Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2004 June Paper 1: Calculator
2 hours

The total number of marks available is 80.
You must write down all the stages in your working.

1. Given that

$$y = \frac{3x - 2}{x^2 + 5},$$

find

(a) an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$, (2)

Solution
Well,

$$u = 3x - 2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 3$$
$$v = x^2 + 5 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x.$$

Quotient rule:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x^2 + 5)(3) - (3x - 2)(2x)}{(x^2 + 5)^2} \\ &= \frac{3x^2 + 15 - (6x^2 - 4x)}{(x^2 + 5)^2} \\ &= \frac{3x^2 + 15 - 6x^2 + 4x}{(x^2 + 5)^2} \\ &= \underline{\underline{\frac{-3x^2 + 4x + 15}{(x^2 + 5)^2}}}.\end{aligned}$$

(b) the x-coordinates of the stationary points. (2)

Solution

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{-3x^2 + 4x + 15}{(x^2+5)^2} = 0$$
$$\Rightarrow -3x^2 + 4x + 15 = 0$$
$$\Rightarrow 3x^2 - 4x - 15 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & (+3) \times (-15) = -45 \end{array}\right\} -9,\ +5$$

e.g.,

$$\Rightarrow 3x^2 - 9x + 5x - 15 = 0$$
$$\Rightarrow 3x(x-3) + 5(x-3) = 0$$
$$\Rightarrow (3x+5)(x-3) = 0$$
$$\Rightarrow 3x + 5 = 0 \text{ or } x - 3 = 0$$
$$\Rightarrow \underline{\underline{x = -1\tfrac{2}{3}}} \text{ or } \underline{\underline{x = 3}}.$$

2. Find the x-coordinates of the three points of intersection of the curve (5)

$$y = x^3$$

with the line

$$y = 5x - 2,$$

expressing non-integer values in the form

$$a \pm \sqrt{b},$$

where a and b are integers.

Solution

Now,

$$x^3 = 5x - 2 \Rightarrow x^3 - 5x + 2 = 0$$

and let

$$\mathrm{f}(x) = x^3 - 5x + 2.$$

Next,

$$\begin{aligned} \mathrm{f}(1) &= 1 - 5 + 2 = -2 \\ \mathrm{f}(-1) &= -1 + 5 + 2 = 6 \\ \mathrm{f}(2) &= 8 - 10 + 2 = 0, \end{aligned}$$

and we know that $(x - 2)$ is a root of $\mathrm{f}(x)$.

Synthetic division:

$$\begin{array}{c|cccc} 2 & 1 & 0 & -5 & 2 \\ & \downarrow & 2 & 4 & -2 \\ \hline & 1 & 2 & -1 & 0 \end{array}$$

and so

$$x^3 - 5x + 2 = (x - 2)(x^2 + 2x - 1).$$

Now, we will complete the square:

$$\begin{aligned} x^2 + 2x - 1 = 0 &\Rightarrow x^2 + 2x = 1 \\ &\Rightarrow x^2 + 2x + 1 = 1 + 1 \\ &\Rightarrow (x + 1)^2 = 2 \\ &\Rightarrow x + 1 = \pm\sqrt{2} \\ &\Rightarrow x = -1 \pm \sqrt{2}. \end{aligned}$$

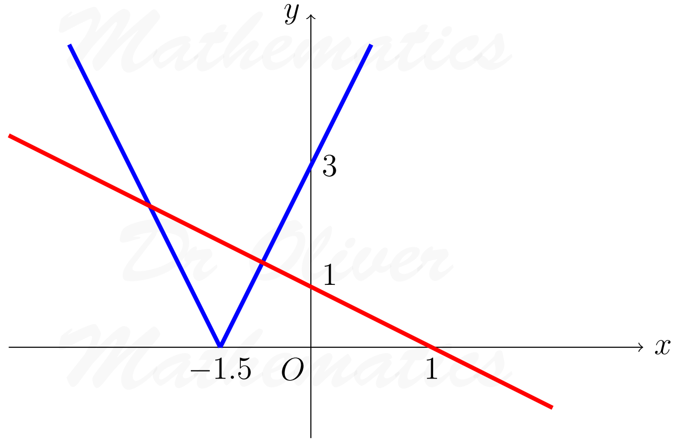
Hence, the solutions are

$$\underline{\underline{x = 2,\ -1 \pm \sqrt{2}}}.$$

3. (a) Sketch on the same diagram the graphs of (3)

$$y = |2x + 3| \text{ and } y = 1 - x.$$

Solution



(b) Find the values of x for which (3)

$$x + |2x+3| = 1.$$

Solution
Well,

$$x + |2x+3| = 1 \Rightarrow |2x+3| = 1 - x$$

and so we look at where the two lines cross.

