}(X = x) = \begin{cases} k(2 - x), & x = 0, 1, 2, \\ k(x - 2), & x = 3, \\ 0, & \text{otherwise,} \end{cases}$$

where k is a positive constant.

(a) Show that $k = 0.25$. (2)

Solution

$$\sum_{k=0}^{3} x\,\mathrm{P}(X = k) = 1 \Rightarrow 2k + k + 0 + k = 1$$
$$\Rightarrow 4k = 1$$
$$\Rightarrow \underline{\underline{k = 0.25}},$$

as required.

(b) Find $\mathrm{E}(X)$ and show that $\mathrm{E}(X^2) = 2.5$. (4)

Solution

$$\begin{aligned}\mathrm{E}(X) &= (0.5 \times 0) + (0.25 \times 1) + (0.25 \times 3) \\ &= 0.25 + 0.75 \\ &= \underline{\underline{1}}\end{aligned}$$

and

$$\begin{aligned}\mathrm{E}(X^2) &= (0.5 \times 0^2) + (0.25 \times 1^2) + (0.25 \times 3^2) \\ &= 0.25 + 2.25 \\ &= \underline{\underline{2.5}},\end{aligned}$$

as required.

(c) Find $\text{Var}(3X - 2)$. (3)

Solution

$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2) - \text{E}(X)^2 \\ &= 2.5 - 1 \\ &= 1.5\end{aligned}$$

and

$$\begin{aligned}\text{Var}(3X - 2) &= \text{Var}(3X) \\ &= 9\,\text{Var}(X) \\ &= 9 \times 1.5 \\ &= \underline{\underline{13.5}}.\end{aligned}$$

Two independent observations X_1 and X_2 are made of X.

(d) Show that $\text{P}(X_1 + X_2 = 5) = 0$. (1)

Solution

x	0	1	3
$\text{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Now,

$$\begin{aligned}\text{P}(X_1 + X_2 = 5) &= \text{P}(X_1 = 3 \text{ and } X_2 = 2) + \text{P}(X_1 = 2 \text{ and } X_2 = 3) \\ &= 0 + 0 \\ &= \underline{\underline{0}}.\end{aligned}$$

(e) Find the complete probability function for $X_1 + X_2$. (3)

Solution

Let $Y = X_1 + X_2$.

Y	0	1	2	3	4	6
$\text{P}(Y = y)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

(f) Find $\mathrm{P}(1.3 \leqslant X_1 + X_2 \leqslant 3.2)$. (3)

Solution

$$\begin{aligned}\mathrm{P}(1.3 \leqslant X_1 + X_2 \leqslant 3.2) &= \mathrm{P}(X_1 + X_2 = 2) + \mathrm{P}(X_1 + X_2 = 3) \\ &= \tfrac{1}{16} + \tfrac{1}{4} \\ &= \underline{\underline{\tfrac{5}{16}}}.\end{aligned}$$

2. A discrete random variable X has a probability function as shown in the table below, where a and b are constants.

x	0	1	2	3
$\mathrm{P}(X = x)$	0.2	0.3	b	a

Given that $\mathrm{E}(X) = 1.7$,

(a) find the value of a and the value of b. (5)

Solution

$$0.2 + 0.3 + b + a = 1 \Rightarrow b = 0.5 - a$$

and

$$\begin{aligned} & (0 \times 0.2) + (1 \times 0.3) + (2 \times b) + (3 \times a) = 1.7 \\ \Rightarrow\quad & 0.3 + 2b + 3a = 1.7 \\ \Rightarrow\quad & 2b = 1.4 - 3a \\ \Rightarrow\quad & b = 0.7 - 1.5a \end{aligned}$$

and eliminate b:

$$0.5 - a = 0.7 - 1.5a \Rightarrow 0.5a = 0.2 \Rightarrow \underline{\underline{a = 0.4}}$$

and

$$b = 0.5 - 0.4 = \underline{\underline{0.1}}.$$

Find

(b) $\mathrm{P}(0 < X < 1.5)$, (1)

Solution

$$\mathrm{P}(0 < X < 1.5) = \mathrm{P}(X = 1) = \underline{\underline{0.3}}.$$

(c) $\mathrm{E}(2X - 3)$. (2)

Solution

$$\mathrm{E}(2X - 3) = 2\,\mathrm{E}(X) - 3 = 2 \times 1.7 - 3 = \underline{\underline{0.4}}$$

(d) Show that $\mathrm{Var}(X) = 1.41$. (3)

Solution

$$\begin{aligned}\mathrm{E}(X^2) &= (0.2 \times 0^2) + (0.3 \times 1^2) + (0.1 \times 2^2) + (0.4 \times 3^2)\\ &= 0.3 + 0.4 + 3.6\\ &= 4.3,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2\\ &= 4.3 - 1.7^2\\ &= \underline{\underline{1.41}},\end{aligned}$$

as required.

(e) Evaluate Var(2X?3). (2)

Solution

$$\begin{aligned}\mathrm{Var}(2X - 3) &= \mathrm{Var}(2X)\\ &= 4\,\mathrm{Var}(X)\\ &= 4 \times 1.41\\ &= \underline{\underline{5.64}}.\end{aligned}$$

3. The discrete random variable X has probability function

$$\mathrm{P}(X = x) = \begin{cases} kx, & x = 1, 2, 3,\\ k(x+1), & x = 4, 5,\end{cases}$$

where k is a constant.

(a) Find the value of k. (2)

Solution

$$\sum_{k=1}^{5} x\,\mathrm{P}(X = k) = 1 \Rightarrow k + 2k + 3k + 5k + 6k = 1$$
$$\Rightarrow 17k = 1$$
$$\Rightarrow \underline{\underline{k = \tfrac{1}{17}}}.$$

(b) Find the exact value of $\mathrm{E}(X)$. (2)

Solution

$$\begin{aligned}\mathrm{E}(X) &= \tfrac{1}{17}[(1 \times 1) + (2 \times 2) + (3 \times 3) + (5 \times 4) + (6 \times 5)\\ &= \tfrac{1}{17}(1 + 4 + 9 + 20 + 30)\\ &= \underline{\underline{3\tfrac{13}{17}}}.\end{aligned}$$

(c) Show that, to 3 significant figures, $\mathrm{Var}(X) = 1.47$. (4)

Solution

$$\begin{aligned}\mathrm{E}(X^2) = k &= \tfrac{1}{17}[(1 \times 1^2) + (2 \times 2^2) + (3 \times 3^2) + (5 \times 4^2) + (6 \times 5^2)\\ &= \tfrac{1}{17}(1 + 8 + 27 + 80 + 150)\\ &= 15\tfrac{11}{17}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2\\ &= 15\tfrac{11}{17} - (3\tfrac{13}{17})^2\\ &= 1\tfrac{137}{289}\\ &= \underline{\underline{1.47\text{ (3 sf)}}},\end{aligned}$$

as required.

(d) Find, to 1 decimal place, $\mathrm{Var}(4 - 3X)$. (2)

Solution

$$\begin{aligned} \text{Var}(4-3X) &= \text{Var}(3X) \\ &= 9\,\text{Var}(X) \\ &= 9 \times 1\tfrac{137}{289} \\ &= 13\tfrac{77}{289} \\ &= \underline{\underline{13.3\ (1\text{ dp})}}. \end{aligned}$$

4. The random variable X has probability distribution

x	1	2	3	4	5
$\text{P}(X = x)$	0.10	p	0.20	q	0.30

(a) Given that $\text{E}(X) = 3.5$, write down two equations involving p and q. (3)

Solution

$$0.1 + p + 0.2 + q + 0.3 = 1 \Rightarrow \underline{\underline{p + q = 0.4}}$$

and

$$\begin{aligned} \text{E}(X) = 3.5 &\Rightarrow (0.1 \times 1) + (p \times 2) + (0.2 \times 3) + (q \times 4) + (0.3 \times 5) = 3.5 \\ &\Rightarrow 0.1 + 2p + 0.6 + 4q + 1.5 = 3.5 \\ &\Rightarrow \underline{\underline{2p + 4q = 1.3}}. \end{aligned}$$

Find

(b) the value of p and the value of q, (3)

Solution

$$\begin{aligned} q = 0.4 - p &\Rightarrow 2p + 4(0.4 - p) = 1.3 \\ &\Rightarrow 2p + 1.6 - 4p = 1.3 \\ &\Rightarrow 2p = 0.3 \\ &\Rightarrow \underline{\underline{p = 0.15}} \end{aligned}$$

and

$$q = 0.4 - 0.15 = \underline{\underline{0.25}}.$$

(c) $\mathrm{Var}(X)$, (4)

