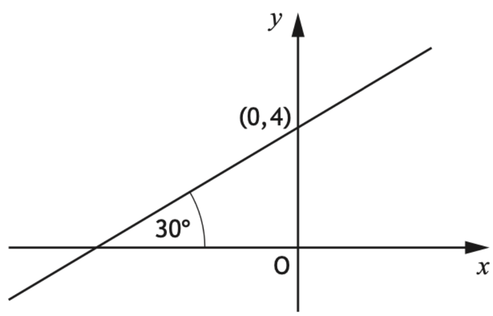
Dr Oliver Mathematics
Mathematics: Higher
2024 Paper 1: Non-Calculator
1 hour 15 minutes

The total number of marks available is 55.
You must write down all the stages in your working.

1. A line passes through the point $(0, 4)$ and makes an angle of $30°$ with the positive direction of the x-axis, as shown in the diagram. (3)



Determine the equation of the line.

Solution

Well,

$$m = \tan 30° = \frac{\sqrt{3}}{3}$$

and the equation of the line is

$$y - 4 = \frac{\sqrt{3}}{3}(x - 0) \Rightarrow \underline{\underline{y = \frac{\sqrt{3}}{3}x + 4}}.$$

2. A sequence is defined by the recurrence relation with

$$u_{n+1} = \frac{1}{5}u_n + 12 \text{ with } u_1 = 20.$$

(a) Calculate the value of u_2. (1)

Solution

Now,

$$\begin{aligned} u_2 &= \tfrac{1}{5}u_1 + 12 \\ &= \tfrac{1}{5}(20) + 12 \\ &= 4 + 12 \\ &= \underline{\underline{16}}. \end{aligned}$$

(b) (i) Explain why this sequence approaches a limit as $n \to \infty$. (1)

Solution

$\underline{\underline{|\tfrac{1}{5}| < 1}}$ so this means this sequence approaches a limit as $n \to \infty$.

(ii) Calculate this limit. (2)

Solution

Let the limit be u. Then

$$\begin{aligned} u = \tfrac{1}{5}u + 12 &\Rightarrow \tfrac{4}{5}u = 12 \\ &\Rightarrow \underline{\underline{u = 15}}. \end{aligned}$$

3. Given that (2)

$$y = (5x^2 + 3)^7,$$

find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 7(5x^2 + 3)^6 \times 10x \\ &= \underline{\underline{70x(5x^2 + 3)^6}}. \end{aligned}$$

4. P and Q have coordinates $(-6, 1, 2)$ and $(-1, 11, -8)$ respectively. (2)

Find the coordinates of the point R which divides PQ in the ratio $2 : 3$.

Solution

Now,

$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OP} + \overrightarrow{PR}\\ &= \overrightarrow{OP} + \tfrac{2}{5}\overrightarrow{PQ}\\ &= \begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix} + \tfrac{2}{5}\begin{pmatrix} -1-(-6) \\ 11-1 \\ -8-2 \end{pmatrix}\\ &= \begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix} + \tfrac{2}{5}\begin{pmatrix} 5 \\ 10 \\ -10 \end{pmatrix}\\ &= \begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix}\\ &= \begin{pmatrix} -4 \\ 5 \\ -2 \end{pmatrix};\end{aligned}$$

hence, the coordinates of the point R are $\underline{\underline{(-4, 5, -2)}}$.

5. A function, h, is defined by where (3)

$$\mathrm{h}(x) = 2x^3 - 7, \text{ where } x \in \mathbb{R}.$$

Find the inverse function, $\mathrm{h}^{-1}(x)$.

