Dr Oliver Mathematics
Advance Level Mathematics
Pure Mathematics 1: Calculator
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. (3)

$$f(x) \equiv 3x^3 + 2ax^2 - 4x + 5a.$$

Given that $(x + 3)$ is a factor of $f(x)$, find the value of the constant a.

Solution

We use synthetic division:

-3	3	$2a$	-4	$5a$
	$\downarrow$	-9	$27 - 6a$	$18a - 69$
	3	$2a - 9$	$23 - 6a$	$23a - 69$

As the remainder is 0,

$$23a - 69 = 0 \Rightarrow \underline{\underline{a = 3}}.$$

2. Figure 1 shows a plot of part of the curve with equation $y = \cos x$ where x is measured in radians.

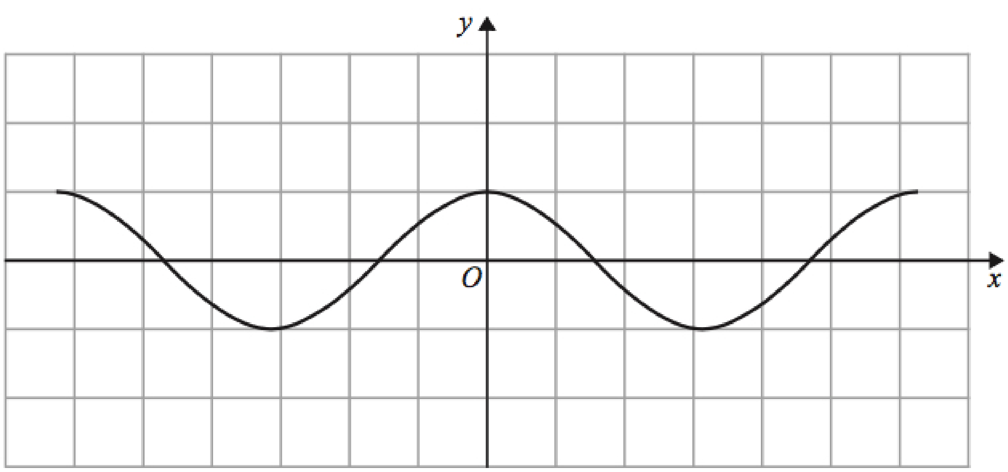


Figure 1: $y = \cos x$

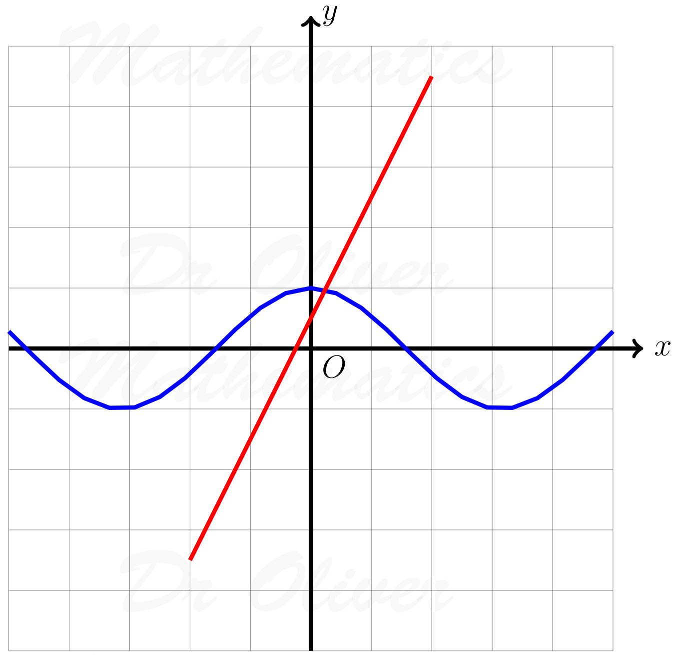
(a) Use Figure 1 to show why the equation (2)

$$\cos x - 2x - \tfrac{1}{2} = 0$$

has only one real root, giving a reason for your answer.

Solution

Plot $y = \cos x$ and $y = 2x + \tfrac{1}{2}$ on the same axes:



As there is one intersection between the graphs, the equation

$$\cos x = 2x + \tfrac{1}{2} \Rightarrow \cos x - 2x - \tfrac{1}{2} = 0$$

has one real root.

