Dr Oliver Mathematics
Mathematics: Higher
2014 Paper 1: Non-Calculator
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.

Section A

1. A sequence is defined by the recurrence relation (2)

$$u_{n+1} = \tfrac{1}{3}u_n + 1, \text{ with } u_2 = 15.$$

What is the value of u_4?

A. $2\frac{1}{9}$

B. $2\frac{1}{3}$

C. 3

D. 30

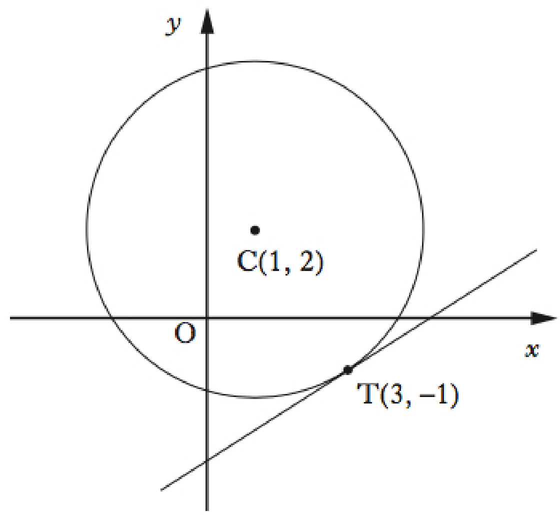
Solution

C

$u_3 = \frac{1}{3}u_2 + 1 = \frac{1}{3} \times 15 + 1 = 6.$

$u_4 = \frac{1}{3}u_3 + 1 = \frac{1}{3} \times 6 + 1 = 3.$

2. The diagram shows a circle with centre $C(1, 2)$ and the tangent at $T(3, -1)$. (2)



What is the gradient of this tangent?

A. $\frac{1}{4}$

B. $\frac{2}{3}$

C. $\frac{3}{2}$

D. 4

Solution

B

$$\text{Gradient of } CT = \frac{2-(-1)}{1-3}$$
$$= -\tfrac{3}{2}$$

and hence the gradient of the tangent is

$$-\frac{1}{-\frac{3}{2}} = \tfrac{2}{3}.$$

3. If (2)

$$\log_4 12 - \log_4 x = \log_4 6,$$

what is the value of x?

A. 2

B. 6

C. 18

D. 72

Solution

A

$$\begin{aligned}\log_4 12 - \log_4 x = \log_4 6 &\Rightarrow \log_4\left(\frac{12}{x}\right) = \log_4 6\\ &\Rightarrow \frac{12}{x} = 6\\ &\Rightarrow x = 2.\end{aligned}$$

4. If (2)

$$3\sin x - 4\cos x$$

is written in the form

$$k\cos(x-a),$$

what are the values of $k\cos a$ and $k\sin a$?

A. $k\cos a = -3$ and $k\sin a = 4$

B. $k\cos a = 3$ and $k\sin a = -4$

C. $k\cos a = 4$ and $k\sin a = -3$

D. $k\cos a = -4$ and $k\sin a = 3$

Solution

D

$$k\cos(x-a) = k\cos x\cos a + k\sin x\sin a.$$

5. Find (2)

$$\int (2x+9)^5\,\mathrm{d}x.$$

A. $10(2x+9)^4 + c$

B. $\frac{1}{4}(2x+9)^4 + c$

C. $10(2x+9)^6 + c$

D. $\frac{1}{12}(2x+9)^6 + c$

Solution

D

$$\int (2x+9)^5 \, \mathrm{d}x = \tfrac{1}{12}(2x+9)^6 + c.$$

6. Given that (2)

$$\mathbf{u} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$

find $2\mathbf{u} - 3\mathbf{v}$ in component form.

