Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2011 November Paper 2 Variant 3: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Solve the inequality (3)

$$x(2x-1) > 15.$$

Solution

Well,

$$\begin{aligned} x(2x-1) > 15 &\Rightarrow 2x^2 - x > 15 \\ &\Rightarrow 2x^2 - x - 15 > 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -1 \\ \text{multiply to:} & (+2) \times (-15) = -30 \end{array}\right\} -6, +5$$

e.g.,

$$\begin{aligned} &\Rightarrow 2x^2 - 6x + 5x - 15 > 0 \\ &\Rightarrow 2x(x-3) + 5(x-3) > 0 \\ &\Rightarrow (2x+5)(x-3) > 0 \end{aligned}$$

We need a 'table of signs':

	$x < -\frac{5}{2}$	$x = -\frac{5}{2}$	$-\frac{5}{2} < x < 3$	$x = 3$	$x > 3$
$2x+5$	$-$	0	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	0	$+$
$(2x+5)(x-3)$	$+$	0	$-$	0	$+$

$$\Rightarrow \underline{\underline{x < -\tfrac{5}{2} \text{ or } x > 3}}.$$

2. (a) Given that (2)

$$y = (12 - 4x)^5,$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

Now,

$$y = (12 - 4x)^5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 5(12 - 4x)^4 \times (-4)$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\underline{-20(12 - 4x)^4}}.$$

(b) Hence find the approximate change in y as x increases from 0.5 to $(0.5 + p)$, where p is small. (2)

Solution

Next,

$$x = 0.5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -200\,000$$

and

$$\delta y \approx \frac{\mathrm{d}y}{\mathrm{d}x} \times \delta x$$
$$= \underline{\underline{-200\,000p}}.$$

3. (a) Find the coefficient of x^3 in the expansion of (2)

$$(1 - 2x)^7.$$

Solution

Well,

$$(1 - 2x)^7 = \ldots + \binom{7}{1}(1)^6(-2x)^1 + \ldots + \binom{7}{3}(1)^4(-2x)^3 + \ldots$$
$$= \ldots - 14x + \ldots - 280x^3 + \ldots;$$

hence, the coefficient of x^3 is $\underline{\underline{-280}}$.

(b) Find the coefficient of x^3 in the expansion of (3)

$$(1 + 3x^2)(1 - 2x)^7.$$

Solution

Now,

$\times$	$-14x$	$-280x^3$
1	$\ldots$	$-280x^3$
$+3x^2$	$-42x^3$	$\ldots$

and the coefficient of x^3 is

$$-280 - 42 = \underline{\underline{-322}}.$$

4. Without using a calculator, find the positive root of the equation (6)

$$(5 - 2\sqrt{2})x^2 - (4 + 2\sqrt{2})x - 2 = 0,$$

giving your answer in the form $a + b\sqrt{2}$, where a and b are integers.

Solution

Well,

$\times$	4	$+2\sqrt{2}$
4	16	$+8\sqrt{2}$
$+2\sqrt{2}$	$+8\sqrt{2}$	$+8$

and $a = 5 - 2\sqrt{2}$, $b = -(4 + 2\sqrt{2})$, and $c = -2$ and we are interested in the **positive** square root:

$$
\begin{aligned}
x &= \frac{(4+2\sqrt{2}) + \sqrt{(4+2\sqrt{2})^2 - 4(5-2\sqrt{2})(-2)}}{2(5-2\sqrt{2})} \\
&= \frac{(4+2\sqrt{2}) + \sqrt{(24+16\sqrt{2}) + 8(5-2\sqrt{2})}}{2(5-2\sqrt{2})} \\
&= \frac{(4+2\sqrt{2}) + \sqrt{64}}{2(5-2\sqrt{2})} \\
&= \frac{(4+2\sqrt{2}) + 8}{2(5-2\sqrt{2})} \\
&= \frac{12+2\sqrt{2}}{2(5-2\sqrt{2})}
\end{aligned}
$$

$\times$	4	$+2\sqrt{2}$
4	16	$+8\sqrt{2}$
$-2\sqrt{2}$	$-8\sqrt{2}$	-8

$$
= \frac{12+2\sqrt{2}}{2(5-2\sqrt{2})} \times \frac{5+2\sqrt{2}}{5+2\sqrt{2}}
$$

$\times$	12	$+2\sqrt{2}$
5	60	$+10\sqrt{2}$
$+2\sqrt{2}$	$+24\sqrt{2}$	$+8$

$\times$	5	$-2\sqrt{2}$
5	25	$-10\sqrt{2}$
$+2\sqrt{2}$	$+10\sqrt{2}$	-8

$$= \frac{68 + 34\sqrt{2}}{2(25 - 8)}$$
$$= \frac{68 + 34\sqrt{2}}{34}$$
$$= \underline{\underline{2 + \sqrt{2}}};$$

hence, $\underline{\underline{a = 2}}$ and $\underline{\underline{b = 1}}$.

