Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2013 June Paper 2 Variant 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Prove that (4)
$$\left(\frac{1+\sin\theta}{\cos\theta}\right)^2+\left(\frac{1-\sin\theta}{\cos\theta}\right)^2\equiv 2+4\tan^2\theta.$$

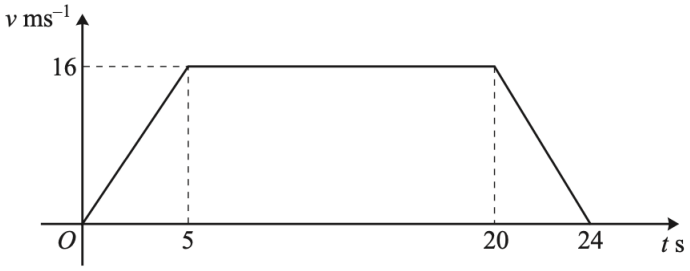
Solution

Well,

$\times$	1	$\pm\sin\theta$
1	1	$\pm\sin\theta$
$\pm\sin\theta$	$\pm\sin\theta$	$+\sin^2\theta$

so
$$\begin{aligned}\left(\frac{1+\sin\theta}{\cos\theta}\right)^2+\left(\frac{1-\sin\theta}{\cos\theta}\right)^2 &\equiv \frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta}+\frac{1-2\sin\theta+\sin^2\theta}{\cos^2\theta}\\ &\equiv \frac{2+2\sin^2\theta}{\cos^2\theta}\\ &\equiv 2\sec^2\theta+2\tan^2\theta\\ &\equiv 2(1+\tan^2\theta)+2\tan^2\theta\\ &\equiv \underline{\underline{2+4\tan^2\theta}},\end{aligned}$$
as required.

2. The velocity-time graph represents the motion of a particle moving in a straight line.



(a) Find the acceleration during the first 5 seconds. (1)

Solution

Well,

$$\begin{aligned}\text{acceleration} &= \frac{16-0}{5-0}\\ &= \underline{\underline{3.2\ \text{ms}^{-2}}}.\end{aligned}$$

(b) Find the length of time for which the particle is travelling with constant velocity. (1)

Solution

$$20-5 = \underline{\underline{15\ \text{seconds}}}.$$

(c) Find the total distance travelled by the particle. (3)

Solution

$$\begin{aligned}\text{Total distance} &= \text{area under the graph}\\ &= \tfrac{1}{2}\times 16\times(15+24)\\ &= \underline{\underline{312\ \text{m}}}.\end{aligned}$$

3. Variables x and y are related by the equation (6)

$$y = 10 - 4\sin^2 x, \text{ where } 0 \leqslant x \leqslant \tfrac{1}{2}\pi.$$

Given that x is increasing at a rate of 0.2 radians per second, find the corresponding rate of change of y when $y = 8$.

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Solution

Well,

$$y = 10 - 4\sin^2 x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -8\sin x \cos x$$

and

$$\begin{aligned} y = 8 &\Rightarrow 10 - 4\sin^2 x = 8 \\ &\Rightarrow -4\sin^2 x = -2 \\ &\Rightarrow \sin^2 x = \tfrac{1}{2} \\ &\Rightarrow \sin x = \tfrac{1}{\sqrt{2}} \\ &\Rightarrow x = \tfrac{1}{4}\pi \end{aligned}$$

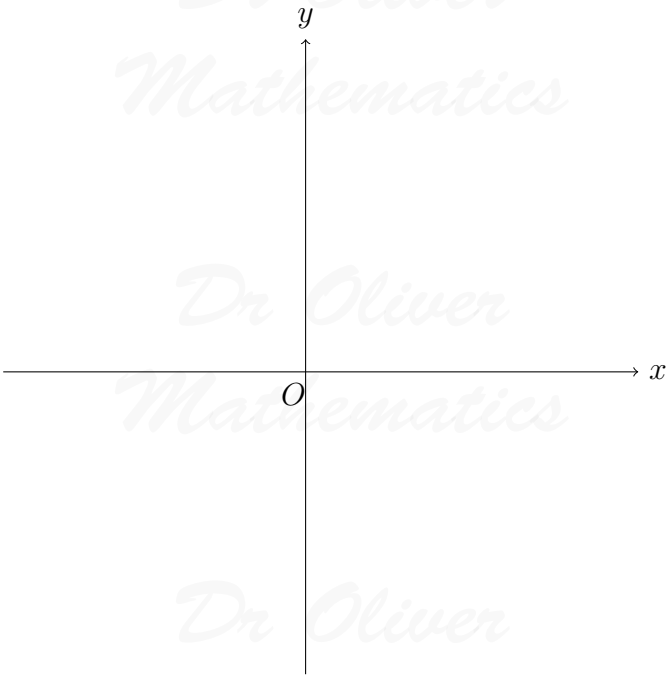
Now,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -8\sin x \cos x\big|_{x=\frac{1}{4}\pi} \times 0.2 \\ &= \underline{\underline{-0.8}}. \end{aligned}$$

