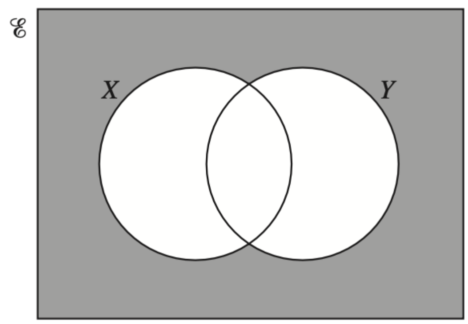
Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2008 June Paper 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. (a) Express, in set notation, the set represented by the shaded region. (1)



Solution
$(X \cup Y)'$.

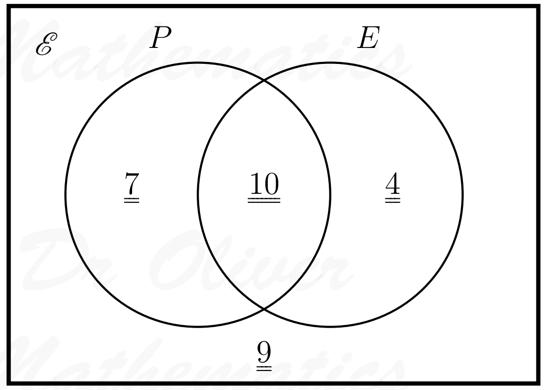
(b) In a class of 30 students, 17 are studying politics, 14 are studying economics, and 10 are studying both of these subjects.

(i) Illustrate this information using a Venn diagram. (1)

Solution

$$\begin{aligned}\text{Neither} &= 30 - (7 + 10 + 4) \\ &= 30 - 21 \\ &= 9.\end{aligned}$$

So,



Find the number of students studying

(ii) neither of these subjects, (1)

Solution

$\underline{\underline{9}}$.

(iii) exactly one of these subjects. (1)

Solution

$$7 + 4 = \underline{\underline{11}}.$$

2. Given that (4)

$$\mathbf{A} = \begin{pmatrix} 7 & 6 \\ 3 & 4 \end{pmatrix},$$

find $\mathbf{A}^{-1}$ and hence solve the simultaneous equations

$$7x + 6y = 17$$
$$3x + 4y = 3.$$

Solution

Well,

$$\det \mathbf{A} = 7 \times 4 - 6 \times 3 = 10$$

and

$$\mathbf{A}^{-1} = \underline{\underline{\tfrac{1}{10}\begin{pmatrix} 4 & -6 \\ -3 & 7 \end{pmatrix}}}.$$

Now,

$$\begin{aligned}
\begin{pmatrix} 7 & 6 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 17 \\ 3 \end{pmatrix} \\
\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} &= \tfrac{1}{10} \begin{pmatrix} 4 & -6 \\ -3 & 7 \end{pmatrix} \begin{pmatrix} 17 \\ 3 \end{pmatrix} \\
\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} &= \tfrac{1}{10} \begin{pmatrix} 50 \\ -30 \end{pmatrix} \\
\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} &= \tfrac{1}{10} \begin{pmatrix} 5 \\ -3 \end{pmatrix};
\end{aligned}$$

hence,

$$\underline{\underline{x = 5, \, y = -3}}.$$

3. Sketch the graph of (4)

$$y = |x^2 - 8x + 12|.$$

Solution

Well,

$$\left.\begin{array}{ll} \text{add to:} & -8 \\ \text{multiply to:} & +12 \end{array}\right\} -6, -2$$