$\underline{2x + 3 = 1 - x}$:

$$\begin{aligned} 2x + 3 = 1 - x &\Rightarrow 3x = -2 \\ &\Rightarrow x = -\tfrac{2}{3}. \end{aligned}$$

$\underline{-(2x+3) = 1 - x}$:

$$\begin{aligned} -(2x+3) = 1 - x &\Rightarrow -2x - 3 = 1 - x \\ &\Rightarrow -x = 4 \\ &\Rightarrow x = -4. \end{aligned}$$

Hence, the values of x are

$$\underline{\underline{x = -4}} \text{ or } \underline{\underline{x = -\tfrac{2}{3}}}.$$

4. The function f is defined, for $0^\circ \leqslant x \leqslant 360^\circ$, by

$$\mathrm{f}(x) = a\sin(bx) + c,$$

where a, b, and c are positive integers.

Given that the amplitude of f is 2 and the period of f is $120°$,

(a) state the value of a and of b. (2)

> **Solution**
> Well, $\underline{\underline{a = 2}}$ and
> $$b = \frac{360}{120} = \underline{\underline{3}}.$$

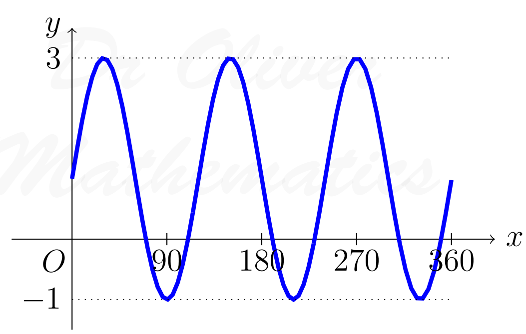
Given further that the minimum value of f is -1,

(b) state the value of c, (1)

> **Solution**
> $\underline{\underline{c = 1}}$.

(c) sketch the graph of f. (3)

> **Solution**



5. The straight line (6)

$$5y + 2x = 1$$

meets the curve

$$xy + 24 = 0$$

at the points A and B.

Find the length of AB, correct to one decimal place.

Solution

Well,

$$\begin{aligned} xy + 24 = 0 \Rightarrow xy &= -24 \\ \Rightarrow y &= -\frac{24}{x} \end{aligned}$$

and let us insert in to the linear equation:

$$5y + 2x = 1 \Rightarrow 5\left(-\frac{24}{x}\right) + 2x = 1$$

multiply by x:

$$\begin{aligned} &\Rightarrow -120 + 2x^2 = x \\ &\Rightarrow 2x^2 - x - 120 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -1 \\ \text{multiply to:} & (+2) \times (-120) = -240 \end{array}\right\} -16,\ +15$$

e.g.,

$$\begin{aligned} &\Rightarrow 2x^2 - 16x + 15x - 120 = 0 \\ &\Rightarrow 2x(x-8) + 15(x-8) = 0 \\ &\Rightarrow (2x+15)(x-8) = 0 \\ &\Rightarrow 2x + 15 = 0 \text{ or } x - 8 = 0 \\ &\Rightarrow x = -7.5 \text{ or } x = 8 \\ &\Rightarrow y = 3.2 \text{ or } y = -3; \end{aligned}$$

so, the two points are $(-7.5, 3.2)$ and $(8, -3)$. Finally,

$$\begin{aligned} \text{length} &= \sqrt{[8-(-7.5)]^2 + (-3-3.2)^2} \\ &= \sqrt{(15.5)^2 + (-6.2)^2} \\ &= \sqrt{240.25 + 38.44} \\ &= \sqrt{278.69} \\ &= 16.694\,010\,9 \text{ (FCD)} \\ &= \underline{\underline{16.7 \text{ cm (1 dp)}}}. \end{aligned}$$

6. The table below shows (6)

   - the daily production, in kilograms, of two types, S_1 and S_2, of sweets from a small company and
   - the percentages of the ingredients A, B, and C required to produce S_1 and S_2.