A. $\begin{pmatrix} -9 \\ 5 \\ -6 \end{pmatrix}$

B. $\begin{pmatrix} -9 \\ -1 \\ -4 \end{pmatrix}$

C. $\begin{pmatrix} -3 \\ -1 \\ 6 \end{pmatrix}$

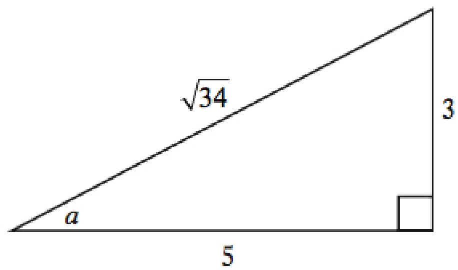
D. $\begin{pmatrix} 11 \\ -5 \\ 4 \end{pmatrix}$

Solution

A

$$\begin{aligned} 2\mathbf{u} - 3\mathbf{v} &= 2\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -9 \\ 5 \\ -6 \end{pmatrix}. \end{aligned}$$

7. A right-angled triangle has sides and angles as shown in the diagram. (2)



What is the value of $\sin 2a$?

A. $\frac{8}{17}$

B. $\frac{3}{\sqrt{34}}$

C. $\frac{15}{17}$

D. $\frac{6}{\sqrt{34}}$

Solution

C

$$\begin{aligned} \sin 2a &= 2 \sin a \cos a \\ &= 2 \times \tfrac{3}{\sqrt{34}} \times \tfrac{5}{\sqrt{34}} \\ &= \tfrac{30}{34} \\ &= \tfrac{15}{17}. \end{aligned}$$

8. What is the derivative of (2)

$$(4 - 9x^4)^{\frac{1}{2}}?$$

A. $-\frac{9}{2}(4 - 9x^4)^{-\frac{1}{2}}$

B. $\frac{1}{2}(4 - 9x^4)^{-\frac{1}{2}}$

C. $2(4 - 9x^4)^{-\frac{1}{2}}$

D. $-18x^3(4 - 9x^4)^{-\frac{1}{2}}$

Solution

D

$$(4 - 9x^4)^{\frac{1}{2}} = -18x^3(4 - 9x^4)^{-\frac{1}{2}}.$$

9. $$\sin x + \sqrt{3}\cos x$$ (2)

cvan be written as

$$2\cos(x - \tfrac{1}{6}\pi).$$

The maximum value of $\sin x + \sqrt{3}\cos x$ is 2.
What is the maximum value of $5\sin 2x + 5\sqrt{3}\cos 2x$?

A. 20

B. 10

C. 5

D. 2

Solution

B

$$5\sin 2x + 5\sqrt{3}\cos 2x = 5(\sin 2x + \sqrt{3}\cos 2x) = 5 \times 2 = 10.$$

10. A sequence is defined by the recurrence relation (2)

$$u_{n+1} = (k-2)u_n + 5, \text{ with } u_0 = 3.$$

For what values of k does this sequence have a limit as $n \to \infty$?

A. $-3 < k < -1$

B. $-1 < k < 1$

C. $1 < k < 3$

D. $k < 3$

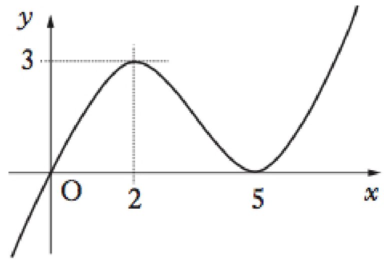
Solution

C

For a limit to exist,

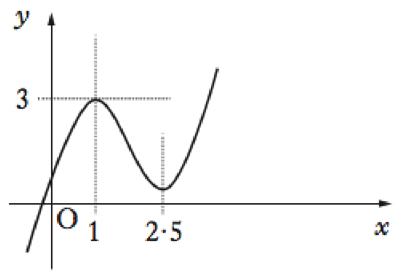
$$-1 < k - 2 < 1 \Rightarrow 1 < k < 3.$$

11. The diagram shows part of the graph of $y = \mathrm{f}(x)$. (2)

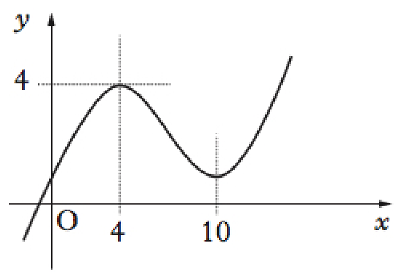


Which of the following diagrams could be the graph of $y = 2\,\mathrm{f}(x) + 1$?