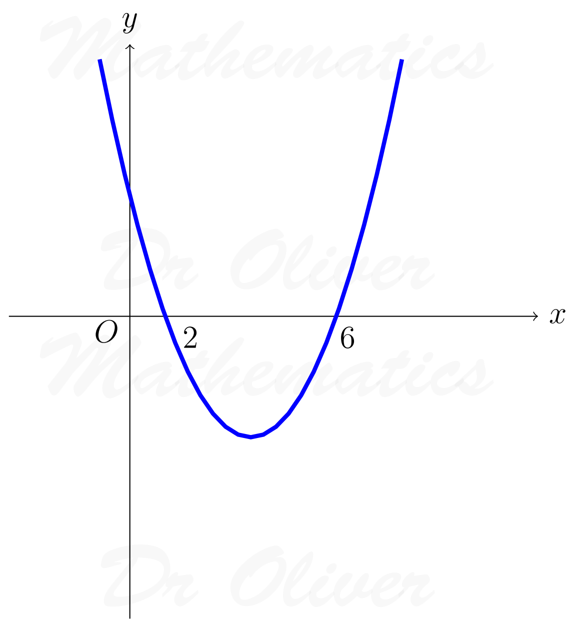
and

$$\begin{aligned}
x^2 - 8x + 12 = 0 &\Rightarrow (x-6)(x-2) = 0 \\
&\Rightarrow x = 6 \text{ or } x = 2.
\end{aligned}$$

So,

$$y = x^2 - 8x + 12$$

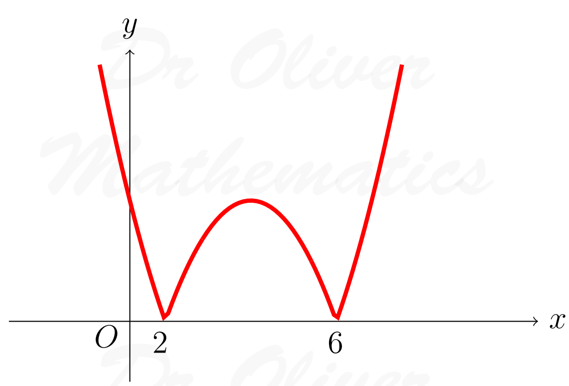
looks like this:



Hence,

$$y = |x^2 - 8x + 12|$$

looks like this:



4. Find the coefficient of x^4 in the expansion of

   (a) $(1 + 2x)^6$, (2)

Solution

Well,

$$(1+2x)^6 = \ldots + \binom{6}{3}(2x)^3 + \binom{6}{4}(2x)^4 + \ldots$$
$$= \ldots + 160x^3 + 240x^4 + \ldots$$

so the coefficient of x^4 is $\underline{\underline{240}}$.

(b) $(1 - \frac{1}{4}x)(1+2x)^6$. (3)

Solution

$\times$	$160x^3$	$+240x^4$
1	$\ldots$	$+240x^4$
$-\frac{1}{4}x$	$-40x^4$	$\ldots$

so the coefficient of x^4 is

$$240 - 40 = \underline{\underline{200}}.$$

5. Two variables, x and y, are related by the equation

$$y = 6x^2 + \frac{32}{x^3}.$$

(a) Obtain an expression for $\frac{dy}{dx}$. (2)

Solution

$$y = 6x^2 + \frac{32}{x^3} \Rightarrow y = 6x^2 + 32x^{-3}$$
$$\Rightarrow \underline{\underline{\frac{dy}{dx} = 12x - 96x^{-4}}}.$$

(b) Use your expression to find the approximate change in the value of y when x increases from 2 to 2.04. (3)

Solution

Well,

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 18$$

and

$$\begin{aligned}\delta y &\approx \frac{\mathrm{d}y}{\mathrm{d}x} \times \delta x \\ &= 18 \times 0.04 \\ &= \underline{\underline{0.72}}.\end{aligned}$$

6. The function f is defined by

$$\mathrm{f}(x) = 2 + \sqrt{x - 3}, \text{ for } x \geqslant 3.$$

Find

(a) the range of f, (1)

Solution
$\underline{\underline{\mathrm{f}(x) \geqslant 2}}$.