5. A school council of 6 people is to be chosen from a group of 8 students and 6 teachers.

Calculate the number of different ways that the council can be selected if

(a) there are no restrictions, (2)

Solution

$$\binom{14}{6} = \underline{\underline{3\,003}}.$$

(b) there must be at least 1 teacher on the council and more students than teachers. (3)

Solution

Now,

$$\begin{aligned}
\text{different ways} &= \text{1 teacher, 5 students} + \text{2 teaches, 4 students} \\
&= \left[\binom{6}{1} \times \binom{8}{5}\right] + \left[\binom{6}{2} \times \binom{8}{4}\right] \\
&= 336 + 1\,050 \\
&= \underline{\underline{1\,386}}.
\end{aligned}$$

After the council is chosen, a chairperson and a secretary have to be selected from the 6 council members.

(c) Calculate the number of different ways in which a chairperson and a secretary can be selected. (1)

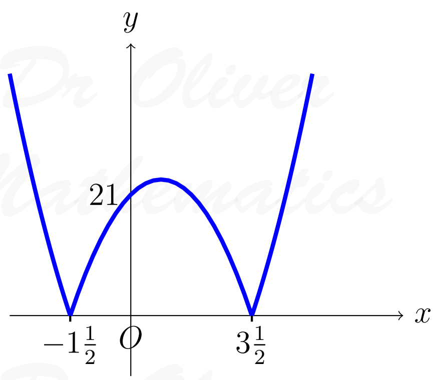
Solution

$$6 \times 5 = \underline{\underline{30}}.$$

6. (a) Sketch the graph of (4)

$$y = |(2x+3)(2x-7)|.$$

Solution



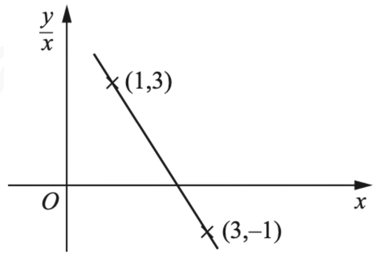
(b) How many values of x satisfy the equation (2)

$$|(2x+3)(2x-7)| = 2x?$$

Solution

Well, $y = 2x$ is a straight line through the origin (with gradient 2) and so the result is $\underline{\underline{2}}$.

7. The variables x and y are related in such a way that when $\frac{y}{x}$ is plotted against x a straight line is obtained, as shown in the graph.



The line passes through the points $(1,3)$ and $(3,-1)$.

(a) Express y in terms of x. (4)

Solution

Well,

$$m = \frac{3-(-1)}{1-3} = \frac{4}{-2} = -2$$

and the equation of the line of best fit is

$$\frac{y}{x} - 3 = -2(x-1) \Rightarrow \frac{y}{x} - 3 = -2x + 2 \Rightarrow \frac{y}{x} = -2x + 5 \Rightarrow \underline{\underline{y = -x(2x+5)}}.$$

(b) Find the value of x and of y such that (2)

$$\frac{y}{x} = -9.$$

Solution

Now,

$$\frac{y}{x} = -9 \Rightarrow -2x + 5 = -9$$
$$\Rightarrow -2x = -14$$
$$\Rightarrow \underline{\underline{x = 7}}$$
$$\Rightarrow \underline{\underline{y = -63}}.$$

8. A sector of a circle, of radius r cm, has a perimeter of 200 cm.

   (a) Express the area, A cm^2, of the sector in terms of r. (3)

Solution
Well, let θ equal the angle and

$$r + r + r\theta = 200 \Rightarrow r\theta = 200 - 2r$$
$$\Rightarrow \theta = \frac{200}{r} - 2.$$

Now,

$$A = \tfrac{1}{2}r^2\theta$$
$$= \tfrac{1}{2}r^2\left(\frac{200}{r} - 2\right)$$
$$= \underline{\underline{100r - r^2}}.$$

   (b) Given that r can vary, find the stationary value of A. (3)