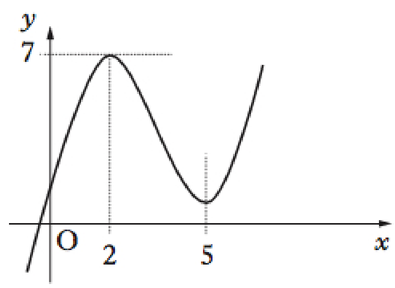
A



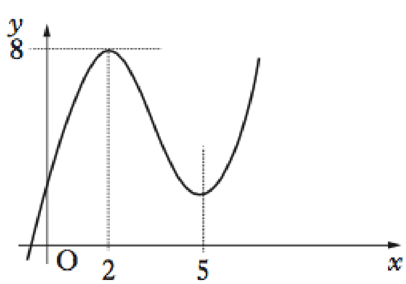
B



C



D



Solution

C

$$2 \times 3 + 1 = 7.$$

12. A function f, defined on a suitable domain, is given by (2)

$$\mathrm{f}(x) = \frac{6x}{x^2 + 6x - 16}.$$

What restrictions are there on the domain of f?

A. $x \neq -8$ or $x \neq 2$

B. $x \neq -4$ or $x \neq 4$

C. $x \neq 0$

D. $x \neq 10$ or $x \neq 16$

Solution

A

$$\frac{6x}{x^2 + 6x - 16} = \frac{6x}{(x+8)(x-2)}$$

and so the points to avoid are $x \neq -8$ and $x \neq 2$.

13. What is the value of (2)

$$\sin \tfrac{1}{3}\pi - \cos \tfrac{5}{4}\pi?$$

A. $\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}$

B. $\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}$

C. $\frac{1}{2} - \frac{1}{\sqrt{2}}$

D. $\frac{1}{2} + \frac{1}{\sqrt{2}}$

Solution

B

$$\begin{aligned} \sin \tfrac{1}{3}\pi - \cos \tfrac{5}{4}\pi &= \sin \tfrac{1}{3}\pi - \cos \tfrac{3}{4}\pi \\ &= \sin \tfrac{1}{3}\pi + \cos \tfrac{1}{4}\pi \\ &= \tfrac{\sqrt{3}}{2} + \tfrac{1}{\sqrt{2}}. \end{aligned}$$

14. The vectors (2)

$$\mathbf{u} = \begin{pmatrix} 1 \\ k \\ k \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix}$$

are perpendicular.

What is the value of k?

A. $-\frac{6}{7}$

B. -1

C. 1

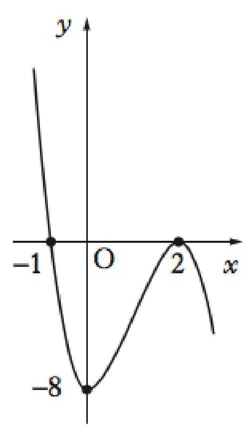
D. $\frac{6}{7}$

Solution

D

$$\begin{aligned} \mathbf{u}.\mathbf{v} = 0 &\Rightarrow -6 + 2k + 5k = 0 \\ &\Rightarrow 7k = 6 \\ &\Rightarrow k = \tfrac{6}{7}. \end{aligned}$$

15. The diagram shows a cubic curve passing through $(-1, 0)$, $(2, 0)$, and $(0, -8)$. (2)



What is the equation of the curve?

A. $y = -2(x+1)^2(x+2)$

B. $y = -2(x+1)(x-2)^2$

C. $y = 4(x+1)^2(x-2)$

D. $y = -8(x+1)(x-2)^2$

Solution

B

The graph goes through $x = -1$, has 'bounce' at $x = 2$, value of x^3 is negative, and

$$x = 0 \Rightarrow y = 4k.$$

16. The unit vectors **a** and **b** are such that (2)

$$\mathbf{a}.\mathbf{b} = \tfrac{2}{3}.$$

Determine the value of

$$\mathbf{a}.(\mathbf{a} + 2\mathbf{b}).$$

A. $\frac{2}{3}$

B. $\frac{4}{3}$

C. $\frac{7}{3}$

D. 3

Solution

C

$$\begin{aligned}\mathbf{a}.(\mathbf{a} + 2\mathbf{b}) &= \mathbf{a}.\mathbf{a} + 2\mathbf{a}.\mathbf{b} \\ &= 1 + 2 \times \tfrac{2}{3} \\ &= \tfrac{7}{3}.\end{aligned}$$

17. (2)

$$3x^2 + 12x + 17$$

is expressed in the form

$$3(x+p)^2 + q.$$

What is the value of q?