(b) an expression for $\mathrm{f}^{-1}(x)$. (2)

Solution
Well,

$$\begin{aligned}y = 2 + \sqrt{x - 3} &\Rightarrow y - 2 = \sqrt{x - 3} \\ &\Rightarrow (y - 2)^2 = x - 3 \\ &\Rightarrow (y - 2)^2 + 3 = x.\end{aligned}$$

Hence,

$$\underline{\underline{\mathrm{f}^{-1}(x) = (x - 2)^2 + 3}}.$$

The function g is defined by

$$\mathrm{g}(x) = \frac{12}{x} + 2, \text{ for } x > 0.$$

Find

(c) $\mathrm{g}\,\mathrm{f}(12)$. (2)

Solution

$$\begin{aligned} \mathrm{g}\,\mathrm{f}(12) &= \mathrm{g}(\mathrm{f}(12)) \\ &= \mathrm{g}(5) \\ &= \underline{\underline{4\tfrac{2}{5}}}. \end{aligned}$$

7. Given that

$$\log_p X = 9 \text{ and } \log_p Y = 6,$$

find

(a) $\log_p \sqrt{X}$, (1)

Solution

$$\begin{aligned} \log_p \sqrt{X} &= \tfrac{1}{2} \log_p X \\ &= \underline{\underline{4\tfrac{1}{2}}}. \end{aligned}$$

(b) $\log_p \left(\dfrac{1}{X}\right)$, (1)

Solution

$$\begin{aligned} \log_p \left(\frac{1}{X}\right) &= -\log_p X \\ &= \underline{\underline{-9}}. \end{aligned}$$

(c) $\log_p(XY)$, (2)

Solution

$$\begin{aligned} \log_p(XY) &= \log_p X + \log_p Y \\ &= \underline{\underline{15}}. \end{aligned}$$

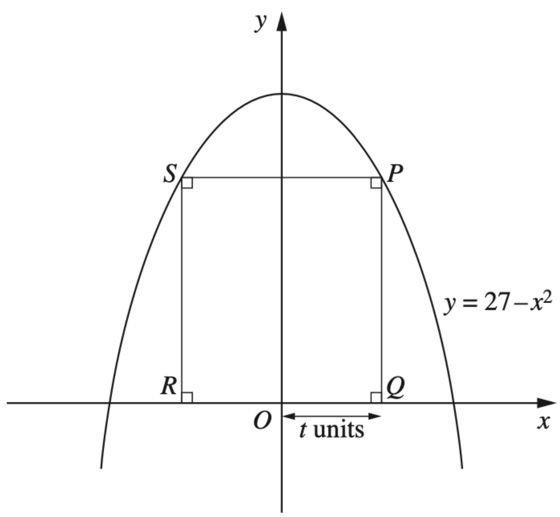
(d) $\log_Y X$ (2)

Solution

$$\log_Y X = \frac{\log_p X}{\log_p Y} = \frac{9}{6} = \underline{\underline{1\tfrac{1}{2}}}.$$

8. The diagram shows part of the curve

$$y = 27 - x^2.$$



- The points P and S lie on this curve.
- The points Q and R lie on the x-axis and $PQRS$ is a rectangle.
- The length of OQ is t units.

(a) Find the length of PQ in terms of t and hence show that the area, A square units, of $PQRS$ is given by (2)

$$A = 54t - 2t^3.$$

Solution

Well,

$$OQ = t \Rightarrow \underline{\underline{PQ = 27 - t^2}}$$

and

$$\begin{aligned}\text{area of } PQRS &= 2 \times t \times (27 - t^2) \\ &= \underline{\underline{54t - 2t^3}},\end{aligned}$$

as required.

(b) Given that t can vary, find the value of t for which A has a stationary value. (3)

Solution

Now,

$$A = 54t - 2t^3 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = 54 - 6t^2$$

and

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} = 0 &\Rightarrow 54 - 6t^2 = 0 \\ &\Rightarrow 6t^2 = 54 \\ &\Rightarrow t^2 = 9 \\ &\Rightarrow t = \pm 3;\end{aligned}$$

given that the length is non-negative, $\underline{\underline{t = 3}}$.