Given that the root of the equation is α, and that α is small,

(b) use the small angle approximation for $\cos x$ to estimate the value of α to 3 decimal places. (3)

Solution

Now,

$$\cos x \approx 1 - \tfrac{1}{2}x^2$$

and

$$\begin{aligned}\cos x - 2x - \tfrac{1}{2} = 0 &\Rightarrow 1 - \tfrac{1}{2}x^2 - 2x - \tfrac{1}{2} = 0 \\ &\Rightarrow x^2 + 4x = 1 \\ &\Rightarrow x^2 + 4x + 4 = 5 \\ &\Rightarrow (x+2)^2 = 5 \\ &\Rightarrow x + 2 = \sqrt{5} \text{ (we ignore the } -\sqrt{5}\text{: why?)} \\ &\Rightarrow \underline{\underline{x = -2 + \sqrt{5}}}.\end{aligned}$$

3.

$$y = \frac{5x^2 + 10x}{(x+1)^2},\ x \neq -1.$$

(a) Show that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{A}{(x+1)^n},$$

where A and n are constants to be found.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x+1)^2 \cdot (10x+10) - (5x^2+10x) \cdot 2(x+1)}{(x+1)^4} \\ &= \frac{10(x+1)^3 - (x+1)(10x^2+20x)}{(x+1)^4} \\ &= \frac{10(x+1)^2 - (10x^2+20x)}{(x+1)^3} \\ &= \frac{(10x^2+20x+10) - (10x^2+20x)}{(x+1)^3} \\ &= \frac{10}{(x+1)^3};\end{aligned}$$

hence, $\underline{\underline{A = 10}}$ and $\underline{\underline{n = 3}}$.

(b) Hence deduce the range of values for x for which (1)

$$\frac{\mathrm{d}y}{\mathrm{d}x} < 0.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} < 0 &\Rightarrow \frac{10}{(x+1)^3} < 0\\ &\Rightarrow (x+1)^3 < 0\\ &\Rightarrow x+1 < 0\\ &\Rightarrow \underline{\underline{x < -1}}.\end{aligned}$$

4. (a) Find the first three terms, in ascending powers of x, of the binomial expansion of (4)

$$\frac{1}{\sqrt{4-x}},$$

giving each coefficient in its simplest form.

Solution

Recall:

$$(a+bx)^n = a^n + na^{n-1}bx + \frac{n(n-1)}{2!}a^{n-2}(bx)^2 + \ldots$$

Now,

$$\begin{aligned}\frac{1}{\sqrt{4-x}} &= [4+(-x)]^{-\frac{1}{2}}\\ &= 4^{-\frac{1}{2}} + (-\tfrac{1}{2})4^{-\frac{3}{2}}(-x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}4^{-\frac{5}{2}}(-x)^2 + \ldots\\ &= \underline{\underline{\tfrac{1}{2} + \tfrac{1}{16}x + \tfrac{3}{256}x^2 + \ldots}}\end{aligned}$$

for

$$\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4.$$

The expansion can be used to find an approximation to $\sqrt{2}$.
Possible values of x that could be substituted into this expansion are:

- $x = -14$ because
$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{18}} = \frac{\sqrt{2}}{6}.$$
- $x = 2$ because
$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$
- $x = -\frac{1}{2}$ because
$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{\frac{9}{2}}} = \frac{\sqrt{2}}{3}.$$

(b) Without evaluating your expansion,

(i) state, giving a reason, which of the three values of x should not be used; (1)

Solution
$\underline{\underline{x = -14}}$ as $|x| \nless 4$.

(ii) state, giving a reason, which of the three values of x would lead to the most accurate approximation to $\sqrt{2}$. (1)

Solution
$\underline{\underline{x = -\frac{1}{2}}}$ as x is close to zero.

5.
$$\mathrm{f}(x) \equiv 2x^2 + 4x + 9,\ x \in \mathbb{R}.$$

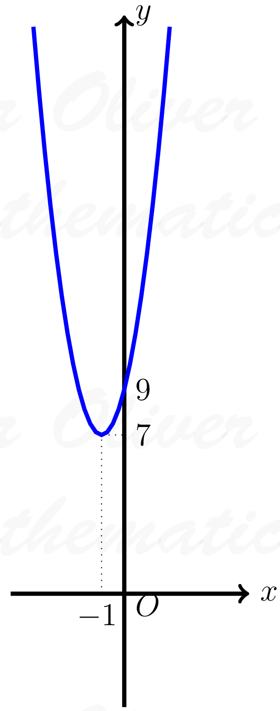
(a) Write $\mathrm{f}(x)$ in the form $a(x+b)^2 + c$, where a, b, and c are integers to be found. (3)

Solution
$$\begin{aligned} 2x^2 + 4x + 9 &= 2(x^2 + 2x) + 9 \\ &= 2\big[(x^2 + 2x + 1) - 1\big] + 9 \\ &= 2(x+1)^2 - 2 + 9 \\ &= \underline{\underline{2(x+1)^2 + 7}}; \end{aligned}$$

hence, $\underline{\underline{a = 2}}$, $\underline{\underline{b = 1}}$, and $\underline{\underline{c = 7}}$.