	A	B	C	Daily Production
Type S_1	60	30	10	300
Type S_2	50	40	10	240

   Given that the costs, in dollars per kilogram, of A, B, and C are 4, 6, and 8 respectively, use matrix multiplication to calculate the total cost of daily production.

Solution

Well,

$$\begin{aligned} & \begin{pmatrix} 300 & 240 \end{pmatrix} \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.5 & 0.4 & 0.1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \\ = & \begin{pmatrix} 300 & 240 \end{pmatrix} \begin{pmatrix} 5 \\ 5.2 \end{pmatrix} \\ = & \begin{pmatrix} 2\,748 \end{pmatrix}. \end{aligned}$$

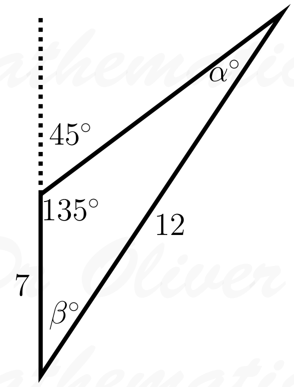
Hence, the total cost of daily production is $\underline{\underline{\$2\,748}}$.

7. To a cyclist travelling due south on a straight horizontal road at 7 ms^{-1}, the wind appears to be blowing from the north-east. (5)

   Given that the wind has a constant speed of 12 ms^{-1}, find the direction from which the wind is blowing.

Solution

We will draw a picture:



Sine rule:

$$\frac{\sin \alpha^\circ}{7} = \frac{\sin 135^\circ}{12} \Rightarrow \sin \alpha^\circ = \frac{7 \sin 135^\circ}{12}$$
$$\Rightarrow \alpha = 24.360\,654\,34 \text{ (FCD)}.$$

Now,

$$\beta = 180 - 135 - \alpha$$
$$= 20.639\,345\,66 \text{ (FCD)}.$$

Finally, the direction is 020.6°.

8. A curve has the equation (7)

$$y = (ax + 3) \ln x,$$

where $x > 0$ and a is a positive constant.

The normal to the curve at the point where the curve crosses the x-axis is parallel to the line

$$5y + x = 2.$$

Find the value of a.

Solution

On the x-axis, $y = 0$. Which means **EITHER** $ax + 3 = 0$ **OR** $\ln x = 0$. Now,

$$ax + 3 = 0 \Rightarrow x = -\frac{3}{a}$$

but $x > 0$. So

$$\ln x = 0 \Rightarrow x = 1.$$

Well,

$$u = ax + 3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = a$$
$$v = \ln x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}.$$

Product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (ax+3)\left(\frac{1}{x}\right) + (a)(\ln x)$$
$$= \frac{ax+3}{x} + a\ln x.$$

Next,

$$5y + x = 2 \Rightarrow 5y = -x + 2$$
$$\Rightarrow y = -\tfrac{1}{5}x + \tfrac{2}{5},$$

which means the the tangent to the curve is $-\frac{1}{5}$. Finally,

$$a = -\frac{1}{-\frac{1}{5}}$$
$$= \underline{\underline{2}}.$$

9. (a) Calculate the term independent of x in the binomial expansion of (3)

$$\left(x - \frac{1}{2x^5}\right)^{18}.$$

Solution
Well,

$$\left(x - \frac{1}{2x^5}\right)^{18} \Rightarrow \left(x - \tfrac{1}{2}x^{-5}\right)^{18}$$

and the general binomial coefficient is

$$\binom{18}{r} x^r (-\tfrac{1}{2}x^{-5})^{18-r}.$$

So the term independent of x is

$$r - 5(18 - r) = 0 \Rightarrow r - 90 + 5r = 0$$
$$\Rightarrow 6r = 90$$
$$\Rightarrow r = 15.$$

Hence, the term independent of x is

$$\binom{18}{15} x^{15} (-\tfrac{1}{2} x^{-5})^3 = \underline{\underline{-102}}.$$

(b) In the binomial expansion of (4)

$$(1 + kx)^n,$$

where $n \geqslant 3$ and k is a constant, the coefficients of x^2 and x^3 are equal.

Express k in terms of n.