Solution

Well,

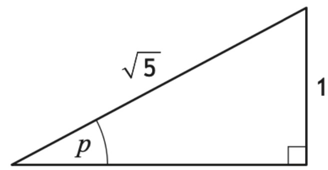
$$\begin{aligned}y = 2x^3 - 7 \Rightarrow y + 7 &= 2x^3\\ \Rightarrow \frac{y+7}{2} &= x^3\\ \Rightarrow \sqrt[3]{\frac{y+7}{2}} &= x;\end{aligned}$$

hence,

$$\underline{\underline{\mathrm{h}^{-1}(x) = \sqrt[3]{\frac{x+7}{2}},\ x \in \mathbb{R}.}}$$

6. The right-angled triangle in the diagram is such that and

$$\sin p = \tfrac{1}{\sqrt{5}} \text{ and } 0 < p < \tfrac{1}{4}\pi.$$



(a) Determine the value of:

(i) $\sin 2p$, (3)

Solution
Well,

$$\begin{aligned} \text{opp}^2 + \text{adj}^2 = \text{hyp}^2 &\Rightarrow 1 + \text{adj}^2 = (\sqrt{5})^2 \\ &\Rightarrow 1 + \text{adj}^2 = 5 \\ &\Rightarrow \text{adj}^2 = 4 \\ &\Rightarrow \text{adj} = 2, \end{aligned}$$

so $\cos p = \frac{2}{\sqrt{5}}$.

Finally,

$$\begin{aligned} \sin 2p &= 2 \sin p \cos p \\ &= 2(\tfrac{1}{\sqrt{5}})(\tfrac{2}{\sqrt{5}}) \\ &= \underline{\underline{\tfrac{4}{5}}}. \end{aligned}$$

(ii) $\cos 2p$. (1)

Solution

$$\begin{aligned} \cos 2p &= 1 - 2\sin^2 p \\ &= 1 - 2(\tfrac{1}{\sqrt{5}})^2 \\ &= 1 - 2(\tfrac{1}{5}) \\ &= \underline{\underline{\tfrac{3}{5}}}. \end{aligned}$$

(b) Hence determine the value of $\sin 4p$. (1)

Solution

$$\begin{aligned}\sin 4p &= 2\sin 2p\cos 2p \\ &= 2(\tfrac{4}{5})(\tfrac{3}{5}) \\ &= \underline{\underline{\tfrac{24}{25}}}.\end{aligned}$$

7. The line (4)

$$y = 2x$$

is a tangent to the circle with equation

$$x^2 + y^2 - 14x - 8y + 45 = 0.$$

Determine the coordinates of the point of contact.

Solution

$$\begin{aligned}x^2 + y^2 - 14x - 8y + 45 = 0 &\Rightarrow x^2 + (2x)^2 - 14x - 8(2x) + 45 = 0 \\ &\Rightarrow x^2 + 4x^2 - 14x - 16x + 45 = 0 \\ &\Rightarrow 5x^2 - 30x + 45 = 0 \\ &\Rightarrow 5(x^2 - 6x + 9) = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -6 \\ \text{multiply to:} & +9\end{array}\right\} -3 \text{ (repeated)}$$

$$\begin{aligned}&\Rightarrow 5(x-3)^2 = 0 \\ &\Rightarrow x = 3 \\ &\Rightarrow y = 6;\end{aligned}$$

the coordinates of the point of contact is $\underline{\underline{(3, 6)}}$.

8. The equation (4)

$$x^2 + (m-4)x + (2m-3) = 0$$

has no real roots.

Determine the range of values for m.
Justify your answer.

Solution

Well,

$$b^2 - 4ac < 0 \Rightarrow (m-4)^2 - 4(1)(2m-3) < 0$$

$\times$	m	-4
m	m^2	$-4m$
-4	$-4m$	$+16$

$$\Rightarrow m^2 - 8m + 16 - 8m + 12 < 0$$
$$\Rightarrow m^2 - 16m + 28 < 0$$

$$\left.\begin{array}{ll}\text{add to:} & -16 \\ \text{multiply to:} & +28\end{array}\right\} -14, -2$$

$$\Rightarrow (m-2)(m-14) < 0.$$

We need a 'table of signs':

	$m < 2$	$m = 2$	$2 < m < 14$	$m = 14$	$m > 14$
$m - 2$	$-$	0	$+$	$+$	$+$
$m - 14$	$-$	$-$	$-$	0	$+$
$(m-2)(m-14)$	$+$	0	$-$	0	$+$

Hence,

$$\underline{\underline{2 < m < 14}}.$$

9. Express (3)

$$\log_a 5 + \log_a 80 - 2\log_a 10$$

in the form

$$\log_a k,$$

where k is a positive integer.