(b) Sketch the curve with equation $y = \mathrm{f}(x)$ showing any points of intersection with the coordinate axes and the coordinates of any turning point. (3)

Solution



(c) (i) Describe fully the transformation that maps the curve with equation $y = \mathrm{f}(x)$ onto the curve with equation $y = \mathrm{g}(x)$ where (4)

$$\mathrm{g}(x) \equiv 2(x-2)^2 + 4x - 3,\ x \in \mathbb{R}.$$

Solution

$$\begin{aligned} 2(x-2)^2 + 4x - 3 &= 2(x-2)^2 + 4[(x-2)+2] - 3 \\ &= 2(x-2)^2 + 4(x-2) + 8 - 3 \\ &= 2(x-2)^2 + 4(x-2) + 5; \end{aligned}$$

so, $\mathrm{g}(x)$ is a translation, by $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

(ii) Find the range of the function

$$\mathrm{h}(x) = \frac{21}{2x^2 + 4x + 9},\ x \in \mathbb{R}.$$

Solution

Now,

$$\frac{21}{2x^2+4x+9} = \frac{21}{2(x+1)^2+7}.$$

The maximum value is

$$\frac{21}{7} = 3$$

and

$$\text{h}(x) \to 0 \text{ as } x \to \pm\infty.$$

So,

$$\underline{\underline{0 < \text{h}(x) \leqslant 3}}.$$

6. (a) Solve, for $-180^\circ \leqslant \theta \leqslant 180^\circ$, the equation (6)

$$5\sin 2\theta = 9\tan\theta,$$

giving your answers, where necessary, to one decimal place.

Solution

$$\begin{aligned}
5\sin 2\theta = 9\tan\theta &\Rightarrow 10\sin\theta\cos\theta = \frac{9\sin\theta}{\cos\theta} \\
&\Rightarrow 10\sin\theta\cos\theta - \frac{9\sin\theta}{\cos\theta} = 0 \\
&\Rightarrow \frac{\sin\theta(10\cos^2\theta - 9)}{\cos\theta} = 0 \\
&\Rightarrow \sin\theta = 0 \text{ or } 10\cos^2\theta - 9 = 0 \\
&\Rightarrow \sin\theta = 0 \text{ or } \cos\theta = \pm\tfrac{3\sqrt{10}}{10}.
\end{aligned}$$

$\underline{\sin\theta = 0}$:

$$\sin\theta = 0 \Rightarrow \underline{\underline{\theta = \pm 180, 0}}.$$

$\underline{\cos\theta = \tfrac{3\sqrt{10}}{10}}$:

$$\begin{aligned}
\cos\theta = \tfrac{3\sqrt{10}}{10} &\Rightarrow \theta = \pm 18.434\,948\,82 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{\theta = \pm 18.4 \text{ (1 dp)}}}.
\end{aligned}$$

$\underline{\cos\theta = -\frac{3\sqrt{10}}{10}}$:

$$\begin{aligned}\cos\theta = -\tfrac{3\sqrt{10}}{10} &\Rightarrow \theta = \pm 161.565\,051\,2 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{\theta = \pm 161.6 \text{ (1 dp)}}}.\end{aligned}$$

(b) Deduce the smallest positive solution to the equation (2)

$$5\sin(2x - 50)^\circ = 9\tan(x - 25)^\circ.$$

Solution

$$\begin{aligned}x - 25 = -18.434\ldots &\Rightarrow x = 6.565\,051\,218 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{x = 6.6 \text{ (1 dp)}}}.\end{aligned}$$

7. In a simple model, the value, £V, of a car depends on its age, t, in years.

The following information is available for car A:

- its value when new is £20 000,
- its value after one year is £16 000.

(a) Use an exponential model to form, for car A, a possible equation linking V with t. (4)

Solution

Well,

$$V = C\mathrm{e}^{-kt}$$

for some C and k. Now,

$$20\,000 = C\mathrm{e}0 \Rightarrow C = 20\,000$$

and

$$\begin{aligned}16\,000 = 20\,000\mathrm{e}^{-k} &\Rightarrow \mathrm{e}^{-k} = \tfrac{4}{5}\\ &\Rightarrow \mathrm{e}^{k} = \tfrac{5}{4}\\ &\Rightarrow k = 0.223\,143\,551\,3 \text{ (FCD)}\\ &\Rightarrow k = 0.223 \text{ (3 sf)}.\end{aligned}$$

Hence,

$$\underline{\underline{V = 20\,000\mathrm{e}^{-0.223\ldots t}}}.$$

The value of car A is monitored over a 10-year period.
Its value after 10 years is £2 000.