Solution

$$\begin{aligned} \mathrm{E}(X^2) &= (0.1 \times 1^2) + (0.15 \times 2^2) + (0.2 \times 3^2) + (0.25 \times 4^2) + (0.3 \times 5^2) \\ &= 0.1 + 0.6 + 1.8 + 4 + 7.5 \\ &= 14 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\ &= 14 - 3.5^2 \\ &= \underline{\underline{1.75}}. \end{aligned}$$

(d) $\mathrm{Var}(3 - 2X)$. (2)

Solution

$$\begin{aligned} \mathrm{Var}(3 - 2X) &= \mathrm{Var}(2X) \\ &= 4\,\mathrm{Var}(X) \\ &= 4 \times 1.75 \\ &= \underline{\underline{7}}. \end{aligned}$$

5. The random variable X has the discrete uniform distribution

$$\mathrm{P}(X = x) = \tfrac{1}{5},\ x = 1, 2, 3, 4, 5.$$

(a) Write down the value of $\mathrm{E}(X)$ and show that $\mathrm{Var}(X) = 2$. (3)

Solution

This is discrete uniform distribution over $\{1, 2, 3, 4, 5\}$ and so

$$\mathrm{E}(X) = \frac{5 + 1}{2} = \underline{\underline{3}}.$$

$$
\begin{aligned}
\mathrm{E}(X^2) &= (0.2 \times 1^2) + (0.2 \times 2^2) + (0.2 \times 3^2) + (0.2 \times 4^2) + (0.2 \times 5^2) \\
&= 0.2 + 0.8 + 1.8 + 3.2 + 5 \\
&= 11
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\
&= 11 - 3^2 \\
&= \underline{\underline{2}}.
\end{aligned}
$$

Find

(b) $\mathrm{E}(3X - 2)$, (2)

Solution

$$\mathrm{E}(3X - 2) = 3\,\mathrm{E}(X) - 2 = 3 \times 3 - 2 = \underline{\underline{7}}.$$

(c) $\mathrm{Var}(4 - 3X)$. (2)

Solution

$$
\begin{aligned}
\mathrm{Var}(4 - 3X) &= \mathrm{Var}(3X) \\
&= 9\,\mathrm{Var}(X) \\
&= 9 \times 2 \\
&= \underline{\underline{18}}.
\end{aligned}
$$

6. The random variable X has probability function

$$\mathrm{P}(X = x) = \frac{2x - 1}{36},\ x = 1, 2, 3, 4, 5, 6.$$

(a) Construct a table giving the probability distribution of X. (3)

Solution

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{36}$	$\frac{1}{12}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{1}{4}$	$\frac{11}{36}$

Find

(b) $P(2 < X \leqslant 5)$, (2)

Solution

$$\begin{aligned} P(2 < X \leqslant 5) &= P(X = 3) + P(X = 4) + P(X = 5) \\ &= \tfrac{5}{36} + \tfrac{7}{36} + \tfrac{1}{4} \\ &= \underline{\underline{\tfrac{7}{12}}}. \end{aligned}$$

(c) the exact value of $E(X)$. (2)

Solution

$$\begin{aligned} E(X) &= (\tfrac{1}{36} \times 1) + (\tfrac{1}{12} \times 2) + (\tfrac{5}{36} \times 3) + (\tfrac{7}{36} \times 4) + (\tfrac{1}{4} \times 5) + (\tfrac{11}{36} \times 6) \\ &= \tfrac{1}{36} + \tfrac{1}{6} + \tfrac{5}{12} + \tfrac{7}{9} + \tfrac{5}{4} + \tfrac{11}{6} \\ &= \underline{\underline{4\tfrac{17}{36}}}. \end{aligned}$$

(d) Show that $\text{Var}(X) = 1.97$ to 3 significant figures. (4)

Solution

$$\begin{aligned} E(X^2) &= (\tfrac{1}{36} \times 1^2) + (\tfrac{1}{12} \times 2^2) + (\tfrac{5}{36} \times 3^2) + (\tfrac{7}{36} \times 4^2) + (\tfrac{1}{4} \times 5^2) + (\tfrac{11}{36} \times 6^2) \\ &= \tfrac{1}{36} + \tfrac{1}{3} + \tfrac{5}{4} + \tfrac{28}{9} + \tfrac{25}{4} + 11 \\ &= 21\tfrac{35}{36} \end{aligned}$$

and

$$\begin{aligned} \text{Var}(X) &= E(X^2) - E(X)^2 \\ &= 21\tfrac{35}{36} - (4\tfrac{17}{36})^2 \\ &= 1.971\,450\,617 \text{ (FCD)} \\ &= \underline{\underline{1.97 \text{ (3 sf)}}}. \end{aligned}$$

(e) Find $\text{Var}(2 - 3X)$. (2)

Solution

$$\begin{aligned}\text{Var}(2-3X) &= \text{Var}(3X)\\ &= 9\,\text{Var}(X)\\ &= 9\times 1.971\ldots\\ &= \underline{\underline{17\tfrac{107}{144}}}.\end{aligned}$$

7. The random variable X has probability distribution

x	1	3	5	7	9
$\text{P}(X = x)$	0.2	p	0.2	q	0.15

(a) Given that $\text{E}(X) = 4.5$, write down two equations involving p and q. (3)

Solution

$$0.2 + p + 0.2 + q + 0.15 = 1 \Rightarrow \underline{\underline{p + q = 0.45}}$$

$$\begin{aligned}\text{E}(X) = 4.5 &\Rightarrow (0.2\times 1) + (p\times 3) + (0.2\times 5) + (q\times 7) + (0.15\times 9) = 4.5\\ &\Rightarrow 0.2 + 3p + 1 + 7q + 1.35 = 4.5\\ &\Rightarrow \underline{\underline{3p + 7q = 1.95}}.\end{aligned}$$

Find

(b) the value of p and the value of q, (3)

Solution

$$\begin{aligned}q = 0.45 - p &\Rightarrow 3p + 7(0.45 - p) = 1.95\\ &\Rightarrow 3p + 3.15 - 7p = 1.95\\ &\Rightarrow 4p = 1.2\\ &\Rightarrow \underline{\underline{p = 0.3}}\end{aligned}$$

and

$$q = 0.45 - 0.3 = \underline{\underline{0.15}}.$$

(c) $\text{P}(4 < X \leqslant 7)$. (2)

Solution

$$\begin{aligned} \mathrm{P}(4 < X \leqslant 7) &= \mathrm{P}(X = 5) + \mathrm{P}(X = 7) \\ &= 0.2 + 0.15 \\ &= \underline{\underline{0.35}}. \end{aligned}$$

Given that $\mathrm{E}(X^2) = 27.4$, find

(d) $\mathrm{Var}(X)$, (2)

Solution

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\ &= 27.4 - 4.5^2 \\ &= \underline{\underline{7.15}}. \end{aligned}$$

(e) $\mathrm{E}(19 - 4X)$, (1)

Solution

$$\mathrm{E}(19 - 4X) = 19 - 4\,\mathrm{E}(X) = \underline{\underline{1}}.$$

(f) $Var(19 - 4X)$. (2)

Solution

$$\begin{aligned} \mathrm{Var}(19 - 4X) &= \mathrm{Var}(4X) \\ &= 16\,\mathrm{Var}(X) \\ &= 16 \times 7.15 \\ &= \underline{\underline{114.4}}. \end{aligned}$$

8. The table below represents the probability distribution of the random variable T.

t	0	1	2	3	4	6	9
$\mathrm{P}(T = t)$	$\frac{7}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

Find the values of

(a) E(T), (2)

Solution

$$\begin{aligned}
\mathrm{E}(T) &= (\tfrac{7}{16} \times 0) + (\tfrac{1}{16} \times 1) + (\tfrac{1}{8} \times 2) + (\tfrac{1}{8} \times 3) + (\tfrac{1}{16} \times 4) + (\tfrac{1}{8} \times 6) + (\tfrac{1}{16} \times 9) \\
&= \tfrac{1}{16} + \tfrac{1}{4} + \tfrac{3}{8} + \tfrac{1}{4} + \tfrac{3}{4} + \tfrac{9}{16} \\
&= \underline{\underline{2\tfrac{1}{4}}}.
\end{aligned}$$