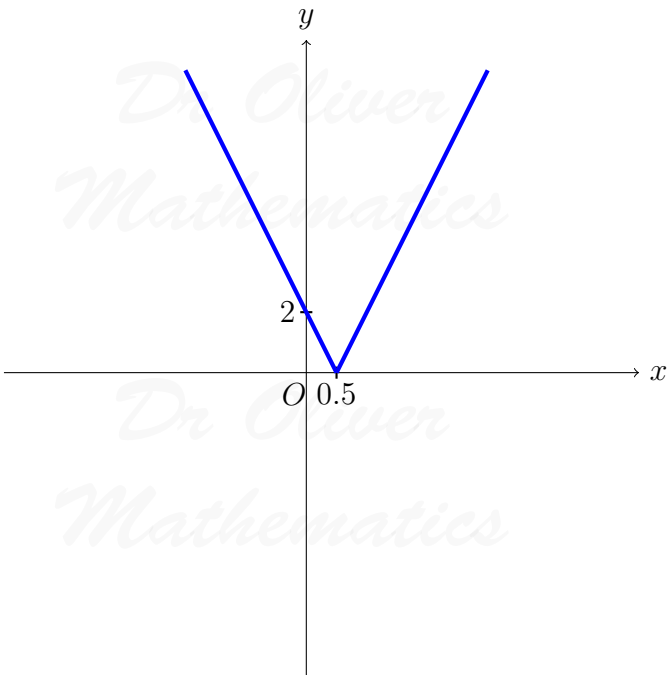
4. (a) Sketch the graph of (3)

$$y = |4x - 2|$$

on the axes below, showing the coordinates of the points where the graph meets the axes.



Solution



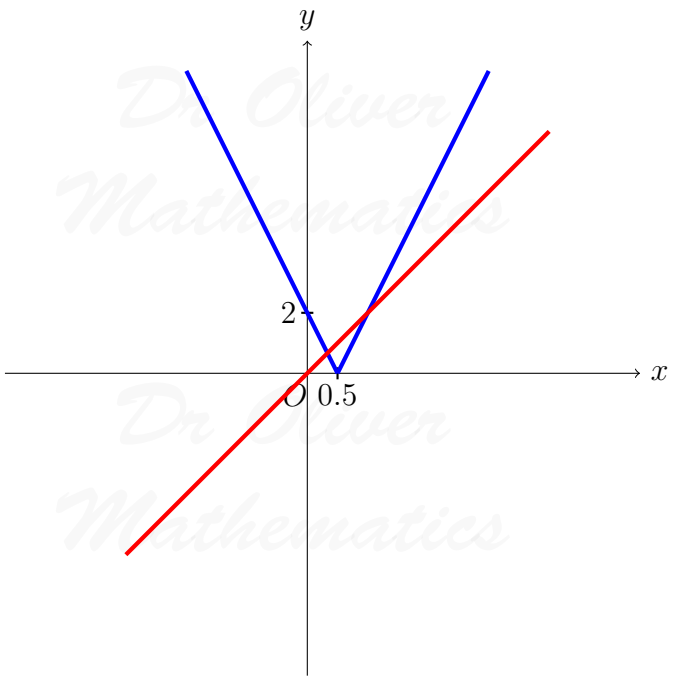
(b) Solve the equation (3)

$$|4x - 2| = x.$$

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Solution

Now,



$\underline{4x - 2 = x}$:

$$4x - 2 = x \Rightarrow 3x = 2$$
$$\Rightarrow x = \tfrac{2}{3}.$$

$\underline{-(4x - 2) = x}$:

$$-(4x - 2) = x \Rightarrow -4x + 2 = x$$
$$\Rightarrow -5x = -2$$
$$\Rightarrow x = \tfrac{2}{5}.$$

Hence,

$$|4x - 2| = x \Rightarrow \underline{\underline{x = \tfrac{2}{5} \text{ or } x = \tfrac{2}{3}}}.$$

5. A piece of wire of length 96 cm is formed into the rectangular shape $PQRSTU$ shown in the diagram.



It is given that $PQ = TU = SR = x$ cm.
It may be assumed that PQ and TU coincide and that TS and QR have the same length.