(b) Evaluate the reliability of your model in light of this information. (2)

Solution

The model predicts a value of

$$20\,000\mathrm{e}^{-0.223\ldots\times 10} = 2\,147.483\,648 \text{ (FCD)}$$

and so the model has a good fit.

The following information is available for car B:

- it has the same value, when new, as car A,
- its value depreciates more slowly than that of car A.

(c) Explain how you would adapt the equation found in (a) so that it could be used to model the value of car B. (1)

Solution

I would make the k less negative, i.e., $\underline{\underline{0 < k < 0.223}}$.

8. Figure 2 shows a sketch of part of the curve with equation

$$y = x(x+2)(x-4).$$

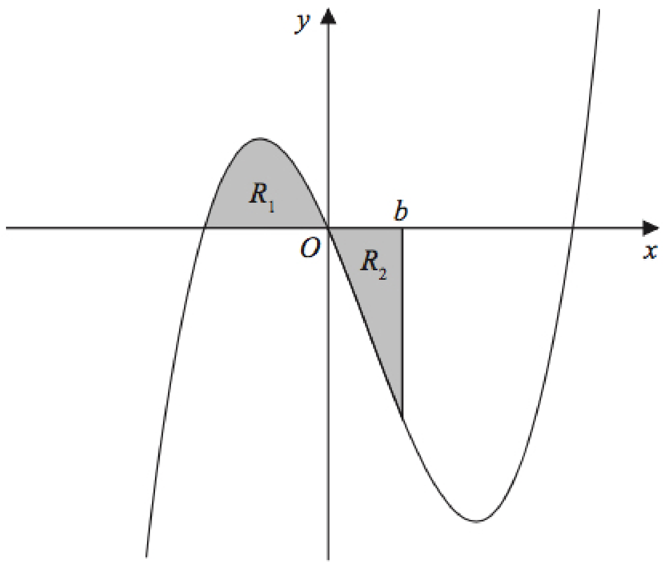


Figure 2: $y = x(x+2)(x-4)$

The region R_1 shown shaded in Figure 2 is bounded by the curve and the negative x-axis.

(a) Show that the exact area of R_1 is $\frac{20}{3}$. (4)

Solution

$$\begin{aligned}\text{Area} &= \int_{-2}^{0} x(x+2)(x-4)\,\mathrm{d}x \\ &= \int_{-2}^{0} x(x^2-2x-8)\,\mathrm{d}x \\ &= \int_{-2}^{0} (x^3-2x^2-8x)\,\mathrm{d}x \\ &= \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - 4x^2\right]_{x=-2}^{0} \\ &= (0-0-0) - \left(4 + \tfrac{16}{3} - 16\right) \\ &= \underline{\underline{\tfrac{20}{3}}},\end{aligned}$$

as required.

The region R_2 also shown shaded in Figure 2 is bounded by the curve, the positive x-axis, and the line with equation $x = b$, where b is a positive constant and $0 < b < 4$.

Given that the area of R_1 is equal to the area of R_2,

(b) verify that b satisfies the equation (4)

$$(b+2)^2(3b^2 - 20b + 20) = 0.$$

Solution

Note how the area is all below the x-axis:

$$\begin{aligned}\int_0^b x(x+2)(x-4)\,\mathrm{d}x = -\tfrac{20}{3} &\Rightarrow \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - 4x^2\right]_{x=0}^{b} = -\tfrac{20}{3} \\ &\Rightarrow \left(\tfrac{1}{4}b^4 - \tfrac{2}{3}b^3 - 4b^2\right) - (0 - 0 - 0) = -\tfrac{20}{3} \\ &\Rightarrow \tfrac{1}{4}b^4 - \tfrac{2}{3}b^3 - 4b^2 + \tfrac{20}{3} = 0 \\ &\Rightarrow 3b^4 - 8b^3 - 48b^2 + 80 = 0.\end{aligned}$$

We use synthetic division twice:

−2	3	−8	−48	0	80
	↓	−6	28	40	−80
	3	−14	−20	40	0

So

$$3b^4 - 8b^3 - 48b^2 + 80 = 0 \Rightarrow (b+2)(3b^3 - 14b^2 - 20b + 40) = 0.$$

−2	3	−14	−20	40
	↓	−6	40	−40
	3	−20	20	0

So

$$3b^3 - 14b^2 - 20b + 40 = 0 \Rightarrow (b+2)(3b^2 - 20 + 20) = 0.$$

Hence, b satisfies the equation.