(b) Var(T). (4)

Solution

$$\begin{aligned}
\mathrm{E}(T^2) &= (\tfrac{7}{16} \times 0^2) + (\tfrac{1}{16} \times 1^2) + (\tfrac{1}{8} \times 2^2) + (\tfrac{1}{8} \times 3^2) + (\tfrac{1}{16} \times 4^2) \\
&\qquad + (\tfrac{1}{8} \times 6^2) + (\tfrac{1}{16} \times 9^2) \\
&= \tfrac{1}{16} + \tfrac{1}{2} + \tfrac{9}{8} + 1 + 4\tfrac{1}{2} + 5\tfrac{1}{16} \\
&= 12
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(T) &= \mathrm{E}(T^2) - \mathrm{E}(T)^2 \\
&= 12\tfrac{1}{4} - (2\tfrac{1}{4})^2 \\
&= \underline{\underline{7\tfrac{3}{16}}}.
\end{aligned}$$

9. The random variable X has probability distribution given in the table below.

x	-1	0	1	2	3
$\mathrm{P}(X = x)$	p	q	0.2	0.15	0.15

Given that $\mathrm{E}(X) = 0.55$, find

(a) the value of p and the value of q, (5)

Solution

$$p + q + 0.2 + 0.15 + 0.15 = 1 \Rightarrow p + q = 0.5$$

and

$$\begin{aligned}
& \mathrm{E}(X) = 0.55 \\
\Rightarrow \quad & (p \times -1) + (q \times 0) + (0.2 \times 1) + (0.15 \times 2) + (0.15 \times 3) = 0.55 \\
\Rightarrow \quad & -p + 0.2 + 0.3 + 0.45 = 0.55 \\
\Rightarrow \quad & \underline{\underline{p = 0.4}} \text{ and } \underline{\underline{q = 0.1}}.
\end{aligned}$$

(b) $\mathrm{Var}(X)$, (4)

Solution

$$\begin{aligned}
\mathrm{E}(X^2) &= (0.4 \times (-1)^2) + (0.1 \times 0^2) + (0.2 \times 1^2) + (0.15 \times 2^2) + (0.15 \times 3^2) \\
&= 0.4 + 0.2 + 0.6 + 1.35 \\
&= 2.55
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\
&= 2.55 - 0.55^2 \\
&= \underline{\underline{2.2475}}.
\end{aligned}$$

(c) $\mathrm{E}(2X - 4)$. (2)

Solution

$$\mathrm{E}(2X - 4) = 2\,\mathrm{E}(X) - 4 = 2 \times 0.55 - 4 = \underline{\underline{-2.9}}.$$

10. The discrete random variable X can take only the values 2, 3, or 4. For these values the cumulative distribution function is defined by

$$\mathrm{F}(x) = \frac{(x+k)^2}{25} \text{ for } x = 2, 3, 4,$$

where k is a positive integer.

(a) Find k. (2)

Solution

$$\mathrm{F}(4) = 1 \Rightarrow \frac{(4+k)^2}{25} = 1 \Rightarrow \underline{\underline{k = 1}}.$$

(b) Find the probability distribution of X. (3)

Solution

$$\mathrm{F}(2) = \frac{(2+1)^2}{25} = \tfrac{9}{25},$$

$$\mathrm{F}(3) = \frac{(3+1)^2}{25} = \tfrac{16}{25} \Rightarrow \mathrm{P}(X = 3) = \tfrac{16}{25} - \tfrac{9}{25} = \tfrac{7}{25},$$

and

$$\mathrm{F}(4) = 1 \Rightarrow \mathrm{P}(X = 4) = 1 - \tfrac{16}{25} = \tfrac{9}{25}.$$

x	2	3	4
$\mathrm{P}(X = x)$	$\frac{9}{25}$	$\frac{7}{25}$	$\frac{9}{25}$

11. When Rohit plays a game, the number of points he receives is given by the discrete random variable X with the following probability distribution.

x	0	1	2	3
$\mathrm{P}(X = x)$	0.4	0.3	0.2	0.1

(a) Find $\mathrm{E}(X)$. (2)

Solution

$$\begin{aligned}\mathrm{E}(X) &= (0.4 \times 0) + (0.3 \times 1) + (0.2 \times 2) + (0.1 \times 3)\\ &= 0 + 0.3 + 0.4 + 0.3\\ &= \underline{\underline{1}}.\end{aligned}$$

(b) Find $\mathrm{F}(1.5)$. (2)

Solution

$$\mathrm{F}(1.5) = \mathrm{P}(X \leqslant 1.5) = \underline{\underline{0.7}}.$$

(c) Show that $\text{Var}(X) = 1$. (4)

Solution

$$\begin{aligned} \text{E}(X^2) &= (0.4 \times 0^2) + (0.3 \times 1^2) + (0.2 \times 2^2) + (0.1 \times 3^2) \\ &= 0 + 0.3 + 0.8 + 0.9 \\ &= 2 \end{aligned}$$

and

$$\begin{aligned} \text{Var}(X) &= \text{E}(X^2) - \text{E}(X)^2 \\ &= 2 - 1^2 \\ &= \underline{\underline{1}}. \end{aligned}$$

(d) Find $\text{Var}(5 - 3X)$. (2)

Solution

$$\begin{aligned} \text{Var}(5 - 3X) &= \text{Var}(3X) \\ &= 9\,\text{Var}(X) \\ &= 9 \times 1 \\ &= \underline{\underline{9}}. \end{aligned}$$

Rohit can win a prize if the total number of points he has scored after 5 games is at least 10. After 3 games he has a total of 6 points. You may assume that games are independent.

(e) Find the probability that Rohit wins the prize. (6)

Solution

$$\begin{aligned} \text{P(Rohit wins)} &= \text{P}(1,3) + \text{P}(2,2) + \text{P}(2,3) + \text{P}(3,3) \\ &= 2 \times 0.3 \times 0.1 + 0.2 \times 0.2 + 2 \times 0.2 \times 0.1 + 0.1 \times 0.1 \\ &= 0.06 + 0.04 + 0.04 + 0.01 \\ &= \underline{\underline{0.15}}. \end{aligned}$$

12. The discrete random variable X has probability function

$$P(X = x) = \begin{cases} a(3-x), & x = 0, 1, 2, \\ b, & x = 3. \end{cases}$$

(a) Find $P(X = 2)$ and complete the table below. (1)

x	0	1	2	3
$P(X = x)$	$3a$	$2a$		b

Solution

$P(X = 2) = a(3 - 2) = \underline{\underline{a}}$ and

x	0	1	2	3
$P(X = x)$	$3a$	$2a$	a	b

Given that $E(X) = 1.6$,

(b) Find the value of a and the value of b. (5)

Solution

$$3a + 2a + a + b = 1 \Rightarrow b = 1 - 6a$$

and

$$\begin{aligned} E(X) = 1.6 &\Rightarrow (3a \times 0) + (2a \times 1) + (a \times 2) + (b \times 3) = 1.6 \\ &\Rightarrow 2a + 2a + 3b = 1.6 \\ &\Rightarrow 4a + 3b = 1.6. \end{aligned}$$

Now,

$$\begin{aligned} 4a + 3(1 - 6a) = 1.6 &\Rightarrow 4a + 3 - 18a = 1.6 \\ &\Rightarrow 14a = 1.4 \\ &\Rightarrow \underline{\underline{a = 0.1}} \text{ and } \underline{\underline{b = 0.4}}. \end{aligned}$$

Find

(c) $P(0.5 < X < 3)$, (2)

Solution

$$P(0.5 < X < 3) = P(X = 1) + P(X = 2) = \underline{\underline{0.3}}.$$

(d) $E(3X - 2)$. (2)

Solution

$$E(3X - 2) = 3\,E(X) - 2 = 3 \times 1.6 - 2 = \underline{\underline{2.8}}.$$

(e) Show that the $\text{Var}(X) = 1.64$. (3)

Solution

$$\begin{aligned} E(X^2) &= (0.3 \times 0^2) + (0.2 \times 1^2) + (0.1 \times 2^2) + (0.4 \times 3^2) \\ &= 0 + 0.2 + 0.4 + 3.6 \\ &= 4.2 \end{aligned}$$

and

$$\begin{aligned} \text{Var}(X) &= E(X^2) - E(X)^2 \\ &= 4.2 - 1.6^2 \\ &= \underline{\underline{1.64}}. \end{aligned}$$

(f) Calculate $\text{Var}(3X - 2)$. (2)

Solution

$$\begin{aligned} \text{Var}(3X - 2) &= \text{Var}(3X) \\ &= 9\,\text{Var}(X) \\ &= 9 \times 1.64 \\ &= \underline{\underline{14.76}}. \end{aligned}$$

13. The probability function of a discrete random variable X is given by

$$p(x) = kx^2,\ x = 1, 2, 3,$$

where k is a positive constant.

