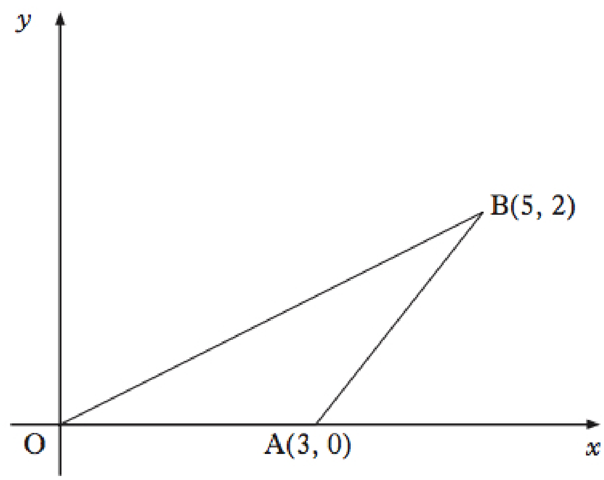
Dr Oliver Mathematics
Mathematics: Higher
2014 Paper 2: Calculator
1 hour 10 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1. $A(3, 0)$, $B(5, 2)$, and the origin are the vertices of a triangle as shown in the diagram.



(a) Obtain the equation of the perpendicular bisector of AB. (4)

Solution

$$\text{Gradient} = \frac{2-0}{5-3} = 1$$

and the gradient of the perpendicular bisector of AB is -1. Now,

$$\left(\frac{3+5}{2}, \frac{0+2}{2}\right) = (4, 1).$$

Finally, the equation of the perpendicular bisector of AB is

$$y - 1 = -1(x-4) \Rightarrow \underline{\underline{y = -x + 5}}. \quad (1)$$

The median from A has equation $y + 2x = 6$.

(b) Find T, the point of intersection of this median and the perpendicular bisector of AB. (2)

Solution

$$y + 2x = 6 \Rightarrow y = -2x + 6. \quad (2)$$

Do (1) − (2):

$$\begin{aligned} 0 = x - 1 &\Rightarrow x = 1 \\ &\Rightarrow y = 4; \end{aligned}$$

hence, $\underline{\underline{T(1, 4)}}$.

(c) Calculate the angle that AT makes with the positive direction of the x-axis. (2)

Solution

$$\begin{aligned} \text{Gradient of } AT &= \frac{4-0}{1-3} \\ &= \frac{4}{-2} \\ &= -2 \end{aligned}$$

and

$$\begin{aligned} \text{angle} &= 180 - \tan^{-1}(-2) \\ &= 116.565\,051\,2 \text{ (FCD)} \\ &= \underline{\underline{117^\circ \text{ (3 sf)}}}. \end{aligned}$$

2. A curve has equation $y = x^4 - 2x^3 + 5$. (4)

Find the equation of the tangent to this curve at the point where $x = 2$.

Solution

Now,

$$x = 2 \Rightarrow y = 5.$$

Next,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 6x^2$$

and

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 8.$$

Finally, the equation of the tangent is

$$\begin{aligned} y - 5 = 8(x-2) &\Rightarrow y - 5 = 8x - 16 \\ &= \underline{\underline{y = 8x - 11}}. \end{aligned}$$

3. Functions f and g are defined on suitable domains by

$$\mathrm{f}(x) = x(x-1) + q \text{ and } \mathrm{g}(x) = x + 3.$$

(a) Find an expression for $\mathrm{f}(\mathrm{g}(x))$. (2)

Solution

$$\begin{aligned} \mathrm{f}(\mathrm{g}(x)) &= \mathrm{f}(x+3) \\ &= (x+3)[(x+3)-1] + q \\ &= (x+3)(x+2) + q \\ &= \underline{\underline{x^2 + 5x + (6+q)}}. \end{aligned}$$

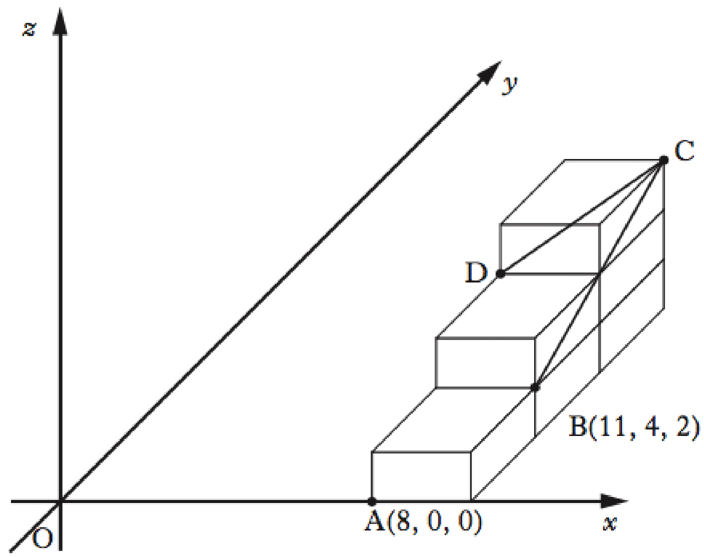
(b) Hence, find the value of q such that the equation $\mathrm{f}(\mathrm{g}(x)) = 0$ has equal roots. (4)

Solution

We want '$b^2 - 4ac = 0$':

$$\begin{aligned} 5^2 - 4 \times 1 \times (6+q) = 0 &\Rightarrow 4(6+q) = 25 \\ &\Rightarrow 6 + q = 6\tfrac{1}{4} \\ &\Rightarrow \underline{\underline{q = \tfrac{1}{4}}}. \end{aligned}$$

4. Six identical cuboids are placed with their edges parallel to the coordinate axes as shown in the diagram.



A and B are the points $(8,0,0)$ and $(11,4,2)$ respectively.