The roots of the equation

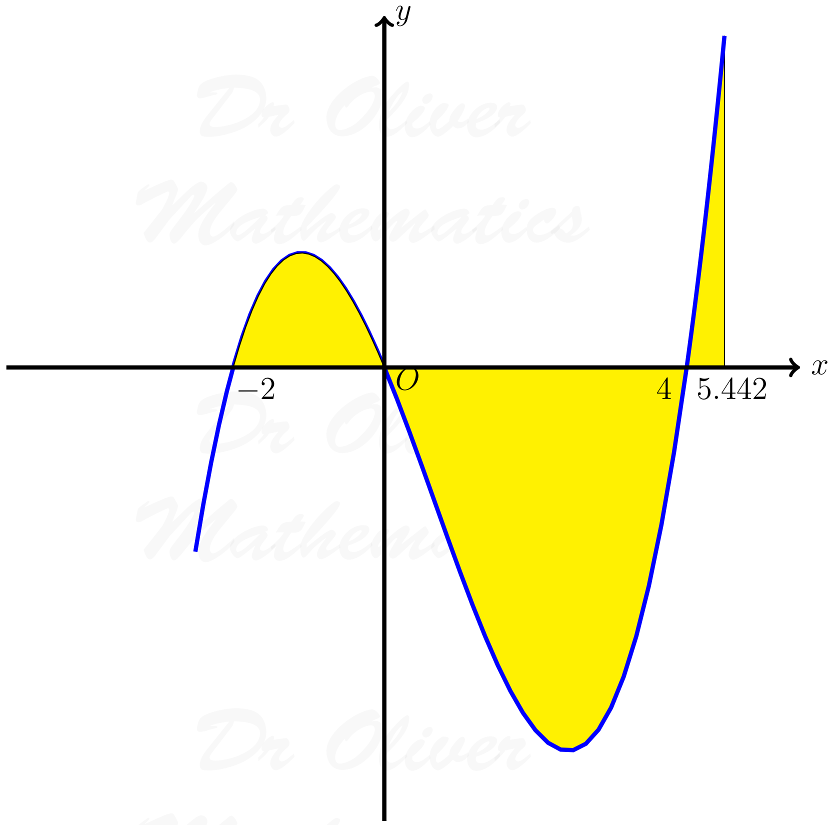
$$3b^2 - 20b + 20 = 0$$

are 1.225 and 5.442 to 3 decimal places.
The value of b is therefore 1.225 to 3 decimal places.

(c) Explain, with the aid of a diagram, the significance of the root 5.442. (2)

Solution

Between $x = -2$ and $x = 5.442$, the area **under** the x-axis equals the area **above** the x-axis.



9. Given that $a > b > 0$ and that a and b satisfy the equation

$$\log a - \log b = \log(a - b),$$

(a) show that (3)

$$a = \frac{b^2}{b-1}.$$

Solution

$$
\begin{aligned}
\log a - \log b = \log(a-b) &\Rightarrow \log\left(\frac{a}{b}\right) = \log(a-b) \\
&\Rightarrow \frac{a}{b} = a - b \\
&\Rightarrow a = b(a-b) \\
&\Rightarrow a = ab - b^2 \\
&\Rightarrow ab - a = b^2 \\
&\Rightarrow a(b-1) = b^2 \\
&\Rightarrow \underline{\underline{a = \frac{b^2}{b-1}}},
\end{aligned}
$$

as required.

(b) Write down the full restriction on the value of b, explaining the reason for this restriction. (2)

Solution

$$
\begin{aligned}
a > 0 &\Rightarrow \frac{b^2}{b-1} > 0 \\
&\Rightarrow b - 1 > 0 \\
&\Rightarrow \underline{\underline{b > 1}}.
\end{aligned}
$$

10. (a) Prove that for all $n \in \mathbb{N}$, (4)

$$n^2 + 2 \text{ is not divisible by } 4.$$

Solution

$\underline{n = 2m, m \in \mathbb{N}}$:

$$
\begin{aligned}
(2m)^2 + 2 &= 4m^2 + 2 \\
&= 4 \times \text{some integer} + 2
\end{aligned}
$$

and so $n^2 + 2$ is not divisible by 4.