Solution
Next,

$$A = 100r - r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}r} = 100 - 2r$$

and

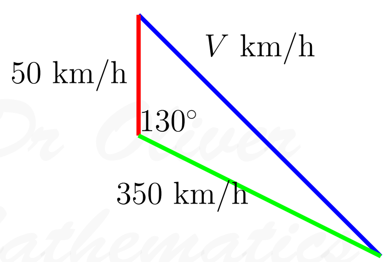
$$\frac{\mathrm{d}A}{\mathrm{d}r} = 0 \Rightarrow 100 - 2r = 0$$
$$\Rightarrow 2r = 100$$
$$\Rightarrow r = 50$$
$$\Rightarrow \underline{\underline{A = 2\,500 \text{ cm}^2}}.$$

9. An aircraft, whose speed in still air is 350 kmh^{-1}, flies in a straight line from A to B, a distance of 480 km.

- There is a wind of 50 kmh^{-1} blowing from the north.
- The pilot sets a course of $130°$.

(a) Calculate the time taken to fly from A to B. (5)

Solution
Let the speed of the aircraft be V kmh^{-1}. We draw a diagram:



Now, cosine rule:

$$V^2 = 50^2 + 350^2 - 2 \times 50 \times 350 \times \cos 130°$$
$$\Rightarrow \quad V^2 = 147\,497.566\,3 \text{ (FCD)}$$
$$\Rightarrow \quad V = 384.054\,119 \text{ (FCD)}.$$

Next,

$$\text{time} = \frac{480}{384.054\ldots}$$
$$= 1.249\,823\,856 \text{ (FCD)}$$
$$= \underline{\underline{1.25 \text{ hours (3 sf)}}}.$$

(b) Calculate the bearing of B from A. (3)

Solution

Sine rule:

$$\frac{\sin\alpha}{350} = \frac{\sin 130^\circ}{V} \Rightarrow \sin\alpha = \frac{350\sin 130^\circ}{384.054\ldots}$$
$$\Rightarrow \sin\alpha = \frac{350\sin 130^\circ}{384.054\ldots}$$
$$\Rightarrow \alpha = 44.276\,301\,41 \text{ (FCD)}.$$

Finally,

$$\begin{aligned}\text{required angle} &= 180 - 44.276\ldots \\ &= 135.723\,698\,6 \text{ (FCD)} \\ &= \underline{\underline{136^\circ \text{ (3 sf)}}}.\end{aligned}$$

10. The line (9)

$$y = 2x + 10$$

intersects the curve

$$2x^2 + 3xy - 5y + y^2 = 218$$

at the points A and B.

Find the equation of the perpendicular bisector of AB.

Solution

Well,

$\times$	$2x$	$+10$
$2x$	$4x^2$	$+20x$
$+10$	$+20x$	$+100$

and

$$\begin{aligned}
& 2x^2 + 3xy - 5y + y^2 = 218 \\
\Rightarrow \quad & 2x^2 + 3x(2x+10) - 5(2x+10) + (2x+10)^2 = 218 \\
\Rightarrow \quad & 2x^2 + (6x^2 + 30x) - 10x - 50 + (4x^2 + 40x + 100) = 218 \\
\Rightarrow \quad & 12x^2 + 60x - 168 = 0 \\
\Rightarrow \quad & 12(x^2 + 5x - 14) = 0
\end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +5 \\ \text{multiply to:} & -14 \end{array}\right\} +7, -2$$

$$\begin{aligned}
\Rightarrow \quad & 12(x+7)(x-2) = 0 \\
\Rightarrow \quad & x = -7 \text{ or } x = 2 \\
\Rightarrow \quad & y = -4 \text{ or } y = 14;
\end{aligned}$$

hence, the two points are $A(-7, -4)$ and $B(2, 14)$.

Now, the midpoint is

$$\left(\frac{-7+2}{2}, \frac{-4+14}{2}\right) = C(-\tfrac{5}{2}, 5).$$

Next,

$$\begin{aligned}
m_{AB} &= \frac{14 - (-4)}{2 - (-7)} \\
&= \frac{18}{9} \\
&= 2
\end{aligned}$$

so

$$m_{\text{normal}} = -\tfrac{1}{2}.$$

Finally, the equation of the perpendicular bisector of AB is

$$\begin{aligned}
y - 5 = -\tfrac{1}{2}(x + \tfrac{5}{2}) &\Rightarrow y - 5 = -\tfrac{1}{2}x - \tfrac{5}{4} \\
&\Rightarrow \underline{\underline{y = -\tfrac{1}{2}x + \tfrac{15}{4}}}.
\end{aligned}$$