Solution

Now,

$$(1 + kx)^n = 1 + knx + \tfrac{1}{2}n(n-1)(kx)^2 + \tfrac{1}{6}n(n-1)(n-2)(kx)^3 + \ldots$$

Next, if the coefficients of x^2 and x^3 are equal, then

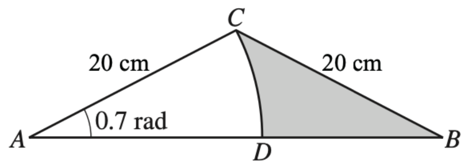
$$\tfrac{1}{2}n(n-1)k^2 = \tfrac{1}{6}n(n-1)(n-2)k^3$$
$$\Rightarrow \tfrac{1}{2}n(n-1)k^2 - \tfrac{1}{6}n(n-1)(n-2)k^3 = 0$$
$$\Rightarrow \tfrac{3}{6}n(n-1)k^2 - \tfrac{1}{6}n(n-1)(n-2)k^3 = 0$$
$$\Rightarrow \tfrac{1}{6}n(n-1)k^2[3 - k(n-2)] = 0.$$

We want to express k in terms of n:

$$3 - k(n-2) = 0 \Rightarrow k(n-2) = 3$$
$$\Rightarrow \underline{\underline{k = \frac{3}{n-2}}}.$$

10. The diagram shows an isosceles triangle ABC in which

- $BC = AC = 20$ cm and
- angle $BAC = 0.7$ radians.



DC is an arc of a circle, centre A.

Find, correct to 1 decimal place,

(a) the area of the shaded region, (4)

Solution

Now, $\angle CBA = 0.7$ radians (isosceles triangle) and $\angle ACB = (\pi - 1.4)$ radians.

Area of $\triangle ABC$:

$$\begin{aligned} \text{area} &= \tfrac{1}{2} \times 20 \times 20 \times \sin(\pi - 1.4) \\ &= 200 \sin(\pi - 1.4). \end{aligned}$$

Area of the shape ACD:

$$\begin{aligned} \text{area} &= \tfrac{1}{2} \times 20 \times 20 \times 0.7 \\ &= 140. \end{aligned}$$

Finally,

$$\begin{aligned} \text{area of the shaded region} &= \text{area of } \triangle ABC - \text{area of the shape} ACD \\ &= 200 \sin(\pi - 1.4) - 140 \\ &= 57.089\,946 \text{ (FCD)} \\ &= \underline{\underline{57.1 \text{ cm}^2 \text{ (1 dp)}}}. \end{aligned}$$

(b) the perimeter of the shaded region. (4)

Solution

Sine rule:

$$\begin{aligned} \frac{AB}{\sin ACB} = \frac{BC}{\sin BAC} &\Rightarrow \frac{AB}{\sin(\pi - 1.4)} = \frac{20}{\sin 0.7} \\ &\Rightarrow AB = \frac{20 \sin(\pi - 1.4)}{\sin 0.7}. \end{aligned}$$

and

$$
\begin{aligned}
BD &= AB - AD \\
&= \frac{20\sin(\pi - 1.4)}{\sin 0.7} - 20.
\end{aligned}
$$

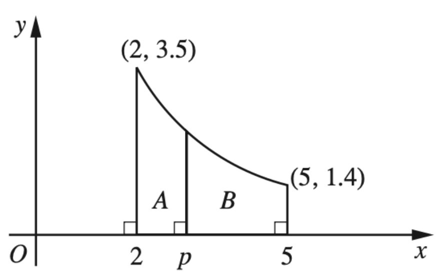
Arc CD:

$$
\begin{aligned}
\text{length} &= 20 \times 0.7 \\
&= 14.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\text{perimeter of the shaded region} &= BC + BD + \text{arc } CD \\
&= 20 + \left(\frac{20\sin(\pi - 1.4)}{\sin 0.7} - 20\right) + 14 \\
&= 44.593\,687\,49 \text{ (FCD)} \\
&= \underline{\underline{44.6 \text{ cm (1 dp)}}}.
\end{aligned}
$$

11. The diagram shows part of a curve, passing through the points $(2, 3.5)$ and $(5, 1.4)$.



The gradient of the curve at any point (x, y) is

$$-\frac{a}{x^3},$$

where a is a positive constant.