$\underline{n = 2m + 1, m \in \mathbb{N}}$:

$$\begin{aligned}(2m+1)^2 + 2 &= (4m^2 + 4m + 1) + 2 \\ &= 4(m^2 + m) + 3 \\ &= 4 \times \text{some integer} + 3\end{aligned}$$

and so $n^2 + 2$ is not divisible by 4.

So, $n^2 + 2$ is not divisible by 4.

(b) "Given $x \in \mathbb{R}$, the value of $|3x - 28|$ is greater than or equal to the value of $(x - 9)$." State, giving a reason, if the above statement is always true, sometimes true, or never true. (2)

Solution

$$\begin{aligned}3x - 28 \geqslant x - 9 &\Rightarrow 2x \geqslant 19 \\ &\Rightarrow x \geqslant 9\tfrac{1}{2}\end{aligned}$$

and

$$\begin{aligned}-(3x - 28) \geqslant x - 9 &\Rightarrow -3x + 28 \geqslant x - 9 \\ &\Rightarrow -4x \geqslant -37 \\ &\Rightarrow x \leqslant 9\tfrac{1}{4}.\end{aligned}$$

so, it is sometimes true (except when $9\frac{1}{4} < x < 9\frac{1}{2}$).

11. A competitor is running a 20 kilometre race.

She runs each of the first 4 kilometres at a steady pace of 6 minutes per kilometre. After the first 4 kilometres, she begins to slow down.

In order to estimate her finishing time, the time that she will take to complete each subsequent kilometre is modelled to be 5% greater than the time that she took to complete the previous kilometre.

Using the model,

(a) show that her time to run the first 6 kilometres is estimated to be 36 minutes 55 seconds, (2)

Solution

$$\begin{aligned}\text{Time} &= (4 \times 6) + (6 \times 1.05) + (6 \times 1.05^2) \\ &= 24 + 6.3 + 6.615 \\ &= 36.915 \text{ mins} \\ &= \underline{\underline{36 \text{ mins } 54.9 \text{ seconds}}}.\end{aligned}$$

(b) show that her estimated time, in minutes, to run the rth kilometre, for $5 \leqslant r \leqslant 20$, is (1)

$$6 \times 1.05^{r-4},$$

Solution

$a = 6$ and $r = 1.05$. Now,
5th km: 6×1.05.
6th km: 6×1.05^2.
7th km: 6×1.05^3 and so on.
Hence the time for the rth km is $\underline{\underline{6 \times 1.05^{r-4}}}$.

(c) estimate the total time, in minutes and seconds, that she will take to complete the race. (4)

Solution

$$\begin{aligned}\text{Overall time} &= (4 \times 6) + (6 \times 1.05) + (6 \times 1.05^2) + \ldots + (6 \times 1.05^{16}) \\ &= 24 + \frac{6.3(1.05^{16} - 1)}{1.05 - 1} \\ &= 24 + 149.042\,198\,1 \text{ (FCD)} \\ &= 173.042\,198\,1 \text{ (FCD)} \\ &= 2 \text{ hours } 53 \text{ mins } 2.531\,888\,167 \text{ seconds (FCD)};\end{aligned}$$

hence, she will complete it in a little under $\underline{\underline{3 \text{ hours}}}$.

12.

$$\mathrm{f}(x) = 10\mathrm{e}^{-0.25x} \sin x,\ x > 0.$$

(a) Show that the x-coordinates of the turning points of the curve with equation $y = \mathrm{f}(x)$ satisfy the equation $\tan x = 4$. (4)

Solution

$$f(x) = 10e^{-0.25x}\sin x \Rightarrow f'(x) = -2.5e^{-0.25x}\sin x + 10e^{-0.25x}\cos x$$
$$\Rightarrow f'(x) = 2.5e^{-0.25x}(4\cos x - \sin x).$$

Now,

$$f'(x) = 0 \Rightarrow 2.5e^{-0.25x}(4\cos x - \sin x) = 0$$
$$\Rightarrow 4\cos x - \sin x = 0$$
$$\Rightarrow 4\cos x = \sin x$$
$$\Rightarrow \underline{\underline{\tan x = 4}},$$

as required.

Figure 3 shows a sketch of part of the curve with equation $y = f(x)$.