11. (a) Solve (3)

$$4\cot \tfrac{1}{2}x = 1, \text{ for } 0^\circ < x < 360^\circ.$$

Solution

Well,

$$0^\circ < x < 360^\circ \Rightarrow 0^\circ < \tfrac{1}{2}x < 180^\circ$$

and

$$\begin{aligned} 4\cot \tfrac{1}{2}x = 1 &\Rightarrow \frac{4}{\tan \frac{1}{2}x} = 1 \\ &\Rightarrow \tan \tfrac{1}{2}x = 4 \\ &\Rightarrow \tfrac{1}{2}x = 75.963\,756\,53 \text{ (FCD)} \\ &\Rightarrow x = 151.927\,7513\,1 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 152 \text{ (3 sf)}}}. \end{aligned}$$

(b) Solve (5)

$$3(1 - \tan y \cos y) = 5\cos^2 y - 2, \text{ for } 0^\circ < y < 360^\circ.$$

Solution

Now,

$$\begin{aligned} 3(1 - \tan y \cos y) = 5\cos^2 y - 2 &\Rightarrow 3(1 - \sin y) = 5(1 - \sin^2 y) - 2 \\ &\Rightarrow 3 - 3\sin y = 5 - 5\sin^2 y - 2 \\ &\Rightarrow 5\sin^2 y - 3\sin y = 0 \\ &\Rightarrow \sin y(5\sin y - 3) = 0 \\ &\Rightarrow \sin y = 0 \text{ or } \sin y = \tfrac{3}{5}. \end{aligned}$$

$\underline{\sin y = 0}$:

$$\sin y = 0 \Rightarrow y = 180.$$

$\underline{\sin y = \tfrac{3}{5}}$:

$$\sin y = \tfrac{3}{5} \Rightarrow y = 36.869\,897\,65,\ 143.130\,102\,4 \text{ (FCD)}.$$

Hence,

$$\underline{\underline{y = 36.9 \text{ (3 sf)}, 143 \text{ (3 sf)}, 180}}.$$

(c) Solve (3)

$$3\sec^2 z = 4, \text{ for } 0 < z < 2\pi \text{ radians.}$$

Solution

$$\begin{aligned} 3\sec^2 z = 4 &\Rightarrow \sec^2 z = \tfrac{4}{3} \\ &\Rightarrow \cos^2 z = \tfrac{3}{4} \\ &\Rightarrow \cos z = \pm\tfrac{\sqrt{3}}{2}. \end{aligned}$$

$\underline{\cos z = \frac{\sqrt{3}}{2}}$:

$$\cos z = \tfrac{\sqrt{3}}{2} \Rightarrow z = \tfrac{1}{6}\pi, \tfrac{11}{6}\pi.$$

$\underline{\cos z = -\frac{\sqrt{3}}{2}}$:

$$\cos z = -\tfrac{\sqrt{3}}{2} \Rightarrow z = \tfrac{5}{6}\pi, \tfrac{7}{6}\pi.$$

Hence,

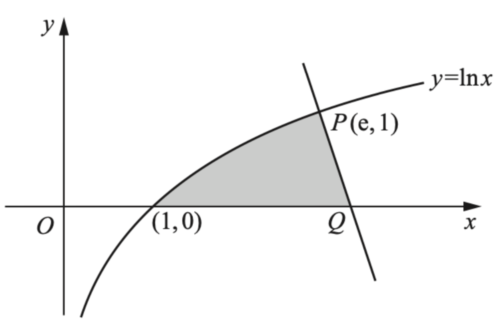
$$\underline{\underline{z = \tfrac{1}{6}\pi, \tfrac{5}{6}\pi, \tfrac{7}{6}\pi, \tfrac{11}{6}\pi.}}$$

EITHER

12. The diagram shows part of the curve

$$y = \ln x$$

cutting the x-axis at the point $(1, 0)$.



The normal to the curve at the point $P(\mathrm{e}, 1)$ cuts the x-axis at the point Q.

(a) Show that Q is the point $\left(\mathrm{e} + \frac{1}{\mathrm{e}}, 0\right)$. (4)

Solution
Well,

$$y = \ln x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}.$$

Now,

$$x = \mathrm{e} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{e}}$$

and

$$m_{\text{normal}} = -\mathrm{e}.$$

Next, the equation of PQ is

$$y - 1 = -\mathrm{e}(x - \mathrm{e})$$

and

$$\begin{aligned} y = 0 &\Rightarrow -\mathrm{e}(x - \mathrm{e}) = -1 \\ &\Rightarrow x - \mathrm{e} = \frac{1}{\mathrm{e}} \\ &\Rightarrow x = \mathrm{e} + \frac{1}{\mathrm{e}}; \end{aligned}$$

hence, $\underline{\underline{\left(\mathrm{e} + \frac{1}{\mathrm{e}}, 0\right)}}$, as required.