(a) Show that $k = \frac{1}{14}$. (2)

Solution

$$k + 4k + 9k = 1 \Rightarrow 14k = 1 \Rightarrow \underline{\underline{k = \tfrac{1}{14}}}.$$

Find

(b) $\mathrm{P}(X \geqslant 2)$, (2)

Solution

$$\mathrm{P}(X \geqslant 2) = 1 - \tfrac{1}{14} = \underline{\underline{\tfrac{13}{14}}}.$$

(c) $\mathrm{E}(X)$, (2)

Solution

$$\begin{aligned}\mathrm{E}(X) &= (\tfrac{1}{14} \times 1) + (\tfrac{2}{7} \times 2) + (\tfrac{9}{14} \times 3) \\ &= \tfrac{1}{14} + \tfrac{4}{7} + \tfrac{27}{14} \\ &= \underline{\underline{2\tfrac{4}{7}}}.\end{aligned}$$

(d) $\mathrm{Var}(1 - X)$. (4)

Solution

$$\begin{aligned}\mathrm{E}(X^2) &= (\tfrac{1}{14} \times 1^2) + (\tfrac{2}{7} \times 2^2) + (\tfrac{9}{14} \times 3^2) \\ &= \tfrac{1}{14} + \tfrac{8}{7} + \tfrac{81}{14} \\ &= 7\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(1 - X) &= \mathrm{Var}(X) \\ &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\ &= 7 - (2\tfrac{4}{7})^2 \\ &= \underline{\underline{\tfrac{19}{49}}}.\end{aligned}$$

14. The discrete random variable X has probability distribution given by

x	-1	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{5}$	a	$\frac{1}{10}$	a	$\frac{1}{5}$

where a is a constant.

(a) Find the value of a. (2)

Solution

$$\tfrac{1}{5} + a + \tfrac{1}{10} + a + \tfrac{1}{5} \Rightarrow 2a = 0.5 \Rightarrow \underline{\underline{a = 0.25}}.$$

(b) Write down $\mathrm{E}(X)$. (1)

Solution

$\mathrm{E}(X) = \underline{\underline{1}}$.

(c) Find $\mathrm{Var}(X)$. (3)

Solution

$$\begin{aligned}\mathrm{E}(X^2) &= (\tfrac{1}{5} \times (-1)^2) + (0.25 \times 0^2) + (\tfrac{1}{10} \times 1^2) + (0.25 \times 2^2) + (\tfrac{1}{5} \times 3^2) \\ &= \tfrac{1}{5} + \tfrac{1}{10} + 1 + \tfrac{9}{5} \\ &= 3\tfrac{1}{10}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\ &= 3\tfrac{1}{10} - 1^2 \\ &= \underline{\underline{2\tfrac{1}{10}}}.\end{aligned}$$

The random variable $Y = 6 - 2X$.

(d) Find $\mathrm{Var}(Y)$. (2)

Solution

$$\begin{aligned}\text{Var}(Y) &= \text{Var}(6-2X)\\ &= 4\,\text{Var}(X)\\ &= 4\times 2\tfrac{1}{10}\\ &= \underline{\underline{8\tfrac{2}{5}}}.\end{aligned}$$

(e) Calculate $\text{P}(X \geqslant Y)$. (3)

Solution

$$\begin{aligned}\text{P}(X \geqslant Y) &= \text{P}(X \geqslant 6-2X)\\ &= \text{P}(3X \geqslant 6)\\ &= \text{P}(X \geqslant 2)\\ &= 0.25+\tfrac{1}{5}\\ &= \underline{\underline{\tfrac{9}{20}}}.\end{aligned}$$

15. The discrete random variable X has the probability distribution

x	1	2	3	4
$\text{P}(X = x)$	k	$2k$	$3k$	$4k$

(a) Show that $k = 0.1$. (1)

Solution

$$k+2k+3k+4k=1 \Rightarrow 10k=1 \Rightarrow \underline{\underline{k=0.1}}.$$

Find

(b) $\text{E}(X)$, (2)

Solution

$$\begin{aligned}\text{E}(X) &= (0.1\times 1)+(0.2\times 2)+(0.3\times 3)+(0.4\times 4)\\ &= 0.1+0.4+0.9+1.6\\ &= \underline{\underline{3}}.\end{aligned}$$

(c) $\mathrm{E}(X^2)$, (2)

Solution

$$\begin{aligned}\mathrm{E}(X^2) &= (0.1 \times 1^2) + (0.2 \times 2^2) + (0.3 \times 3^2) + (0.4 \times 4^2) \\ &= 0.1 + 0.8 + 2.7 + 6.4 \\ &= \underline{\underline{10}}.\end{aligned}$$

(d) $\mathrm{Var}(2 - 5X)$. (3)

Solution

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\ &= 10 - 3^2 \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(2 - 5X) &= 25\,\mathrm{Var}(X) \\ &= 25 \times 1 \\ &= \underline{\underline{25}}.\end{aligned}$$

Two independent observations X_1 and X_2 are made of X.

(e) Show that $\mathrm{P}(X_1 + X_2 = 4) = 0.1$. (2)

Solution

$$\begin{aligned}\mathrm{P}(X_1 + X_2 = 4) &= \mathrm{P}(1,3) + \mathrm{P}(2,2) + \mathrm{P}(3,1) \\ &= 0.1 \times 0.3 + 0.2 \times 0.2 + 0.3 \times 0.1 \\ &= 0.03 + 0.04 + 0.03 \\ &= \underline{\underline{0.1}}.\end{aligned}$$

(f) Complete the probability distribution table for $X_1 + X_2$. (2)

y	2	3	4	5	6	7	8
$\mathrm{P}(X_1 + X_2 = y)$	0.01	0.04	0.10		0.25	0.24	

Solution

y	2	3	4	5	6	7	8
$\mathrm{P}(X_1 + X_2 = y)$	0.01	0.04	0.10	0.2	0.25	0.24	0.16

(g) Find $\mathrm{P}(1.5 < X_1 + X_2 \leqslant 3.5)$. (2)

Solution

$$\mathrm{P}(1.5 < X_1 + X_2 \leqslant 3.5) = \mathrm{P}(X_1 + X_2 = 2) + \mathrm{P}(X_1 + X_2 = 3) = \underline{\underline{0.05}}.$$

16. The discrete random variable Y has the probability distribution

y	1	2	3	4
$\mathrm{P}(Y = y)$	a	b	0.3	c

where a, b, and c are constants.

The cumulative distribution function $\mathrm{F}(y)$ of Y is given in the following table

y	1	2	3	4
$\mathrm{F}(y)$	0.1	0.5	d	1.0

where d is a constant.

(a) Find the value of a, the value of b, the value of c, and the value of d. (5)

Solution

$\underline{\underline{a = 0.1}}$. Now,

$$\mathrm{F}(2) = 0.5 \Rightarrow 0.1 + b = 0.5 \Rightarrow \underline{\underline{b = 0.4}}$$

and

$$0.1 + 0.4 + 0.3 + c = 1 \Rightarrow \underline{\underline{c = 0.2}}.$$

Finally,

$$\mathrm{F}(3) = d \Rightarrow d = 0.1 + 0.4 + 0.3 = \underline{\underline{0.8}}.$$

(b) Find $\mathrm{P}(3Y + 2 \geqslant 8)$. (2)

Solution

$$\begin{aligned}\mathrm{P}(3Y + 2 \geqslant 8) &= \mathrm{P}(3Y \geqslant 6)\\ &= \mathrm{P}(Y \geqslant 2)\\ &= 1 - \mathrm{P}(Y \leqslant 1)\\ &= 1 - 0.1\\ &= \underline{\underline{0.9}}.\end{aligned}$$

17. A spinner is designed so that the score S is given by the following probability distribution.

s	0	1	2	4	5
$\mathrm{P}(S = s)$	p	0.25	0.25	0.20	0.20

(a) Find the value of p. (2)

Solution

$$p + 0.25 + 0.25 + 0.2 + 0.2 = 1 \Rightarrow \underline{\underline{p = 0.1}}.$$

(b) Find $\mathrm{E}(S)$. (2)