(a) how that $a = 20$ and obtain the equation of the curve. (5)

Solution

Well,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{a}{x^3} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -ax^{-3}$$
$$\Rightarrow y = -a(-\tfrac{1}{2}x^{-2}) + c$$
$$\Rightarrow y = \tfrac{1}{2}ax^{-2} + c,$$

where c is a constant. Now,

$$x = 2, y = 3.5 \Rightarrow 3.5 = \tfrac{1}{2}a(2^{-2}) + c$$
$$\Rightarrow 3.5 = \tfrac{1}{8}a + c$$
$$\Rightarrow 3.5 - \tfrac{1}{8}a = c.$$

Next,

$$x = 5, y = 1.4 \Rightarrow 1.4 = \tfrac{1}{2}a(5^{-2}) + 3.5 - \tfrac{1}{8}a$$
$$\Rightarrow -2.1 = \tfrac{1}{50}a - \tfrac{1}{8}a$$
$$\Rightarrow -2.1 = -\tfrac{21}{200}a$$
$$\Rightarrow \underline{\underline{a = 20}},$$

as required.

Well,

$$3.5 - \tfrac{1}{8}(20) = 1$$

and the equation of the curve is

$$y = \tfrac{1}{2}(20)x^{-2} + 1 \Rightarrow \underline{\underline{y = \frac{10}{x^2} + 1}}.$$

The diagram also shows lines perpendicular to the x-axis at $x = 2$, $x = p$, and $x = 5$.

Given that the areas of the regions A and B are equal,

(b) find the value of p. (5)

Solution

Well, as the two areas are the same,

$$\begin{aligned}
& \int_2^p (10x^{-2}+1)\,\mathrm{d}x = \int_p^5 (10x^{-2}+1)\,\mathrm{d}x \\
\Rightarrow\quad & \left[-10x^{-1}+x\right]_{x=2}^{p} = \left[-10x^{-1}+x\right]_{x=2}^{5} \\
\Rightarrow\quad & \left(-\frac{10}{p}+p\right) - (-5+2) = (-2+5) - \left(-\frac{10}{p}+p\right) \\
\Rightarrow\quad & 2\left(-\frac{10}{p}+p\right) + 5 - 2 = -2 + 5 \\
\Rightarrow\quad & -\frac{10}{p} + p = 0 \\
\Rightarrow\quad & p = \frac{10}{p} \\
\Rightarrow\quad & p^2 = 10 \\
\Rightarrow\quad & p = \pm\sqrt{10};
\end{aligned}$$

but $p > 0$ (why?). Hence,

$$\underline{\underline{p = \sqrt{10}}}.$$

EITHER

12. An examination paper contains 12 different questions of which

- 3 are on trigonometry,
- 4 are on algebra, and
- 5 are on calculus.

Candidates are asked to answer 8 questions.

Calculate

(a) (i) the number of different ways in which a candidate can select 8 questions if there is no restriction, (2)

Solution
There are

$$\binom{12}{8} = \underline{\underline{495 \text{ different ways.}}}$$

(ii) the number of these selections which contain questions on only 2 of the 3 topics, trigonometry, algebra, and calculus. (2)

> **Solution**
> He can answer
>
> - 3 are on trigonometry and 4 are on algebra: unfortunately, there are only 7 questions so that is discounted;
> - 3 are on trigonometry and 5 are on calculus: they must answer all of them which is a total of 1 way;
> - 4 are on algebra and 5 are on calculus: they has a spare question which is a total of 9 ways.
>
> So, to answer 8 questions,
>
> $$0 + 1 + 9 = \underline{\underline{10 \text{ different ways.}}}$$

A fashion magazine runs a competition, in which 8 photographs of dresses are shown, lettered A, B, C, D, E, F, G, and H.

Competitors are asked to submit an arrangement of 5 letters showing their choice of dresses in descending order of merit.

The winner is picked at random from those competitors whose arrangement of letters agrees with that chosen by a panel of experts.

(b) (i) Calculate the number of possible arrangements of 5 letters chosen from the 8. (2)

> **Solution**
> We have permutations:
>
> $${}_8P_5 = \underline{\underline{6\,720 \text{ different ways.}}}$$

Calculate the number of these arrangements

(ii) in which A is placed first, (2)

> **Solution**
> We have an one-eighth of (b)(i):
>
> $$\tfrac{1}{8} \times 6\,720 = \underline{\underline{840 \text{ different ways.}}}$$

(iii) which contain A. (2)

Solution
We have an five-eighths of (b)(i):

$$\tfrac{5}{8} \times 6\,720 = \underline{\underline{4\,200 \text{ different ways}}}.$$

OR

13. The table shows experimental values of the variables x and y which are related by the equation

$$y = Ab^x,$$

where A and b are constants.