(a) Show that the area, A cm^2, enclosed by the wire is given by (2)

$$A = \frac{96 - 3x^2}{2}.$$

Solution

Well, let the width of y cm. Then

$$\begin{aligned} 2x + y + x + y = 96 &\Rightarrow 2y = 96 - 3x \\ &\Rightarrow y = \frac{96 - 3x}{2} \end{aligned}$$

and

$$\begin{aligned} A &= xy \\ &= x\left(\frac{96 - 3x}{2}\right) \\ &= \frac{96x - 3x^2}{2}, \end{aligned}$$

as required.

(b) Given that x can vary, find the stationary value of A and determine the nature of this stationary value. (4)

Solution

Well,

$$A = \frac{96x - 3x^2}{2} \Rightarrow A = 48x - \tfrac{3}{2}x^2$$
$$\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}x} = 48 - 3x$$
$$\Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -3.$$

Now,

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \Rightarrow 48 - 3x = 0$$
$$\Rightarrow 3x = 48$$
$$\Rightarrow \underline{\underline{x = 16 \text{ cm}}}$$
$$\Rightarrow A = 48(16) - \tfrac{3}{2}(16^2)$$
$$\Rightarrow \underline{\underline{A = 384 \text{ cm}^2}}.$$

Next,

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -3 < 0$$

and so this stationary value is a $\underline{\underline{\text{maximum}}}$.

6. Find the equation of the normal to the curve (6)

$$y = \frac{x^2 + 8}{x - 2}$$

at the point on the curve where $x = 4$.

Solution

Quotient rule:

$$u = x^2 + 8 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$v = x - 2 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-2)(2x)-(1)(x^2+8)}{(x-2)^2}$$
$$= \frac{2x^2-4x-x^2-8}{(x-2)^2}$$
$$= \frac{x^2-4x-8}{(x-2)^2}.$$

Now,

$$x = 4 \Rightarrow y = \frac{4^2+8}{4-2} = 12.$$

Next,

$$x = 4 \Rightarrow m_{\text{tangent}} = -2$$
$$\Rightarrow m_{\text{normal}} = \tfrac{1}{2}.$$

Hence, the equation of the normal is

$$y - 12 = \tfrac{1}{2}(x-4) \Rightarrow y - 12 = \tfrac{1}{2}x - 2$$
$$\Rightarrow \underline{\underline{y = \tfrac{1}{2}x + 10}}.$$

7. (a) Find the first four terms in the expansion of (3)

$$(2+x)^6$$

in ascending powers of x.

Solution

Now,

$$(2+x)^6 = (2)^6 + \binom{6}{1}(2)^5(x) + \binom{6}{2}(2)^4(x)^2 + \binom{6}{3}(2)^3(x)^3 + \ldots$$
$$= \underline{\underline{64 + 192x + 240x^2 + 160x^3 + \ldots}}.$$

(b) Hence find the coefficient of x^3 in the expansion of (4)

$$(1+3x)(1-x)(2+x)^6.$$

Solution

Well,

$\times$	1	$+3x$
1	1	$+3x$
$-x$	$-x$	$-3x^2$

and so

$$(1+3x)(1-x)(2+x)^6 = (1+2x-3x^2)(64+192x+240x^2+160x^3+\ldots)$$

$\times$	$+192x$	$+240x^2$	$+160x^3$
1	$\ldots$	$\ldots$	$+160x^3$
$+2x$	$\ldots$	$+480x^3$	$\ldots$
$-3x^2$	$-576x^3$	$\ldots$	$\ldots$

$$= \ldots + 64x^3 + \ldots;$$

the coefficient of x^3 is $\underline{\underline{64}}$.

8. The line (7)

$$y = 2x - 8$$

cuts the curve

$$2x^2 + y^2 - 5xy + 32 = 0$$

at the points A and B.

Find the length of the line AB.