Solution

$$\begin{aligned}\mathrm{E}(S) &= (0.1 \times 0) + (0.25 \times 1) + (0.25 \times 2) + (0.2 \times 4) + (0.2 \times 5)\\ &= 0 + 0.25 + 0.5 + 0.8 + 1\\ &= \underline{\underline{2.55}}.\end{aligned}$$

(c) Show that $\mathrm{E}(S^2) = 9.45$. (2)

Solution

$$\begin{aligned}\mathrm{E}(S^2) &= (0.1 \times 0^2) + (0.25 \times 1^2) + (0.25 \times 2^2) + (0.2 \times 4^2) + (0.2 \times 5^2)\\ &= 0 + 0.25 + 1 + 3.2 + 5\\ &= \underline{\underline{9.45}}.\end{aligned}$$

(d) Find Var(S). (2)

Solution

$$\begin{aligned}\text{Var}(S) &= \text{E}(S^2) - \text{E}(S)^2 \\ &= 9.45 - 2.55^2 \\ &= \underline{\underline{2.9475}}.\end{aligned}$$

18. The discrete random variable X can take only the values 2, 3, 4, or 6. For these values the probability distribution function is given by

x	2	3	4	6
$\text{P}(X = x)$	$\frac{5}{21}$	$\frac{2k}{21}$	$\frac{7}{21}$	$\frac{k}{21}$

where k is a positive integer.

(a) Show that $k = 3$. (2)

Solution

$$\begin{aligned}& \tfrac{5}{21} + \tfrac{2k}{21} + \tfrac{7}{21} + \tfrac{k}{21} = 1 \\ \Rightarrow\ & 3k + 12 = 21 \\ \Rightarrow\ & 3k = 9 \\ \Rightarrow\ & \underline{\underline{k = 3}}.\end{aligned}$$

Find

(b) F(3), (1)

Solution

$$\text{F}(3) = \text{P}(X \leqslant 3) = \tfrac{5}{21} + \tfrac{6}{21} = \underline{\underline{\tfrac{11}{21}}}.$$

(c) E(X), (2)

Solution

$$
\begin{aligned}
\mathrm{E}(X) &= \left(\tfrac{5}{21} \times 2\right) + \left(\tfrac{6}{21} \times 3\right) + \left(\tfrac{7}{21} \times 4\right) + \left(\tfrac{3}{21} \times 6\right) \\
&= \tfrac{10}{21} + \tfrac{18}{21} + \tfrac{28}{21} + \tfrac{18}{21} \\
&= \tfrac{74}{21} \\
&= \underline{\underline{3\tfrac{11}{21}}}.
\end{aligned}
$$

(d) $\mathrm{E}(X^2)$, (2)

Solution

$$
\begin{aligned}
\mathrm{E}(X^2) &= \left(\tfrac{5}{21} \times 2^2\right) + \left(\tfrac{6}{21} \times 3^2\right) + \left(\tfrac{7}{21} \times 4^2\right) + \left(\tfrac{3}{21} \times 6^2\right) \\
&= \tfrac{20}{21} + \tfrac{54}{21} + \tfrac{112}{21} + \tfrac{108}{21} \\
&= \tfrac{294}{21} \\
&= \underline{\underline{14}}.
\end{aligned}
$$

(e) $\mathrm{Var}(7X - 5)$. (4)

Solution

$$
\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\
&= 14 - \left(\tfrac{74}{21}\right)^2 \\
&= \tfrac{698}{441}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}(7X - 5) &= 49\,\mathrm{Var}(X) \\
&= 49 \times \tfrac{698}{441} \\
&= \underline{\underline{77\tfrac{5}{9}}}.
\end{aligned}
$$

19. A discrete random variable X has probability function

$$
\mathrm{P}(X = x) = \begin{cases} k(1-x)^2, & x = -1, 0, 1, \text{ and } 2, \\ 0, & \text{otherwise.} \end{cases}
$$

(a) Show that $k = \frac{1}{6}$. (3)

Solution

$$k(4+1+0+1) = 1 \Rightarrow 6k = 1 \Rightarrow \underline{\underline{k = \tfrac{1}{6}}}.$$

(b) Find $\mathrm{E}(X)$. (2)

Solution

$$\begin{aligned}\mathrm{E}(X) &= \tfrac{1}{6}[(4\times(-1)) + (1\times 0) + (0\times 1) + (1\times 2)] \\ &= \tfrac{1}{6}\times(-2k) \\ &= \underline{\underline{-\tfrac{1}{3}}}.\end{aligned}$$

(c) Show that $\mathrm{E}(X^2) = \frac{4}{3}$. (2)

Solution

$$\begin{aligned}\mathrm{E}(X) &= \tfrac{1}{6}[(2\times(-1)^2) + (1\times 0^2) + (0\times 1^2) + (1\times 2^2)] \\ &= \tfrac{1}{6}(4+0+0+4) \\ &= \underline{\underline{\tfrac{4}{3}}}.\end{aligned}$$

(d) Find $\mathrm{Var}(1-3X)$. (3)

Solution

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\ &= \tfrac{4}{3} - (-\tfrac{1}{3})^2 \\ &= \tfrac{11}{9}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(1-3X) &= 9\,\mathrm{Var}(X) \\ &= 9\times\tfrac{11}{9} \\ &= \underline{\underline{11}}.\end{aligned}$$

20. The discrete random variable X can take only the values 1, 2, and 3. For these values

the cumulative distribution function is defined by

$$\mathrm{F}(x) = \frac{x^3 + k}{40},\ x = 1, 2, 3.$$

(a) Show that $k = 13$. (2)

Solution

$$\mathrm{F}(3) = 1 \Rightarrow \frac{3^3 + k}{40} = 1 \Rightarrow \underline{\underline{k = 13}}.$$

(b) Find the probability distribution of X. (4)

Solution

Now,

$$\mathrm{F}(1) = \mathrm{P}(X = 1) = \tfrac{1^3+13}{40} = \tfrac{7}{20},$$
$$\mathrm{F}(2) = \tfrac{21}{40} \Rightarrow \tfrac{7}{20} + \mathrm{P}(X = 2) = \tfrac{21}{40} \Rightarrow \mathrm{P}(X = 2) = \tfrac{7}{40},$$

and

$$\mathrm{P}(X = 3) = 1 - \tfrac{7}{20} - \tfrac{7}{40} = \tfrac{19}{40}.$$

x	1	2	3
$\mathrm{P}(X = x)$	$\frac{7}{20}$	$\frac{7}{40}$	$\frac{19}{40}$

Given that $\mathrm{Var}(X) = \frac{259}{320}$,

(c) find the exact value of $\mathrm{Var}(4X - 5)$. (2)

Solution

$$\begin{aligned}\mathrm{Var}(4X - 5) &= 16\,\mathrm{Var}(X)\\ &= 16 \times \tfrac{259}{320}\\ &= \underline{\underline{12.95}}.\end{aligned}$$

21. A fair blue die has faces numbered 1, 1, 3, 3, 5, and 5. The random variable B represents the score when the blue die is rolled.

(a) Write down the probability distribution for B. (2)

Solution

b	1	3	5
$\mathrm{P}(B = b)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(b) State the name of this probability distribution. (1)

Solution

Discrete uniform distribution

(c) Write down the value of $\mathrm{E}(B)$. (1)

Solution

$\underline{\underline{3}}$.

A second die is red and the random variable R represents the score when the red die is rolled. The probability distribution of R is

r	2	4	6
$\mathrm{P}(R = r)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

(d) Find $\mathrm{E}(R)$. (2)

Solution

$$\begin{aligned}\mathrm{E}(R) &= (\tfrac{2}{3} \times 2) + (\tfrac{1}{6} \times 4) + (\tfrac{1}{6} \times 6)\\ &= \tfrac{4}{3} + \tfrac{2}{3} + 1\\ &= \underline{\underline{3}}.\end{aligned}$$

(e) Find $\mathrm{Var}(R)$. (3)

Solution

$$\begin{aligned}\mathrm{E}(R^2) &= (\tfrac{2}{3} \times 2^2) + (\tfrac{1}{6} \times 4^2) + (\tfrac{1}{6} \times 6^2)\\ &= \tfrac{8}{3} + \tfrac{8}{3} + 6\\ &= \tfrac{32}{3}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\ &= \tfrac{32}{3} - 3^2 \\ &= \underline{\underline{\tfrac{7}{3}}}.\end{aligned}$$

22. A biased die with six faces is rolled. The discrete random variable X represents the score on the uppermost face. The probability distribution of X is shown in the table below.