A. 1

B. 5
C. 17
D. -19

Solution

B

$$\begin{aligned}3x^2 + 12x + 17 &= 3[x^2 + 4x] + 17 \\ &= 3[(x^2 + 4x + 4) - 4] + 17 \\ &= 3[(x + 2)^2 - 4] + 17 \\ &= 3(x + 2)^2 + 5.\end{aligned}$$

18. What is the value of (2)

$$1 - 2\sin^2 15°?$$

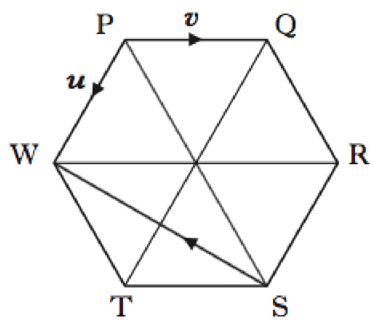
A. $\frac{1}{2}$
B. $\frac{3}{4}$
C. $\frac{\sqrt{3}}{2}$
D. $\frac{7}{8}$

Solution

C

$$\begin{aligned}1 - 2\sin^2 15° &= \cos 30° \\ &= \tfrac{\sqrt{3}}{2}.\end{aligned}$$

19. The diagram shows a regular hexagon $PQRSTW$. (2)
$\overrightarrow{PW}$ and $\overrightarrow{PQ}$ represent vectors $\mathbf{u}$ and $\mathbf{v}$ respectively.



What is $\overrightarrow{SW}$ in terms of $\mathbf{u}$ and $\mathbf{v}$?

A. $-\mathbf{u} - 2\mathbf{v}$

B. $-\mathbf{u} - \mathbf{v}$

C. $\mathbf{u} - \mathbf{v}$

D. $\mathbf{u} + 2\mathbf{v}$

Solution

A

$$\begin{aligned}\overrightarrow{SW} &= 2\overrightarrow{ST} + \overrightarrow{WP} \\ &= -2\mathbf{v} - \mathbf{u}.\end{aligned}$$

20. Evaluate (2)

$$2 - \log_5 \tfrac{1}{25}.$$

A. -3

B. 0

C. $\frac{3}{2}$

D. 4

Solution

D

$$\begin{aligned} 2 - \log_5 \tfrac{1}{25} &= 2 + \log_5 25 \\ &= 2 + \log_5 5^2 \\ &= 2 + 2\log_5 5 \\ &= 2 + 2 \\ &= 4. \end{aligned}$$

Section B

21. A curve has equation $y = 3x^2 - x^3$.

(a) Find the coordinates of the stationary points on this curve and determine their nature. (6)

Solution

$$\begin{aligned} y = 3x^2 - x^3 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6x - 3x^2 \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6 - 6x. \end{aligned}$$

Now,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 = &\Rightarrow 6x - 3x^2 = 0 \\ &\Rightarrow 3x(2 - x) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 2. \end{aligned}$$