(c) Find this stationary value of A and determine its nature. (3)

Solution

Now,

$$t = 3 \Rightarrow \underline{\underline{A = 108}}.$$

Next,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 54 - 6t^2 \Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}t^2} = -12t$$

and

$$t = 3 \Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}t^2} = -36 < 0.$$

Hence, it is a $\underline{\underline{\text{maximum}}}$.

9. A musician has to play 4 pieces from a list of 9.
Of these 9 pieces,

- 4 were written by Beethoven,

- 3 by Handel, and
- 2 by Sibelius.

Calculate the number of ways the 4 pieces can be chosen if

(a) there are no restrictions, (2)

Solution

$$\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = \underline{\underline{126 \text{ ways.}}}$$

(b) there must be 2 pieces by Beethoven, 1 by Handel, and 1 by Sibelius, (3)

Solution

$$\begin{aligned}\binom{4}{2} \times \binom{3}{1} \times \binom{2}{1} &= 6 \times 3 \times 2 \\ &= \underline{\underline{36 \text{ ways.}}}\end{aligned}$$

(c) there must be at least one piece by each composer. (4)

Solution

$$\begin{aligned}\text{At least one piece} &= \text{2B,H,S} + \text{B,2H,S} + \text{B,H,2S} \\ &= 36 + (4 \times 3 \times 2) + (4 \times 3 \times 1) \\ &= 36 + 24 + 12 \\ &= \underline{\underline{72 \text{ ways.}}}\end{aligned}$$

10. The line (9)

$$2x + y = 12$$

intersects the curve

$$x^2 + 3xy + y^2 = 176$$

at the points A and B.

Find the equation of the perpendicular bisector of AB.

Solution

$\times$	12	$-2x$
12	144	$-24x$
$-2x$	$-24x$	$+4x^2$

and

$$2x + y = 12 \Rightarrow y = 12 - 2x.$$

Now,

$$\begin{aligned} x^2 + 3xy + y^2 = 176 &\Rightarrow x^2 + 3x(12 - 2x) + (12 - 2x)^2 = 176 \\ &\Rightarrow x^2 + 36x - 6x^2 + 144 - 48x + 4x^2 = 176 \\ &\Rightarrow -x^2 - 12x - 32 = 0 \end{aligned}$$

multiply by (-1):

$$\Rightarrow x^2 + 12x + 32 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & +12 \\ \text{multiply to:} & +32 \end{array}\right\} +8,\ +4$$

$$\begin{aligned} &\Rightarrow (x + 8)(x + 4) = 0 \\ &\Rightarrow x = -8 \text{ or } x = -4 \\ &\Rightarrow y = 28 \text{ or } y = 20; \end{aligned}$$

so, $A(-8, 28)$ and $B(-4, 20)$.

The midpoint of AB is

$$\left(\frac{-8 + (-4)}{2}, \frac{28 + 20}{2}\right) = (-6, 24).$$

Next,

$$\begin{aligned} m_{AB} &= \frac{28 - 20}{-8 - (-4)} \\ &= \frac{8}{-4} \\ &= -2 \end{aligned}$$

and

$$m_{\text{normal}} = \tfrac{1}{2}.$$

Finally, the equation of the perpendicular bisector of AB is

$$\begin{aligned} y - 24 = \tfrac{1}{2}(x+6) &\Rightarrow y - 24 = \tfrac{1}{2}x + 3 \\ &\Rightarrow \underline{\underline{y = \tfrac{1}{2}x + 27}}. \end{aligned}$$

11. (a) Find all the angles between 0° and 360° which satisfy

(i) $2\sin x - 3\cos x = 0$, (3)