Solution

Well,

$\times$	$2y$	-8
$2y$	$4y^2$	$-16y$
-8	$-16y$	$+64$

and so

$$\begin{aligned}
2x^2 + y^2 - 5xy + 32 = 0 &\Rightarrow 2x^2 + (2x-8)^2 - 5x(2x-8) + 32 = 0 \\
&\Rightarrow 2x^2 + (4x^2 - 32x + 64) - 10x^2 + 40x + 32 = 0 \\
&\Rightarrow -4x^2 + 8x + 96 = 0 \\
&\Rightarrow -4(x^2 - 2x - 24) = 0
\end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -2 \\ \text{multiply to:} & -24 \end{array}\right\} -6,\ +4$$

$$\begin{aligned}
&\Rightarrow -4(x-6)(x+4) = 0 \\
&\Rightarrow x - 6 = 0 \text{ or } x + 4 = 0 \\
&\Rightarrow x = 6 \text{ or } x = -4 \\
&\Rightarrow y = 4 \text{ or } y = -16;
\end{aligned}$$

so, the points A and B are $(6, 4)$ and $(-4, -16)$.

Finally,

$$\begin{aligned}
AB &= \sqrt{[6-(-4)]^2 + [4-(-16)]^2} \\
&= \sqrt{10^2 + 20^2} \\
&= \sqrt{500} \\
&= \underline{\underline{10\sqrt{5}}}.
\end{aligned}$$

9. It is given that $x \in \mathbb{R}$ and that

   - $\mathscr{E} = \{x : -5 < x < 12\}$,
   - $S = \{x : 5x + 24 > x^2\}$, and
   - $T = \{x : 2x + 7 > 15\}$.

   Find the values of x such that

   (a) $x \in S$, (3)

Solution

What is S? Well,

$$5x + 24 > x^2 \Rightarrow 0 > x^2 - 5x - 24$$

add to: -5
multiply to: -24
$\Big\}$ $-8, +3$

$$\Rightarrow 0 > (x-8)(x+3).$$

We need a 'table of signs':

	$x < -3$	$x = -3$	$-3 < x < 8$	$x = 8$	$x > 8$
$(x+3)$	$-$	0	$+$	$+$	$+$
$(x-8)$	$-$	$-$	$-$	0	$+$
$(x+3)(x-8)$	$+$	0	$-$	0	$+$

So

$$S = \underline{\underline{\{x : -3 < x < 8\}}}.$$

(b) $x \in S \cup T$, (2)

Solution

What is T? Well,

$$2x + 7 > 15 \Rightarrow 2x > 8$$
$$\Rightarrow x > 4.$$

So

$$S \cup T = \underline{\underline{\{x : -3 < x < 12\}}}.$$

(c) $x \in (S \cap T)'$. (3)

Solution

Now,

$$S \cap T = \{x : 4 < x < 8\}.$$

Next,

$$S' = \{x : -5 < x \leqslant -3 \text{ and } 8 \leqslant x < 12\} \text{ and } T' = \{x : -5 < x \leqslant 4\}.$$

So,

$$(S \cap T)' = \{x : -5 < x \leqslant 4\}.$$

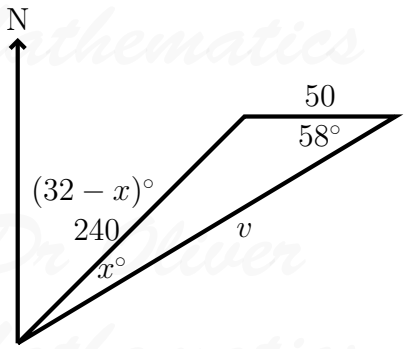
10. A plane, whose speed in still air is 240 kmh^{-1}, flies directly from A to B, where B is 500 km from A on a bearing of 032°.

There is a constant wind of 50 kmh^{-1} blowing from the west.

(a) Find the bearing on which the plane is steered. (4)

Solution

Let v kmh^{-1} be the actual speed of the aircraft.



Sine rule:

$$\frac{\sin x^\circ}{50} = \frac{\sin 58^\circ}{240} \Rightarrow \sin x^\circ = \frac{50 \sin 58^\circ}{240}$$
$$\Rightarrow x = 10.176\,245\,7 \text{ (FCD)}$$
$$\Rightarrow 32 - x = 21.823\,754\,3 \text{ (FCD)};$$

hence, the bearing on which the plane is steered is $\underline{\underline{022^\circ \text{ (FCD)}}}$.