Next,

$$x = 0 \Rightarrow y = 0 \text{ and } x = 2 \Rightarrow y = 12 - 8 = 4.$$

Now,

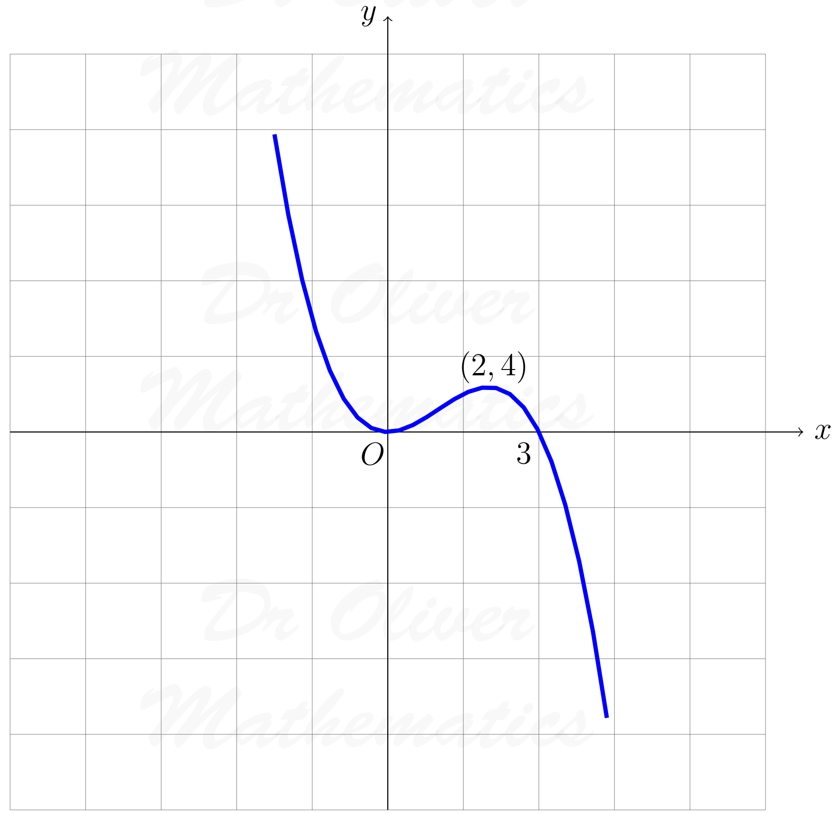
$$x = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6 \text{ and } x = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -6.$$

Finally, $\underline{\underline{(0, 0)}}$ is $\underline{\underline{\text{minimum turning point}}}$ and $\underline{\underline{(2, 4)}}$ is $\underline{\underline{\text{maximum turning point}}}$.

(b) State the coordinates of the points where the curve meets the coordinate axes and sketch the curve. (2)

Solution

$$\begin{aligned} y = 0 &\Rightarrow 3x^2 - x^3 = 0 \\ &\Rightarrow x^2(3 - x) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 3. \end{aligned}$$



22. For the polynomial $6x^3 + 7x^2 + ax + b$,

- $(x + 1)$ is a factor, and
- 72 is the remainder when it is divided by $(x - 2)$.

(a) Determine the values of a and b. (4)

Solution

-1	6	7	a	b
	$\downarrow$	-6	-1	$1 - a$
	6	1	$a - 1$	$b + 1 - a$

So

$$b + 1 - a = 0 \Rightarrow a - b = 1. \quad (1)$$

$$\begin{array}{c|cccc}
2 & 6 & 7 & a & b \\
 & \downarrow & 12 & 38 & 2(a+38) \\
\hline
 & 6 & 19 & a+38 & b+2(a+38)
\end{array}$$

So

$$\begin{aligned} b + 2(a+38) = 72 &\Rightarrow b + 2a + 76 = 72 \\ &\Rightarrow 2a + b = -4. \quad (2) \end{aligned}$$

Finally, (1) + (2):

$$\begin{aligned} 3a = -3 &\Rightarrow \underline{\underline{a = -1}} \\ &\Rightarrow -1 - b = 1 \\ &\Rightarrow \underline{\underline{b = -2}}. \end{aligned}$$

(b) Hence factorise the polynomial completely. (3)

Solution

$$\begin{array}{c|cccc}
-1 & 6 & 7 & -1 & -2 \\
 & \downarrow & -6 & -1 & 2 \\
\hline
 & 6 & 1 & -2 & 0
\end{array}$$

$$\left.\begin{array}{ll} \text{add to:} & +1 \\ \text{multiply to:} & (+6) \times (-2) = -12 \end{array}\right\} + 4, -3$$

Now, e.g.,

$$\begin{aligned} 6x^2 + x - 2 &= 6x^2 + 4x - 3x - 2 \\ &= 2x(3x+2) - (3x+2) \\ &= (2x-1)(3x+2). \end{aligned}$$