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	a	a	a	b	b	0.3

(a) Given that $\mathrm{E}(X) = 4.2$, find the value of a and the value of b. (5)

Solution

$$3a + 2b + 0.3 = 1 \Rightarrow b = 0.35 - 1.5a$$

and

$$\begin{aligned} & (a \times 1) + (a \times 2) + (a \times 3) + (b \times 4) + (b \times 5) + (0.3 \times 6) = 4.2 \\ \Rightarrow\ & 6a + 9b + 1.8 = 4.2 \\ \Rightarrow\ & 9b = 2.4 - 6a \\ \Rightarrow\ & b = \tfrac{4}{15} - \tfrac{2}{3}a \end{aligned}$$

and eliminate b:

$$0.35 - 1.5a = \tfrac{4}{15} - \tfrac{2}{3}a \Rightarrow \tfrac{5}{6}a = \tfrac{1}{12} \Rightarrow \underline{\underline{a = 0.1}}$$

and

$$b = 0.35 - 1.5 \times 0.1 = \underline{\underline{0.2}}.$$

(b) Show that $\mathrm{E}(X^2) = 20.4$. (1)

Solution

$$
\begin{aligned}
\mathrm{E}(X^2) &= (0.1 \times 1^2) + (0.1 \times 2^2) + (0.1 \times 3^2) + (0.2 \times 4^2) \\
&\quad + (0.2 \times 5^2) + (0.3 \times 6^2) \\
&= 0.1 + 0.4 + 0.9 + 3.2 + 5 + 10.8 \\
&= \underline{\underline{20.4}}.
\end{aligned}
$$

(c) Find $\mathrm{Var}(5 - 3X)$. (3)

Solution

$$
\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\
&= 20.4 - 4.2^2 \\
&= 2.76
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}(5 - 3X) &= 9\,\mathrm{Var}(X) \\
&= 9 \times 2.76 \\
&= \underline{\underline{29.84}}.
\end{aligned}
$$

A biased die with five faces is rolled. The discrete random variable Y represents the score which is uppermost. The cumulative distribution function of Y is shown in the table below.

y	1	2	3	4	5
$\mathrm{F}(y)$	$\frac{1}{10}$	$\frac{2}{10}$	$3k$	$4k$	$5k$

(d) Find the value of k. (1)

Solution

$$5k = 1 \Rightarrow \underline{\underline{k = 0.2}}.$$

(e) Find the probability distribution of Y. (3)

Solution

$$\mathrm{F}(2) = 0.2 \Rightarrow \mathrm{P}(X = 2) = \mathrm{P}(X \leqslant 2) - \mathrm{P}(X \leqslant 1) = 0.1,$$

$$\mathrm{F}(3) = 0.6 \Rightarrow \mathrm{P}(X = 3) = \mathrm{P}(X \leqslant 3) - \mathrm{P}(X \leqslant 2) = 0.4,$$

$$\mathrm{F}(4) = 0.8 \Rightarrow \mathrm{P}(X = 4) = \mathrm{P}(X \leqslant 4) - \mathrm{P}(X \leqslant 3) = 0.2,$$

and

$$\mathrm{F}(5) = 1 \Rightarrow \mathrm{P}(X = 5) = \mathrm{P}(X \leqslant 5) - \mathrm{P}(X \leqslant 4) = 0.2.$$

y	1	2	3	4	5
$\mathrm{P}(Y = y)$	0.1	0.1	0.4	0.2	0.2

Each die is rolled once. The scores on the two dice are independent.

(f) Find the probability that the sum of the two scores equals 2. (2)

Solution

$$\mathrm{P}(\text{sum of the two scores equals 2}) = 0.1 \times 0.1 = \underline{\underline{0.01}}.$$

23. The discrete random variable X takes the values 1, 2, and 3 and has cumulative distribution function $\mathrm{F}(x)$ given by

x	1	2	3
$\mathrm{F}(x)$	0.4	0.65	1

(a) Find the probability distribution of X. (3)

Solution

$$\mathrm{F}(2) = 0.4 \Rightarrow \mathrm{P}(X = 1) = 0.4,$$

$$\mathrm{F}(2) = 0.65 \Rightarrow \mathrm{P}(X = 2) = \mathrm{P}(X \leqslant 2) - \mathrm{P}(X \leqslant 1) = 0.25,$$

and

$$\mathrm{F}(3) = 1 \Rightarrow \mathrm{P}(X = 3) = \mathrm{P}(X \leqslant 3) - \mathrm{P}(X \leqslant 2) = 0.35.$$

x	1	2	3
$\mathrm{P}(X = x)$	0.4	0.25	0.35

(b) Write down the value of $\mathrm{F}(1.8)$. (1)

Solution

$$F(1.8) = P(X \leqslant 1.8) = P(X = 1) = \underline{\underline{0.4}}.$$

24. The score S when a spinner is spun has the following probability distribution.

s	0	1	2	4	5
$P(S = s)$	0.2	0.2	0.1	0.3	0.2

(a) Find $E(S)$. (2)

Solution

$$\begin{aligned} E(S) &= (0.2 \times 0) + (0.2 \times 1) + (0.1 \times 2) + (0.3 \times 4) + (0.2 \times 5) \\ &= 0.2 + 0.2 + 1.2 + 1 \\ &= \underline{\underline{2.6}}. \end{aligned}$$

(b) Show that $E(S^2) = 10.4$. (2)

Solution

$$\begin{aligned} E(S^2) &= (0.2 \times 0^2) + (0.2 \times 1^2) + (0.1 \times 2^2) + (0.3 \times 4^2) + (0.2 \times 5^2) \\ &= 0.2 + 0.4 + 4.8 + 5 \\ &= \underline{\underline{10.4}}, \end{aligned}$$

as required.

(c) Hence find $\text{Var}(S)$. (2)

Solution

$$\begin{aligned} \text{Var}(X) &= E(X^2) - E(X)^2 \\ &= 10.4 - 2.6^2 \\ &= \underline{\underline{3.64}}. \end{aligned}$$

(d) Find (4)

(i) $\mathrm{E}(5S-3)$,

Solution

$$\mathrm{E}(5S-3) = 5\,\mathrm{E}(S) - 3 = 5 \times 2.6 - 3 = \underline{\underline{10}}.$$

(ii) $\mathrm{Var}(5S-3)$.

Solution

$$\mathrm{Var}(5S-3) = 25\,\mathrm{Var}(S) = 25 \times 3.64 = \underline{\underline{91}}.$$

(e) Find $\mathrm{P}(5S-3 > S+3)$. (3)

Solution

$$\begin{aligned}\mathrm{P}(5S-3 > S+3) &= \mathrm{P}(4S > 6)\\ &= \mathrm{P}(S > \tfrac{3}{2})\\ &= \underline{\underline{0.6}}.\end{aligned}$$

The spinner is spun twice. The score from the first spin is S_1 and the score from the second spin is S_2. The random variables S_1 and S_2 are independent and the random variable $X = S_1 \times S_2$.

(f) Show that $\mathrm{P}(\{S_1 = 1\} \cap X < 5) = 0.16$. (2)

Solution

$$\mathrm{P}(\{S_1 = 1\} \cap X < 5) = 0.2 \times 0.8 = \underline{\underline{0.16}},$$

as required.

(g) Find $\mathrm{P}(X < 5)$. (3)

Solution

$$\begin{aligned}\mathrm{P}(X<5) &= \mathrm{P}(\{S_1 = 0\} \cap S_2 = \text{anything}) + \mathrm{P}(\{S_1 = 1\} \cap S_2 \leqslant 4)\\ &\qquad + \mathrm{P}(\{S_1 = 2\} \cap S_2 \leqslant 2) + \mathrm{P}(\{S_1 = 4\} \cap S_2 \leqslant 1)\\ &\qquad + \mathrm{P}(\{S_1 = 5\} \cap S_2 = 0)\\ &= (0.2 \times 1) + (0.2 \times 0.8) + (0.1 \times 0.5) + (0.3 \times 0.4) + (0.2 \times 0.2)\\ &= 0.2 + 0.16 + 0.05 + 0.12 + 0.04\\ &= \underline{\underline{0.57}}.\end{aligned}$$

25. The discrete random variable X has the probability function

$$\mathrm{P}(X=x)=\begin{cases} kx, & x=2,4,6, \\ k(x-2), & x=8, \\ 0, & \text{otherwise}, \end{cases}$$

where k is a constant.