(b) Find, to the nearest minute, the time taken for the flight. (4)

Solution

We need the third angle:

$$180 - 58 - x = 111.823\,754\,3 \text{ (FCD)}.$$

Cosine rule:

$$v^2 = 240^2 + 50^2 - 2 \times 240 \times 50 \times \cos 111.823\ldots^\circ$$
$$\Rightarrow \quad v^2 = 69\,022\,065\,89 \text{ (FCD)}$$
$$\Rightarrow \quad v = 262.720\,509\,1 \text{ (FCD)}.$$

Finally,

$$\begin{aligned}\text{time taken} &= \frac{500}{262.720\ldots} \\ &= 1.903\,163\,182 \text{ (FCD)} \\ &= \underline{\underline{1 \text{ hour } 54 \text{ minutes (nearest minute)}}}.\end{aligned}$$

11. A one-one function f is defined by

$$\mathrm{f}(x) = (x-1)^2 - 5 \text{ for } x \geqslant k.$$

(a) State the least value that k can take. (1)

Solution

$\underline{\underline{k = 1}}$.

For this least value of k,

(b) write down the range of f, (1)

Solution

Well,

$$\begin{aligned}x = 1 &\Rightarrow \mathrm{f}(x) = (1-1)^2 - 5 \\ &\Rightarrow \mathrm{f}(x) = -5\end{aligned}$$

so $\underline{\underline{\mathrm{f}(x) \geqslant -5}}$.

(c) find $\mathrm{f}^{-1}(x)$, (2)

Solution

Now,

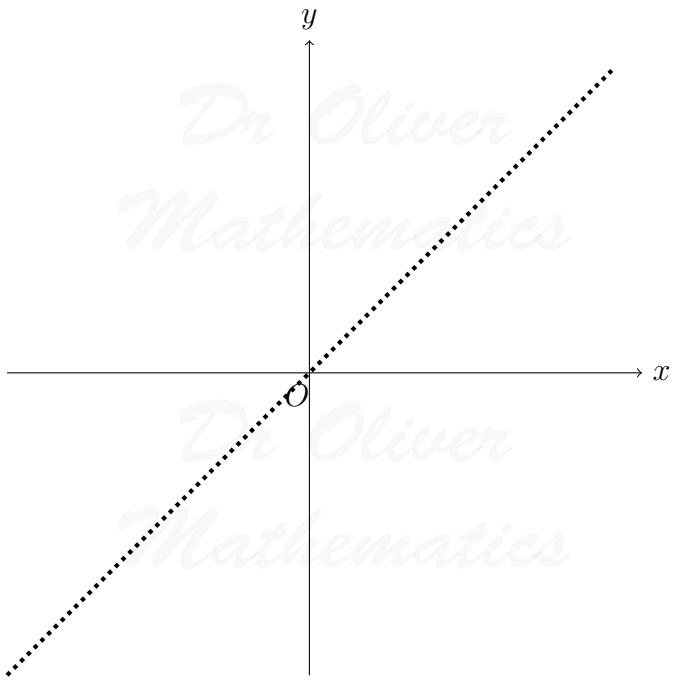
$$\begin{aligned}y = (x-1)^2 - 5 &\Rightarrow y + 5 = (x-1)^2 \\ &\Rightarrow \sqrt{y+5} = x - 1 \\ &\Rightarrow 1 + \sqrt{y+5} = x\end{aligned}$$