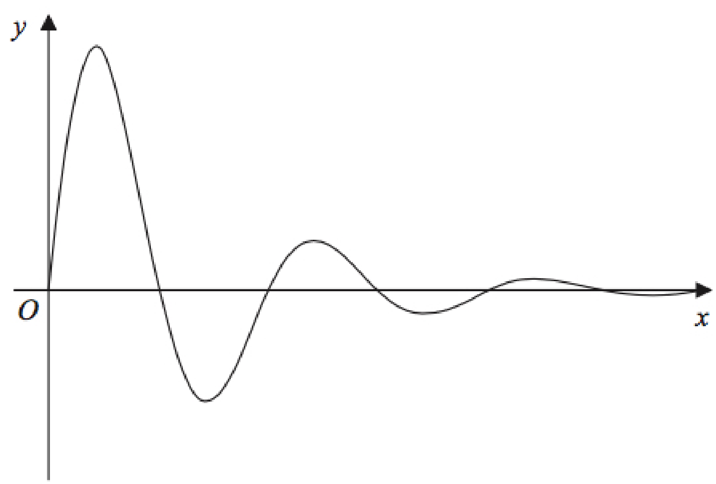


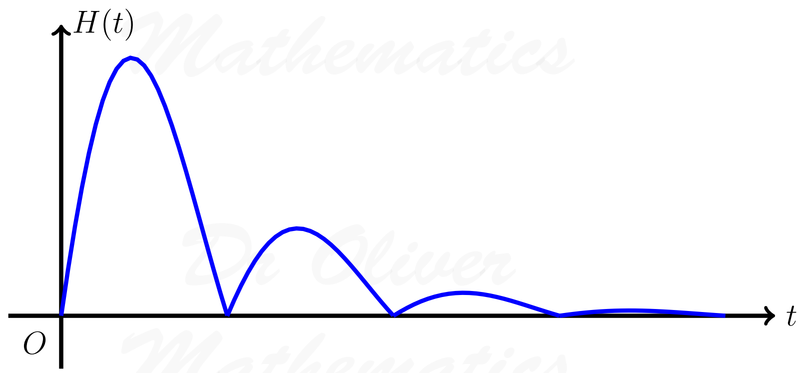
Figure 3: $f(x) = 10e^{-0.25x}\sin x$

(b) Sketch the graph of H against t where (2)

$$H(t) = \left|10e^{-0.25t}\sin t\right|,\ t \geqslant 0,$$

showing the long-term behaviour of this curve.

Solution



The function $H(t)$ is used to model the height, in metres, of a ball above the ground t seconds after it has been kicked.

Using this model, find

(c) the maximum height of the ball above the ground between the first and second bounce. (3)

Solution

$$\begin{aligned}\tan x = 4 &\Rightarrow x = 1.325\,817\,664 \text{ (no)}, 4.467\,410\,317 \text{ (yes)}\\ &\Rightarrow H(4.467\ldots) = 3.175\,357\,431 \text{ (FCD)}\\ &\Rightarrow H(4.467\ldots) = \underline{\underline{3.18 \text{ m (3 sf)}}}.\end{aligned}$$

(d) Explain why this model should not be used to predict the time of each bounce. (1)

Solution

E.g., the times between each bounce should not stay the same when the heights of each bounce is getting smaller, the bounces would get faster and faster, bounces would become more frequent, etc.

13. The curve C with equation

$$y = \frac{p - 3x}{(2x - q)(x + 3)}, \ x \in \mathbb{R}, \ x \neq -3, \ x \neq 2,$$

where p and q are constants, passes through the point $(3, \frac{1}{2})$ and has two vertical asymptotes with equations $x = 2$ and $x = -3$.

(a) (i) Explain why you can deduce that $q = 4$. (3)

Solution

Well, $(x+3)$ goes with the vertical asymptote at $x = -3$. So what goes with $x = 2$?

$$x - 2 \Rightarrow 2x - 4$$

and $\underline{\underline{q = 4}}$.

(ii) Show that $p = 15$.

Solution

Make $x = 3$ and $y = \frac{1}{2}$:

$$\begin{aligned} \frac{p-9}{(6-4)(3+3)} = \tfrac{1}{2} &\Rightarrow \frac{p-9}{12} = \tfrac{1}{2} \\ &\Rightarrow p - 9 = 6 \\ &\Rightarrow \underline{\underline{p = 15}}, \end{aligned}$$

as required.

Figure 4 shows a sketch of part of the curve C.