Solution

$$\begin{aligned} 2\sin x - 3\cos x = 0 &\Rightarrow 2\sin x = 3\cos x \\ &\Rightarrow \tan x = \tfrac{3}{2} \\ &\Rightarrow x = 56.309\,932\,47,\ 236.309\,932\,47 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 56.3^\circ,\ 236^\circ \text{ (3 sf)}}}. \end{aligned}$$

(ii) $2\sin^2 y - 3\cos y = 0$. (5)

Solution

$$\begin{aligned} & 2\sin^2 y - 3\cos y = 0 \\ \Rightarrow\ & 2(1 - \cos^2 y) - 3\cos y = 0 \\ \Rightarrow\ & 2 - 2\cos^2 y - 3\cos y = 0 \\ \Rightarrow\ & 2\cos^2 y + 3\cos y - 2 = 0 \end{aligned}$$

e.g.,

$$\left.\begin{array}{ll} \text{add to:} & +3 \\ \text{multiply to:} & (+2)\times(-2) = -4 \end{array}\right\} -1,\ +4$$

$$\begin{aligned} \Rightarrow\ & 2\cos^2 y + 4\cos y - \cos y - 2 = 0 \\ \Rightarrow\ & 2\cos y(\cos y + 2) - 1(\cos y + 2) = 0 \\ \Rightarrow\ & (2\cos y - 1)(\cos y + 2) = 0 \\ \Rightarrow\ & \cos y = \tfrac{1}{2} \text{ (as } \cos y \neq -2) \\ \Rightarrow\ & \underline{\underline{y = 60^\circ,\ 300^\circ}}. \end{aligned}$$

(b) Given that $0 \leqslant z \leqslant 3$ radians, find, correct to 2 decimal places, all the values of z for which (3)

$$\sin(2z+1) = 0.9.$$

Solution

Well,

$$0 \leqslant z \leqslant 3 \Rightarrow 0 \leqslant 2z \leqslant 6$$

and

$$\begin{aligned}\sin(2z+1) = 0.9 &\Rightarrow 2z+1 = 1.119\,769\,515,\ 2.021\,823\,139\ \text{(FCD)} \\ &\Rightarrow 2z = 0.119\,769\,515,\ 1.021\,823\,139\ \text{(FCD)} \\ &\Rightarrow z = 0.059\,884\,757\,5,\ 0.510\,911\,569\,3\ \text{(FCD)} \\ &\Rightarrow \underline{\underline{z = 0.059\,9,\ 0.511\ \text{(3 sf)}}}.\end{aligned}$$

EITHER

12. The point $P(0, 5)$ lies on the curve for which

$$\frac{dy}{dx} = e^{\frac{1}{2}x}.$$

The point Q, with x-coordinate 2, also lies on the curve.

(a) Find, in terms of e, the y-coordinate of Q. (5)

Solution

Well,

$$\frac{dy}{dx} = e^{\frac{1}{2}x} \Rightarrow y = 2e^{\frac{1}{2}x} + c,$$

for some constant c. Now,

$$\begin{aligned}x = 0, y = 5 &\Rightarrow 5 = 2e^0 + c \\ &\Rightarrow 5 = 2 + c \\ &\Rightarrow c = 3,\end{aligned}$$

so

$$y = 2e^{\frac{1}{2}x} + 3.$$

Finally,

$$x = 2 \Rightarrow \underline{\underline{y_Q = 2e + 3}}.$$

The tangents to the curve at the points P and Q intersect at the point R.