(a) Show that $k=\frac{1}{18}$. (2)

Solution

$$2k+4k+6k+6k=1 \Rightarrow 18k=1 \Rightarrow \underline{\underline{k=\tfrac{1}{18}}}.$$

(b) Find the exact value of F(5). (1)

Solution

$$\mathrm{F}(5)=\mathrm{P}(X \leqslant 5)=\tfrac{2}{18}+\tfrac{4}{18}=\underline{\underline{\tfrac{1}{3}}}.$$

(c) Find the exact value of $\mathrm{E}(X)$. (2)

Solution

$$\begin{aligned}\mathrm{E}(X)&=(\tfrac{2}{18}\times 2)+(\tfrac{4}{18}\times 4)+(\tfrac{6}{18}\times 6)+(\tfrac{6}{18}\times 8)\\ &=\tfrac{4}{18}+\tfrac{16}{18}+\tfrac{36}{18}+\tfrac{48}{18}\\ &=\underline{\underline{5\tfrac{7}{9}}}.\end{aligned}$$

(d) Find the exact value of $\mathrm{E}(X^2)$. (2)

Solution

$$\begin{aligned}\mathrm{E}(X^2)&=(\tfrac{2}{18}\times 2^2)+(\tfrac{4}{18}\times 4^2)+(\tfrac{6}{18}\times 6^2)+(\tfrac{6}{18}\times 8^2)\\ &=\tfrac{8}{18}+\tfrac{64}{18}+\tfrac{216}{18}+\tfrac{384}{18}\\ &=\underline{\underline{37\tfrac{1}{3}}}.\end{aligned}$$

(e) Calculate $\mathrm{Var}(3-4X)$, giving your answer to 3 significant figures. (3)

Solution

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\ &= 37\tfrac{1}{3} - (5\tfrac{7}{9})^2 \\ &= 3\tfrac{77}{81}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(3-4X) &= 16\,\mathrm{Var}(X) \\ &= 16 \times 3\tfrac{77}{81} \\ &= 63\tfrac{17}{81} \\ &= \underline{\underline{63.2\text{ (3 sf)}}}.\end{aligned}$$

26. The discrete random variable X has probability distribution

x	-4	-2	1	3	5
$\mathrm{P}(X=x)$	0.4	p	0.05	0.15	p

(a) Show that $p = 0.2$. (2)

Solution

$$0.4 + p + 0.05 + 0.15 + p = 1 \Rightarrow 2p = 0.4 \Rightarrow \underline{\underline{p = 0.2}}.$$

Find

(b) $\mathrm{E}(X)$, (2)

Solution

$$\begin{aligned}\mathrm{E}(X) &= (0.4 \times -4) + (0.2 \times -2) + (0.05 \times 1) + (0.15 \times 3) + (0.2 \times 5) \\ &= -1.6 - 0.4 + 0.05 + 0.45 + 1 \\ &= \underline{\underline{-0.5}}.\end{aligned}$$

(c) $\mathrm{F}(0)$, (1)

Solution

$$F(0) = P(X \leqslant 0) = 0.4 + 0.2 = \underline{\underline{0.6}}.$$

(d) $P(3X + 2 > 5)$. (2)

Solution

$$\begin{aligned} P(3X + 2 > 5) &= P(3X > 3) \\ &= P(X > 1) \\ &= 0.15 + 0.2 \\ &= \underline{\underline{0.35}}. \end{aligned}$$

Given that $\text{Var}(X) = 13.35$,

(e) find the possible values of a such that $\text{Var}(aX + 3) = 53.4$. (2)

Solution

$$\begin{aligned} \text{Var}(aX + 3) = 53.4 &\Rightarrow \text{Var}(aX) = 53.4 \\ &\Rightarrow a^2 \text{Var}(X) = 53.4 \\ &\Rightarrow 13.35a^2 = 53.4 \\ &\Rightarrow a^2 = 4 \\ &\Rightarrow \underline{\underline{a = \pm 2}}. \end{aligned}$$

27. The discrete random variable X has probability distribution

$$P(X = x) = \tfrac{1}{10},\ x = 1, 2, 3, \ldots, 10.$$

(a) Write down the name given to this distribution. (2)

Solution

Discrete uniform distribution.

(b) Write down the value of (2)

(i) $P(X = 10)$,

Solution
$\underline{\underline{\frac{1}{10}}}$.

(ii) $\mathrm{P}(X \leqslant 10)$.

Solution
$\underline{\underline{\frac{9}{10}}}$.

28. In a quiz, a team gains 10 points for every question it answers correctly and loses 5 points for every question it does not answer correctly. The probability of answering a question correctly is 0.6 for each question. One round of the quiz consists of 3 questions.

The discrete random variable X represents the total number of points scored in one round. The table shows the incomplete probability distribution of X

x	30	15	0	-15
$\mathrm{P}(X = x)$	0.216			0.064

(a) Show that the probability of scoring 15 points in a round is 0.432. (2)

Solution

$$\mathrm{P}(2 \text{ right, } 1 \text{ wrong}) = 3 \times 0.6^2 \times 0.4 = \underline{\underline{0.432}},$$

as required.

(b) Find the probability of scoring 0 points in a round. (1)

Solution

$$0.216 + 0.432 + p + 0.064 = 1 \Rightarrow \underline{\underline{p = 0.288}}.$$

(c) Find the probability of scoring a total of 30 points in 2 rounds. (3)

Solution

$$\begin{aligned}\mathrm{P}(30 \text{ points}) &= \mathrm{P}(2 \text{ right, } 1 \text{ wrong})^2 + \mathrm{P}(3 \text{ right and } 1 \text{ right, } 2 \text{ wrong}) \\ &= 0.432^2 + 2 \times 0.216 \times 0.288 \\ &= \underline{\underline{0.311\,04}}.\end{aligned}$$

(d) Find E(X). (2)

Solution

$$\begin{aligned}\mathrm{E}(X) &= (0.216 \times 30) + (0.432 \times 15) + (0.288 \times 0) + (0.064 \times -15) \\ &= 6.48 + 6.48 + 0 - 0.96 \\ &= \underline{\underline{12}}.\end{aligned}$$

(e) Find Var(X). (3)

Solution

$$\begin{aligned}\mathrm{E}(X) &= (0.216 \times 30^2) + (0.432 \times 15^2) + (0.288 \times 0^2) + (0.064 \times (-15)^2) \\ &= 194.4 + 97.2 + 0 + 14.4 \\ &= 306\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\ &= 306 - 12^2 \\ &= \underline{\underline{162}}.\end{aligned}$$

In a bonus round of 3 questions, a team gains 20 points for every question it answers correctly and loses 5 points for every question it does not answer correctly.

(f) Find the expected number of points scored in the bonus round. (3)

Solution
Let B represents be the bonus points.

b	60	35	10	-15
$\mathrm{P}(B = b)$	0.216	0.432	0.288	0.064

$$\begin{aligned}\mathrm{E}(B) &= (0.216 \times 60) + (0.432 \times 35) + (0.288 \times 10) + (0.064 \times -15) \\ &= 12.96 + 15.12 + 28.8 - 0.96 \\ &= \underline{\underline{30}}.\end{aligned}$$

29. The discrete random variable X has the following probability distribution, where p and q are constants.

x	-2	-1	$\frac{1}{2}$	$\frac{3}{2}$	2
$\mathrm{P}(X = x)$	p	q	0.2	0.3	p

(a) Write down an equation in p and q. (1)

Solution

$$p + q + 0.2 + 0.3 + p = 1 \Rightarrow \underline{\underline{2p + q = 0.5}}.$$

Given that $\mathrm{E}(X) = 0.4$,

(b) find the value of q, (3)

Solution

$$\begin{aligned}\mathrm{E}(X) = 0.4 &\Rightarrow (p \times -2) + (q \times -1) + (0.2 \times \tfrac{1}{2}) + (0.3 \times \tfrac{3}{2}) + (p \times 2) = 0.5\\ &\Rightarrow -2p - q + 0.1 + 0.45 + 2p = 0.4\\ &\Rightarrow \underline{\underline{q = 0.15}}.\end{aligned}$$

(c) hence find the value of p. (2)

Solution

$$2p + 0.15 = 0.5 \Rightarrow 2p = 0.35 \Rightarrow \underline{\underline{p = 0.175}}.$$

Given also that $\mathrm{E}(X^2) = 2.275$,

(d) find $\mathrm{Var}(X)$. (2)

Solution

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = 2.275 - 0.4^2 = \underline{\underline{2.115}}.$$

Sarah and Rebecca play a game. A computer selects a single value of X using the probability distribution above. Sarah's score is given by the random variable $S = X$ and Rebecca's score is given by the random variable $R = \dfrac{1}{X}$.