Solution

$$\begin{aligned}\log_a 5 + \log_a 80 - 2\log_a 10 &= \log_a(5 \times 80) - \log_a 10^2 \\ &= \log_a 400 - \log_a 100 \\ &= \log_a\left(\frac{400}{100}\right) \\ &= \underline{\underline{\log_a 4}}.\end{aligned}$$

10. (a) Show that $(x-1)$ is a factor of (2)

$$2x^4 + 3x^3 - 4x^2 - 3x + 2.$$

Solution

We use synthetic division:

1	2	3	−4	−3	2
	↓	2	5	1	−2
	2	5	1	−2	0

Hence, the remainder is 0 and $(x-1)$ is a $\underline{\underline{\text{factor}}}$:

$$2x^4 + 3x^3 - 4x^2 - 3x + 2 = \underline{\underline{(x-1)(2x^3 + 5x^2 + x - 2)}}.$$

(b) Hence, or otherwise, factorise (4)

$$2x^4 + 3x^3 - 4x^2 - 3x + 2$$

fully.

Solution

Let

$$\mathrm{f}(x) = 2x^3 + 5x^2 + x - 2.$$

Then

$$f(-1) = -2 + 5 - 1 - 2 = 0,$$

and $(x+1)$ is a factor:

$$\begin{array}{r|rrrr} -1 & 2 & 5 & 1 & -2 \\ & \downarrow & -1 & -3 & 2 \\ \hline & 2 & 3 & -2 & 0 \end{array}$$

So,

$$f(x) = (x+1)(2x^2 + 3x - 2).$$

Now,

$$\left.\begin{array}{ll} \text{add to:} & +3 \\ \text{multiply to:} & (+2) \times (-2) = -4 \end{array}\right\} -1,\ +4$$

Next, e.g.,

$$\begin{aligned} 2x^2 + 3x - 2 &= 2x^2 + 4x - x - 2 \\ &= 2x(x+2) - 1(x+2) \\ &= (2x-1)(x+2). \end{aligned}$$

Finally

$$2x^4 + 3x^3 - 4x^2 - 3x + 2 = \underline{\underline{(x-1)(x+1)(2x-1)(x+2)}}.$$

11. (a) Express (4)

$$\cos x^\circ + \sqrt{3}\sin x^\circ$$

in the form

$$k\cos(x-\alpha)^\circ,$$

where $k > 0$ and $0 < \alpha < 360$.

Solution

Well,

$$\begin{aligned} k\cos(x-\alpha)^\circ &= k[\cos x^\circ \cos\alpha^\circ + \sin x^\circ \sin\alpha^\circ] \\ &= k\cos x^\circ \cos\alpha^\circ + k\sin x^\circ \sin\alpha^\circ \end{aligned}$$

and so

$$k \cos \alpha^\circ = 1 \text{ and } k \sin \alpha^\circ = \sqrt{3}.$$

Now,

$$\begin{aligned} k &= \sqrt{k^2} \\ &= \sqrt{k^2(\cos^2 \alpha^\circ + \sin^2 \alpha^\circ)} \\ &= \sqrt{k^2 \cos^2 \alpha^\circ + k^2 \sin^2 \alpha^\circ} \\ &= \sqrt{(k \cos \alpha^\circ)^2 + (k \sin \alpha^\circ)^2} \\ &= \sqrt{1^2 + (\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

and

$$\tan \alpha^\circ = \frac{k \sin \alpha^\circ}{k \cos \alpha^\circ} \Rightarrow \tan \alpha^\circ = \frac{\sqrt{3}}{1}$$
$$\Rightarrow \alpha = 60.$$