(b) Find, in terms of e, the x-coordinate of R. (5)

Solution

At P,

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

and so the tangent at P is

$$y - 5 = 1(x - 0) \Rightarrow \boxed{y = x + 5 \quad (1).}$$

At Q,

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}$$

and so the tangent at Q is

$$\begin{aligned} y - (2\mathrm{e} + 3) = \mathrm{e}(x - 2) &\Rightarrow y - 2\mathrm{e} - 3 = \mathrm{e}x - 2\mathrm{e} \\ &\Rightarrow \boxed{y = \mathrm{e}x + 3 \quad (2).} \end{aligned}$$

Finally, do (1) = (2):

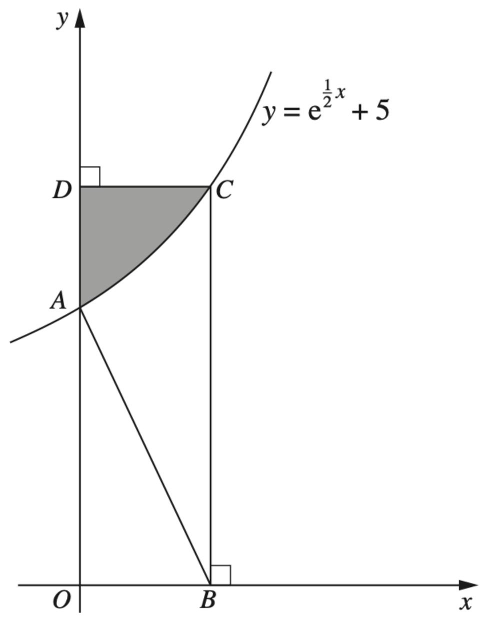
$$\begin{aligned} x_R + 5 = \mathrm{e}x_R + 3 &\Rightarrow 2 = \mathrm{e}x_R - x_R \\ &\Rightarrow 2 = x_R(\mathrm{e} - 1) \\ &\Rightarrow \underline{\underline{x_R = \frac{2}{\mathrm{e} - 1}}}. \end{aligned}$$

OR

13. The diagram shows part of the curve

$$y = \mathrm{e}^{\frac{1}{2}x} + 5$$

crossing the y-axis at A.



The normal to the curve at A meets the x-axis at B.

(a) Find the coordinates of B. (4)

Solution

Well,

$$y = e^{\frac{1}{2}x} + 5 \Rightarrow \frac{dy}{dx} = \tfrac{1}{2}e^{\frac{1}{2}x}.$$

Now, at the point A,

$$x = 0 \Rightarrow \frac{dy}{dx} = \tfrac{1}{2}$$

which means

$$m_{\text{normal}} = -2.$$

Next,

$$x = 0 \Rightarrow y = 6$$

and $A(0, 6)$.

The equation of this normal is

$$y - 5 = -2(x - 0)$$

and

$$\begin{aligned} y = 0 &\Rightarrow -6 = -2x \\ &\Rightarrow x = 3, \end{aligned}$$

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so $\underline{\underline{B(3,0)}}$.

The line through B, parallel to the y-axis, meets the curve at C.

The line through C, parallel to the x-axis, meets the y-axis at D.

(b) Find the area of the shaded region. (6)

Solution

Well,

$$x = 3 \Rightarrow y = \mathrm{e}^{\frac{3}{2}} + 5$$

and $C(3, \mathrm{e}^{\frac{3}{2}} + 5)$. Now,

$$\begin{aligned}
\text{shaded region} &= \text{area of } OBCD - \text{area under the graph} \\
&= [3 \times (\mathrm{e}^{\frac{3}{2}} + 5)] - \int_0^3 (\mathrm{e}^{\frac{1}{2}x} + 5)\,\mathrm{d}x \\
&= 3(\mathrm{e}^{\frac{3}{2}} + 5) - \left[2\mathrm{e}^{\frac{1}{2}x} + 5x\right]_{x=0}^{3} \\
&= 3(\mathrm{e}^{\frac{3}{2}} + 5) - \left\{\left(2\mathrm{e}^{\frac{3}{2}} + 15\right) - (2 + 0)\right\} \\
&= 3\mathrm{e}^{\frac{3}{2}} + 15 - 2\mathrm{e}^{\frac{3}{2}} - 15 + 2 \\
&= \underline{\underline{\mathrm{e}^{\frac{3}{2}} + 2}}.
\end{aligned}$$

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