(b) Show that (1)

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \ln x) = 1 + \ln x.$$

Solution
Product rule:

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = \ln x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (x)\left(\frac{1}{x}\right) + (1)(\ln x) \\ &= \underline{\underline{1 + \ln x}}, \end{aligned}$$

as required.

(c) Hence find (5)

$$\int \ln x \, \mathrm{d}x$$

and the area of the shaded region.

Solution

Well,

$$\begin{aligned}\int \ln x \, \mathrm{d}x &= \int \left[\frac{\mathrm{d}}{\mathrm{d}x}(x \ln x) - 1\right] \mathrm{d}x \\ &= \underline{\underline{x \ln x - x + c}}.\end{aligned}$$

Finally,

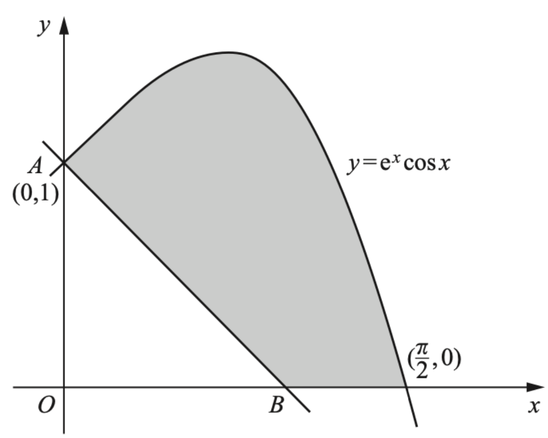
$$\begin{aligned}&\text{area of the shaded region} \\ = \quad & \int_0^{\mathrm{e}} \ln x \, \mathrm{d}x + \text{area under } PQ \\ = \quad & [(\mathrm{e} \ln \mathrm{e} - \mathrm{e} + c) - (1 \ln 1 - 1 + c)] + \left[\tfrac{1}{2} \times 1 \times \frac{1}{\mathrm{e}}\right] \\ = \quad & \underline{\underline{1 + \frac{1}{2\mathrm{e}}}}.\end{aligned}$$

OR

13. The diagram shows part of the curve

$$y = \mathrm{e}^x \cos x,$$

cutting the x-axis at the point $(\frac{1}{2}\pi, 0)$.



The normal to the curve at the point $A(0,1)$ cuts the x-axis at the point B.

(a) Find the coordinates of B. (4)

Solution

Product rule:

$$u = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$$
$$v = \cos x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = -\sin x$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (\mathrm{e}^x)(-\sin x) + (\mathrm{e}^x)(\cos x)\\ &= \mathrm{e}^x\cos x - \mathrm{e}^x \sin x.\end{aligned}$$

Now,

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

and

$$m_{\text{normal}} = -1.$$

Next, the equation of AB is

$$y - 1 = -(x - 0).$$

Finally,

$$y = 0 \Rightarrow x = 1,$$

and $\underline{\underline{B(1,0)}}$.

(b) Show that (2)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x(\cos x + \sin x)) = 2\mathrm{e}^x \cos x.$$

Solution

Product rule:

$$u = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$$
$$v = \cos x + \sin x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = -\sin x + \cos x$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x(\cos x + \sin x) = (\mathrm{e}^x)(-\sin x + \cos x) + (\mathrm{e}^x)(\cos x + \sin x)$$
$$= \underline{\underline{2\mathrm{e}^x \cos x}},$$

as required.

(c) Hence find (4)

$$\int \mathrm{e}^x \cos x \,\mathrm{d}x$$

and the area of the shaded region.

Solution
Well,

$$\int \mathrm{e}^x \cos x \,\mathrm{d}x = \tfrac{1}{2}\int 2\mathrm{e}^x \cos x \,\mathrm{d}x$$
$$= \underline{\underline{\tfrac{1}{2}\mathrm{e}^x(\cos x + \sin x) + c}}.$$

Now,

$$\text{area of the shaded region} = \text{integral} - \text{area of } OAB$$
$$= \left[\tfrac{1}{2}\mathrm{e}^x(\cos x + \sin x)\right]_{x=0}^{\frac{1}{2}\pi} - (\tfrac{1}{2} \times 1 \times 1)$$
$$= \tfrac{1}{2}\mathrm{e}^{\frac{1}{2}\pi} - \tfrac{1}{2} - \tfrac{1}{2}$$
$$= \underline{\underline{\tfrac{1}{2}\mathrm{e}^{\frac{1}{2}\pi} - 1}}.$$