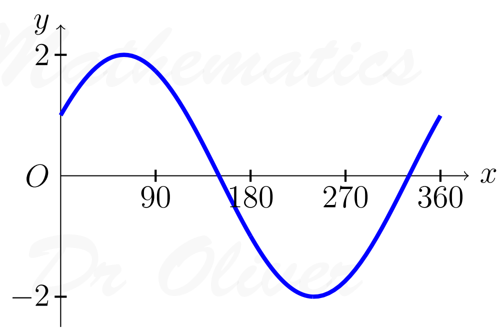
Hence,

$$\cos x^\circ + \sqrt{3} \sin x^\circ = \underline{\underline{2 \cos(x - 60)^\circ}}.$$

(b) Hence, or otherwise, sketch the graph with equation (3)

$$y = \cos x^\circ + \sqrt{3} \sin x^\circ,\ 0 \leqslant x \leqslant 360.$$

Solution



12. The function f is given by (4)

$$\mathrm{f}(x) = 12\sqrt[3]{x},\ x > 0.$$

When $x = a$, the rate of change of f with respect to x is 1.

Determine the value of a.

Solution

$$\begin{aligned}\mathrm{f}(x) = 12\sqrt[3]{x} \Rightarrow \mathrm{f}(x) &= 12x^{\frac{1}{3}} \\ \Rightarrow \mathrm{f}'(x) &= 4x^{-\frac{2}{3}}\end{aligned}$$

and

$$\begin{aligned}x = a,\ \mathrm{f}'(a) = 1 &\Rightarrow 4a^{-\frac{2}{3}} = 1 \\ &\Rightarrow a^{-\frac{2}{3}} = \tfrac{1}{4} \\ &\Rightarrow a^{\frac{2}{3}} = 4 \\ &\Rightarrow \sqrt[3]{a^2} = 4 \\ &\Rightarrow a^2 = 64 \\ &\Rightarrow \underline{\underline{a = 8}}.\end{aligned}$$

13. P and Q are the points $(4, 10)$ and $(6, 2)$ respectively.

(a) Find the equation of the perpendicular bisector of PQ. (4)

Solution

The midpoint is

$$\left(\frac{4+6}{2}, \frac{10+2}{2}\right) = (5, 6).$$

Now,

$$\begin{aligned}m_{PQ} &= \frac{2-10}{6-4} \\ &= \frac{-8}{2} \\ &= -4\end{aligned}$$

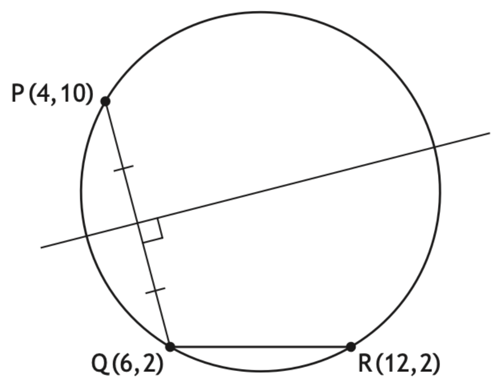
and

$$m_{\text{normal}} = \tfrac{1}{4}.$$

Finally, the equation of the perpendicular bisector of PQ is

$$\begin{aligned} y - 6 = \tfrac{1}{4}(x-5) &\Rightarrow y - 6 = \tfrac{1}{4}x - \tfrac{5}{4} \\ &\Rightarrow \underline{\underline{y = \tfrac{1}{4}x + \tfrac{19}{4}}}. \end{aligned}$$

The point R has coordinates $(12, 2)$.



- A circle passes through the points P, Q, and R.
- The chord QR is horizontal.

(b) Find the equation of the circle. (4)

Solution

Well, the midpoint of QR is $(9, 2)$.

Now, the y-coordinate of the circle centre is

$$\begin{aligned} y &= \tfrac{1}{4}(9) + \tfrac{19}{4} \\ &= 7; \end{aligned}$$

so, the circle centre is $(9, 7)$.

Finally, the equation of the circle is

$$\begin{aligned} (x-9)^2 + (y-7)^2 &= (4-9)^2 + (10-7)^2 \\ &= 25 + 9 \\ &= 34; \end{aligned}$$

hence,

$$\underline{\underline{(x-9)^2 + (y-7)^2 = 34}}.$$