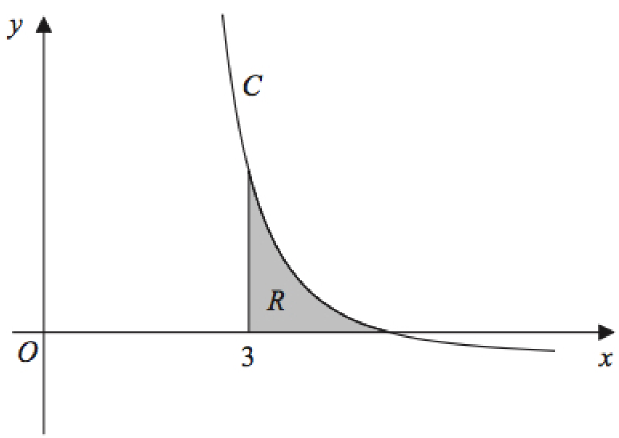


Figure 4: $y = \dfrac{p - 3x}{(2x - q)(x + 3)}$

The region R, shown shaded in Figure 4, is bounded by the curve C, the x-axis, and the line with equation $x = 3$.

(b) Show that the exact value of the area of R is (8)

$$a \ln 2 + b \ln 3,$$

where a and b are rational constants to be found.

Solution

$$\frac{15-3x}{(2x-4)(x+3)} \equiv \frac{A}{2x-4} + \frac{B}{x+3} \equiv \frac{A(x+3)+B(2x-4)}{(2x-4)(x+3)}$$

and so

$$15-3x \equiv A(x+3)+B(2x-4).$$

$\underline{x=2}$: $9 = 5A \Rightarrow A = \frac{9}{5}$.
$\underline{x=-3}$: $24 = -10B \Rightarrow B = -\frac{12}{5}$.
Hence,

$$\frac{15-3x}{(2x-4)(x+3)} \equiv \frac{\frac{9}{5}}{2x-4} - \frac{\frac{12}{5}}{x+3}.$$

Now,

$$\frac{15-3x}{(2x-4)(x+3)} = 0 \Rightarrow 15-3x = 0 \Rightarrow x = 5.$$

Finally,

$$\begin{aligned}\text{area} &= \int_3^5 \frac{15-3x}{(2x-4)(x+3)}\,\mathrm{d}x \\ &= \int_3^5 \left(\frac{\frac{9}{5}}{2x-4} - \frac{\frac{12}{5}}{x+3}\right)\mathrm{d}x \\ &= \left[\tfrac{9}{10}\ln|2x-4| - \tfrac{12}{5}\ln|x+3|\right]_{x=3}^{5} \\ &= \left(\tfrac{9}{10}\ln 6 - \tfrac{12}{5}\ln 8\right) - \left(\tfrac{9}{10}\ln 2 - \tfrac{12}{5}\ln 6\right) \\ &= \left(\tfrac{9}{10}\ln 2 + \tfrac{9}{10}\ln 3 - \tfrac{36}{5}\ln 2\right) - \left(\tfrac{9}{10}\ln 2 - \tfrac{12}{5}\ln 2 - \tfrac{12}{5}\ln 3\right) \\ &= \underline{\underline{-\tfrac{24}{5}\ln 2 + \tfrac{33}{10}\ln 3}};\end{aligned}$$

hence, $\underline{\underline{a = -\frac{24}{5}}}$ and $\underline{\underline{b = \frac{33}{10}}}$.

14. The curve C, in the standard Cartesian plane, is defined by the equation

$$x = 4\sin 2y,\ -\tfrac{1}{4}\pi < y < \tfrac{1}{4}\pi.$$

The curve C passes through the origin O.

(a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the origin. (2)

Solution

$$x = 4\sin 2y \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = 8\cos 2y$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{8\cos 2y}$$

and, at the origin,

$$\underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{8}}}.$$

(b) (i) Use the small angle approximation for $\sin 2y$ to find an equation linking x and y for points close to the origin. (2)

Solution
When y is small,

$$\sin 2y \approx 2y$$

and we have

$$\underline{\underline{x \approx 8y}}.$$

(ii) Explain the relationship between the answers to (a) and (b)(i).

Solution
The value found in (a) is the gradient of the line found in (b)(i).

(c) Show that, for all points (x, y) lying on C, (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a\sqrt{b - x^2}},$$

where a and b are constants to be found.

Solution

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{8\cos 2y} \\
&= \frac{1}{8\sqrt{\cos^2 2y}} \\
&= \frac{1}{8\sqrt{1-\sin^2 2y}} \\
&= \frac{1}{8\sqrt{1-(\frac{x}{4})^2}} \\
&= \frac{1}{8\sqrt{1-\frac{x^2}{16}}} \\
&= \frac{1}{8\sqrt{\frac{1}{16}(16-x^2)}} \\
&= \underline{\underline{\frac{1}{2\sqrt{16-x^2}}}};
\end{aligned}
$$

hence, $\underline{\underline{a = 2}}$ and $\underline{\underline{b = 16}}$.