so

$$\underline{\underline{\mathrm{f}^{-1}(x) = 1 + \sqrt{x+5}}}.$$

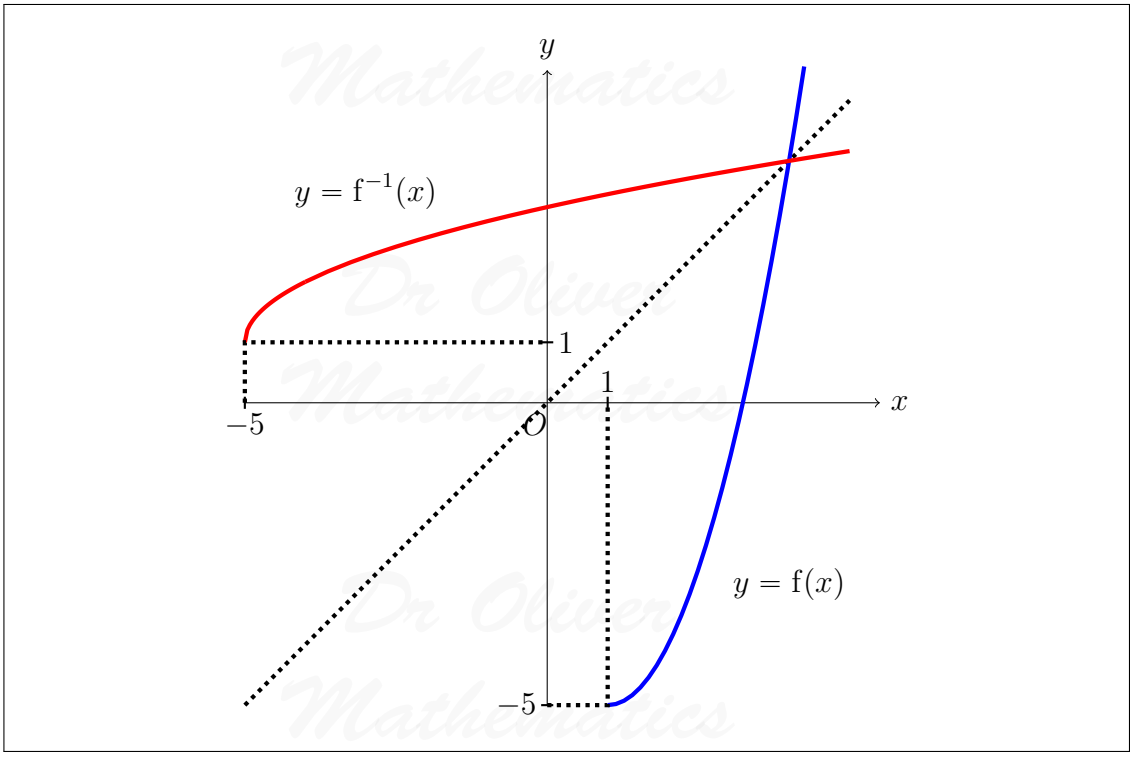
(d) sketch and label, on the axes below, the graph of (2)

$$y = \mathrm{f}(x) \text{ and of } y = \mathrm{f}^{-1}(x).$$



Solution

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(e) find the value of x for which (2)

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x).$$

Solution

Now,

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x) \Rightarrow (x-1)^2 - 5 = x$$

×	x	-1
x	x^2	$-x$
-1	$-x$	$+1$

$$\Rightarrow (x^2 - 2x + 1) - 5 = x$$
$$\Rightarrow x^2 - 3x - 4 = 0$$

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add to: -3
multiply to: -4
$\Big\}$ $-4, +1$

$$\Rightarrow (x-4)(x+1) = 0$$
$$\Rightarrow x - 4 = 0 \text{ or } x + 1 = 0$$
$$\Rightarrow x = 4 \text{ or } x = -1.$$

But $x = -1$ (see the above graph) so $\underline{\underline{x = 4}}$.

12. The function

$$\mathrm{f}(x) = x^3 + x^2 + ax + b$$

is divisible by $(x - 3)$ and leaves a remainder of 20 when divided by $(x + 1)$.

(a) Show that $b = 6$ and find the value of a. (4)

Solution

We use synthetic division twice:

3	1	1	a	b
	$\downarrow$	3	12	$3a + 36$
	1	4	$a + 12$	$3a + b + 36$

so

$$3a + b + 36 = 0 \Rightarrow 3a + b = -36 \quad (1)$$

and

-1	1	1	a	b
	$\downarrow$	-1	0	$-a$
	1	0	a	$-a + b$

so

$$-a + b = 20 \quad (2).$$

Next, (2) − (1):

$$-4a = 56 \Rightarrow \underline{\underline{a = -14}}$$

insert a into (2):

$$\Rightarrow -(-14) + b = 20$$
$$\Rightarrow \underline{\underline{b = 6}}.$$

(b) Using your value of a and taking b as 6, find the non-integer roots of the equation (5)

$$\mathrm{f}(x) = 0$$

in the form

$$p \pm \sqrt{q},$$

where p and q are integers.

Solution

Well,

3	1	1	-14	6
	$\downarrow$	3	12	-6
	1	4	-2	0

so

$$x^3 + x^2 - 14x + 6 = (x - 3)(x^2 + 4x - 2).$$

Now, non-integer roots:

$$\begin{aligned} x^2 + 4x - 2 = 0 &\Rightarrow x^2 + 4x = 2 \\ &\Rightarrow x^2 + 4x + 4 = 2 + 4 \\ &\Rightarrow (x + 2)^2 = 6 \\ &\Rightarrow x + 2 = \pm\sqrt{6} \\ &\Rightarrow \underline{\underline{x = -2 \pm \sqrt{6}}}; \end{aligned}$$

hence, $\underline{\underline{p = -2}}$ and $\underline{\underline{q = 6}}$.