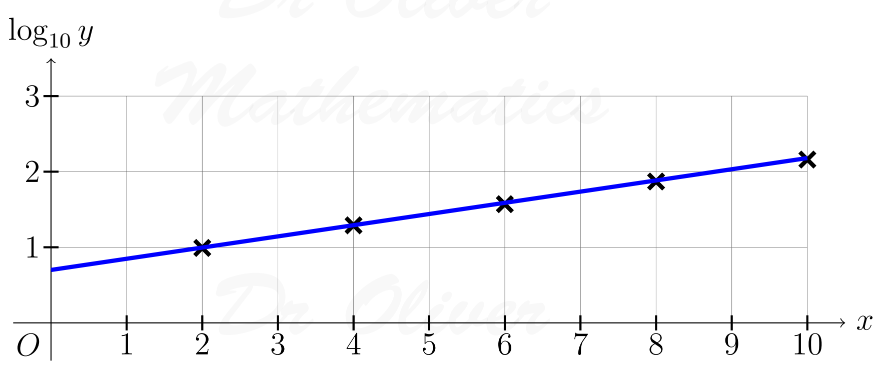
x	2	4	6	8	10
y	9.8	19.4	37.4	74.0	144.4

(a) Use the data above in order to draw, on graph paper, the straight line graph of $\log_{10} y$ against x. (2)

Solution
Use 3 significant figures to get $\log_{10} y$:

x	2	4	6	8	10
y	9.8	19.4	37.4	74.0	144.4
$\log_{10} y$	0.991	1.29	1.57	1.87	2.16

Now, we plot the points and a draw a best of fit:



We have a very good fit!

(b) Use your graph to estimate the value of A and of b. (5)

Solution
Suppose

$$\log_{10} y = mx + c,$$

for some constants m and c. Now,

$$\begin{aligned} m &= \frac{1.57 - 1.29}{6 - 4} \\ &= 0.14 \end{aligned}$$

and

$$\begin{aligned} x = 4, \log_{10} y = 1.29 &\Rightarrow 1.29 = 4 \times 0.14 + c \\ &\Rightarrow 1.29 = 0.56 + c \\ &\Rightarrow c = 0.73. \end{aligned}$$

Putting it together,

$$\begin{aligned} \log_{10} y = 0.14x + 0.73 &\Rightarrow y = 10^{0.14x+0.73} \\ &\Rightarrow y = 10^{0.73}(10^{0.14})^x \\ &\Rightarrow \underline{\underline{y = 5.37 \cdot 1.38^x}}. \end{aligned}$$

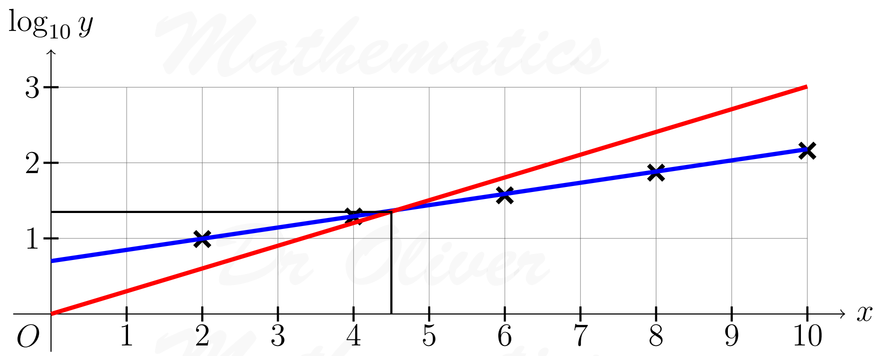
(c) On the same diagram, draw the straight line representing $y = 2^x$ and hence find the value of x for which (3)

$$Ab^x = 2^x.$$

Solution
Well,

$$\begin{aligned} y = 2^x &\Rightarrow \log_{10} y = \log_{10} 2^x \\ &\Rightarrow \log_{10} y = x \log_{10} 2 \end{aligned}$$

and we plot the line:



Hence, $\underline{\underline{x = 4.5}}$.