(e) Find $\mathrm{E}(R)$. (3)

Solution

r	$-\frac{1}{2}$	-1	2	$\frac{2}{3}$	$\frac{1}{2}$
$\mathrm{P}(R = r)$	0.175	0.15	0.2	0.3	0.175

$$\begin{aligned}\mathrm{E}(R) &= (0.175 \times -\tfrac{1}{2}) + (0.15 \times -1) + (0.2 \times 2) + (0.3 \times \tfrac{2}{3}) + (0.175 \times \tfrac{1}{2}) \\ &= -0.0875 - 0.15 + 0.4 + 0.2 + 0.0875 \\ &= \underline{\underline{0.45}}.\end{aligned}$$

(f) Find the probability that (4)

(i) Sarah is the winner,

Solution

$$\begin{aligned}\mathrm{P}(S > R) &= \mathrm{P}(X = 2) + \mathrm{P}(X = \tfrac{3}{2}) \\ &= 0.175 + 0.3 \\ &= \underline{\underline{0.475}}.\end{aligned}$$

(ii) Rebecca is the winner.

Solution

$$\begin{aligned}\mathrm{P}(R > S) &= \mathrm{P}(X = -2) + \mathrm{P}(X = \tfrac{1}{2}) \\ &= 0.175 + 0.2 \\ &= \underline{\underline{0.375}}.\end{aligned}$$

30. The discrete random variable X has probability distribution

x	-1	0	1	2
$\mathrm{P}(X = x)$	a	b	b	c

The cumulative distribution function of X is given by

x	-1	0	1	2
$\mathrm{F}(x)$	$\frac{1}{3}$	d	$\frac{5}{6}$	e

(a) Find the values of a, b, c, d, and e. (5)

Solution

$$\mathrm{F}(-1) = \tfrac{1}{3} \Rightarrow \mathrm{P}(X = -1) = \underline{\underline{\tfrac{1}{3}}}.$$

$$\mathrm{F}(1) = \tfrac{5}{6} \Rightarrow \tfrac{1}{3} + 2\,\mathrm{P}(X = 0) = \tfrac{5}{6} \Rightarrow \mathrm{P}(X = 0) = \underline{\underline{\tfrac{1}{4}}}.$$

$$\mathrm{P}(X = 2) = 1 - \tfrac{1}{3} - \tfrac{1}{4} - \tfrac{1}{4} = \underline{\underline{\tfrac{1}{6}}}.$$

$$\mathrm{F}(0) = \mathrm{P}(X = -1) + \mathrm{P}(X = 0) = \underline{\underline{\tfrac{7}{12}}}.$$

$$\mathrm{F}(2) = \underline{\underline{1}}.$$

Hence, the discrete random variable X has probability distribution

x	-1	0	1	2
$\mathrm{P}(X = x)$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$

and the cumulative distribution function of X is given by

x	-1	0	1	2
$\mathrm{F}(x)$	$\frac{1}{3}$	$\frac{7}{12}$	$\frac{5}{6}$	1

(b) Write down the value of $\mathrm{P}(X^2 = 1)$. (1)

Solution

$$\begin{aligned}\mathrm{P}(X^2 = 1) &= \mathrm{P}(X = -1) + \mathrm{P}(X = 1)\\ &= \tfrac{1}{3} + \tfrac{1}{4}\\ &= \underline{\underline{\tfrac{7}{12}}}.\end{aligned}$$

31. The score, X, for a biased spinner is given by the probability distribution

x	0	3	6
$\mathrm{P}(X = x)$	$\frac{1}{12}$	$\frac{2}{3}$	$\frac{1}{4}$

Find

(a) $\mathrm{E}(X)$, (2)

Solution

$$\begin{aligned}\mathrm{E}(X) &= (\tfrac{1}{12} \times 0) + (\tfrac{2}{3} \times 3) + (\tfrac{1}{4} \times 6) \\ &= 0 + 2 + \tfrac{3}{2} \\ &= \underline{\underline{3.5}}.\end{aligned}$$

(b) $\mathrm{Var}(X)$. (3)

Solution

$$\begin{aligned}\mathrm{E}(X^2) &= (\tfrac{1}{12} \times 0^2) + (\tfrac{2}{3} \times 3^2) + (\tfrac{1}{4} \times 6^2) \\ &= 0 + 6 + 9 \\ &= 15\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\ &= 15 - 3.5^2 \\ &= \underline{\underline{2.75}}.\end{aligned}$$

A biased coin has one face labelled 2 and the other face labelled 5. The score, Y, when the coin is spun has

$$\mathrm{P}(Y = 5) = p \text{ and } E(Y) = 3.$$

(c) Form a linear equation in p and show that $p = \frac{1}{3}$. (3)

Solution

$$\underline{\underline{2(1 - p) + 5p = 3}}$$

and

$$2 - 2p + 5p = 3 \Rightarrow 3p = 1 \Rightarrow \underline{\underline{p = \tfrac{1}{3}}}.$$

(d) Write down the probability distribution of Y. (1)

Solution

y	2	5
$\mathrm{P}(Y=y)$	$\frac{2}{3}$	$\frac{1}{3}$

Sam plays a game with the spinner and the coin. Each is spun once and Sam calculates his score, S, as follows

$$\text{if } X = 0, \text{ then } S = Y^2$$
$$\text{if } X \neq 0, \text{ then } S = XY.$$

(e) Show that $\mathrm{P}(S = 30) = \frac{1}{12}$. (2)

Solution

$$\begin{aligned}\mathrm{P}(S = 30) &= \mathrm{P}(X = 6)\,\mathrm{P}(Y = 5)\\ &= \tfrac{1}{4} \times \tfrac{1}{3}\\ &= \underline{\underline{\tfrac{1}{12}}}.\end{aligned}$$

(f) Find the probability distribution of S. (3)

Solution

$$\begin{aligned}\mathrm{P}(X = 0, Y = 2) &= \tfrac{1}{12} \times \tfrac{2}{3} = \tfrac{1}{18}\\ \mathrm{P}(X = 0, Y = 5) &= \tfrac{1}{12} \times \tfrac{1}{3} = \tfrac{1}{36}\\ \mathrm{P}(X = 3, Y = 2) &= \tfrac{2}{3} \times \tfrac{2}{3} = \tfrac{4}{9}\\ \mathrm{P}(X = 3, Y = 5) &= \tfrac{2}{3} \times \tfrac{1}{3} = \tfrac{2}{9}\\ \mathrm{P}(X = 6, Y = 2) &= \tfrac{1}{4} \times \tfrac{2}{3} = \tfrac{1}{6},\end{aligned}$$

and the probability distribution of S is

S	4	6	12	15	25	30
$\mathrm{P}(S=s)$	$\frac{1}{18}$	$\frac{4}{9}$	$\frac{1}{6}$	$\frac{2}{9}$	$\frac{1}{36}$	$\frac{1}{12}$

(g) Find $\mathrm{E}(S)$. (2)

Solution

$$\begin{aligned}\mathrm{E}(S) &= (\tfrac{1}{18} \times 4) + (\tfrac{4}{9} \times 6) + (\tfrac{1}{6} \times 12) + (\tfrac{2}{9} \times 15) + (\tfrac{1}{36} \times 25) + (\tfrac{1}{12} \times 30) \\ &= \tfrac{2}{9} + \tfrac{8}{3} + 2 + \tfrac{10}{3} + \tfrac{25}{36} + \tfrac{5}{2} \\ &= \underline{\underline{11\tfrac{5}{12}}}.\end{aligned}$$

Charlotte also plays the game with the spinner and the coin. Each is spun once and Charlotte ignores the score on the coin and just uses X^2 as her score. Sam and Charlotte each play the game a large number of times.

(h) State, giving a reason, which of Sam and Charlotte should achieve the higher total score. (2)

Solution

Well,

$$\mathrm{E}(X^2) = 15 \text{ and } \mathrm{E}(S) = 11\tfrac{5}{12}$$

and Charlotte will win.