Finally,

$$6x^3 + 7x^2 - x - 2 = \underline{\underline{(x+1)(2x-1)(3x+2)}}.$$

23. (a) Find P and Q, the points of intersection of the line (4)

$$y = 3x - 5$$

and the circle C_1 with equation

$$x^2 + y^2 + 2x - 4y - 15 = 0.$$

Solution

$$\begin{aligned} x^2 + y^2 + 2x - 4y - 15 = 0 &\Rightarrow x^2 + (3x-5)^2 + 2x - 4(3x-5) - 15 = 0 \\ &\Rightarrow x^2 + (9x^2 - 30x + 25) + 2x - 12x + 20 - 15 = 0 \\ &\Rightarrow 10x^2 - 40x + 30 = 0 \\ &\Rightarrow x^2 - 4x + 3 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & +3 \end{array}\right\} -1, -3$$

$$\begin{aligned} &\Rightarrow (x-1)(x-3) = 0 \\ &\Rightarrow x = 1 \text{ or } x = 3 \\ &\Rightarrow y = -2 \text{ or } y = 4; \end{aligned}$$

hence, P and Q are $\underline{\underline{(1, -2) \text{ and } (3, 4)}}$ (in some order).

T is the centre of C_1.

(b) Show that PT and QT are perpendicular. (3)

Solution

$$\begin{aligned} x^2 + y^2 + 2x - 4y - 15 = 0 &\Rightarrow x^2 + 2x + 1 + y^2 - 4y + 4 = 15 + 1 + 4 \\ &\Rightarrow (x+1)^2 + (y-2)^2 = 20 \end{aligned}$$

and hence $T(-1, 2)$. Now,

$$\begin{aligned} m_{PT} \times m_{QT} &= \frac{-2-2}{1-(-1)} \times \frac{4-2}{3-(-1)} \\ &= (-2) \times \tfrac{1}{2} \\ &= -1 \end{aligned}$$

and, hence, PT and QT are $\underline{\underline{\text{perpendicular}}}$.

A second circle C_2 passes through P, Q, and T.

(c) Find the equation of C_2. (3)

Solution

Well, $\triangle PQT$ is a right-angled triangle with the right-angle at T (why?). Now, the centre of the circle is at

$$\left(\frac{1+3}{2}, \frac{-2+4}{2}\right) = (2,1)$$

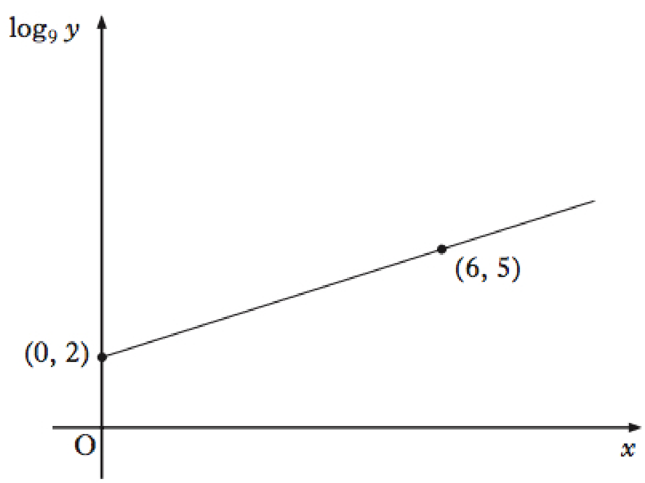
and, hence, the equation of C_2 is

$$(x-2)^2 + (y-1)^2 = [2-(-1)]^2 + (1-2)^2 \Rightarrow \underline{\underline{(x-2)^2 + (y-1)^2 = 10}}.$$

24. Two variables, x and y, are related by the equation (5)

$$y = ka^x.$$

When $\log_9 y$ is plotted against x, a straight line passing through the points $(0,2)$ and $(6,5)$ is obtained, as shown in the diagram.



Find the values of k and a.

Solution

$$\text{Gradient} = \frac{5-2}{6-0} = \tfrac{1}{2}.$$

Now, the equation of the line is

$$\begin{aligned}\log_9 y - 2 = \tfrac{1}{2}x &\Rightarrow y = 9^{\frac{1}{2}x+3}\\ &\Rightarrow y = 9^{\frac{1}{2}x} \times 9^2\\ &\Rightarrow y = (9^{\frac{1}{2}})^x \times 81\\ &\Rightarrow \underline{\underline{y = 81 \cdot 3^x}};\end{aligned}$$

hence, $\underline{\underline{k = 81}}$ and $\underline{\underline{a = 3}}$.