(a) State the coordinates of C and D. (2)

Solution
$\underline{\underline{C(11,12,6)}}$ and $\underline{\underline{D(8,8,4)}}$.

(b) Determine the components of $\overrightarrow{CB}$ and $\overrightarrow{CD}$. (2)

Solution

$$\overrightarrow{CB} = \begin{pmatrix} 11 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 11 \\ 12 \\ 6 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 \\ -8 \\ -4 \end{pmatrix}}}$$

and

$$\overrightarrow{CD} = \begin{pmatrix} 8 \\ 8 \\ 4 \end{pmatrix} - \begin{pmatrix} 11 \\ 12 \\ 6 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -3 \\ -4 \\ -2 \end{pmatrix}}}.$$

(c) Find the size of the angle BCD. (5)

Solution

$$\begin{aligned}
&\overrightarrow{CB}.\overrightarrow{CD} = |\overrightarrow{CB}|\,|\overrightarrow{CD}| \cos BCD \\
\Rightarrow\quad & 0 + 32 + 8 = \sqrt{0 + (-8)^2 + (-4)^2}\,\sqrt{(-3)^2 + (-4)^2 + (-2)^2}\, \cos BCD \\
\Rightarrow\quad & 40 = \sqrt{80}\,\sqrt{29}\, \cos BCD \\
\Rightarrow\quad & \cos BCD = \tfrac{2\sqrt{145}}{29} \\
\Rightarrow\quad & \angle BCD = 33.854\,514\,81 \text{ (FCD)} \\
\Rightarrow\quad & \underline{\underline{\angle BCD = 33.9^\circ \text{ (3 sf)}}}.
\end{aligned}$$

5. Given that (5)

$$\int_4^t (3x+4)^{-\frac{1}{2}}\,\mathrm{d}x = 2,$$

find the value of t.

Solution

$$\begin{aligned}
\int_4^t (3x+4)^{-\frac{1}{2}}\,\mathrm{d}x = 2 &\Rightarrow \left[\tfrac{2}{3}(3x+4)^{\frac{1}{2}}\right]_{x=4}^{t} = 2 \\
&\Rightarrow \tfrac{2}{3}(3t+4)^{\frac{1}{2}} - \tfrac{8}{3} = 2 \\
&\Rightarrow \tfrac{2}{3}(3t+4)^{\frac{1}{2}} = \tfrac{14}{3} \\
&\Rightarrow (3t+4)^{\frac{1}{2}} = 7 \\
&\Rightarrow 3t + 4 = 49 \\
&\Rightarrow 3t = 45 \\
&\Rightarrow \underline{\underline{t = 15}}.
\end{aligned}$$

6. Solve the equation (5)

$$\sin x - 2\cos 2x = 1,$$

for $0 \leqslant x < 2\pi$.

Solution

$$\begin{aligned}
\sin x - 2\cos 2x = 1 &\Rightarrow \sin x - 2(1 - 2\sin^2 x) = 1 \\
&\Rightarrow 4\sin^2 x + \sin x - 3 = 0
\end{aligned}$$

$$\begin{array}{ll} \text{add to:} & +1 \\ \text{multiply to:} & (+4)\times(-3) = -12 \end{array} \Bigg\} -3,\ +4$$

$$\begin{aligned} &\Rightarrow 4\sin^2 x + 4\sin x - 3\sin x - 3 = 0 \\ &\Rightarrow 4\sin x(\sin x + 1) - 3(\sin x + 1) = 0 \\ &\Rightarrow (4\sin x - 3)(\sin x + 1) = 0 \\ &\Rightarrow \sin x = \tfrac{3}{4} \text{ or } \sin x = -1. \end{aligned}$$

$\underline{\sin x = \frac{3}{4}}$:

$$\begin{aligned} \sin x = \tfrac{3}{4} &\Rightarrow x = 0.848\,062\,079,\ 2.293\,530\,575 \text{ (FCD)} \\ \sin x = \tfrac{3}{4} &\Rightarrow \underline{\underline{x = 0.848,\ 2.29 \text{ (3 sf)}}}. \end{aligned}$$

$\underline{\sin x = \frac{3}{4}}$:

$$\sin x = -1 \Rightarrow \underline{\underline{x = \tfrac{3}{2}\pi}}.$$

7. Land enclosed between a path and a railway line is being developed for housing. This land is represented by the shaded area shown in Diagram 1.

- The path is represented by a parabola with equation $y = 6x - x^2$.
- The railway is represented by a line with equation $y = 2x$.
- One square unit in the diagram represents 300 m^2 of land.

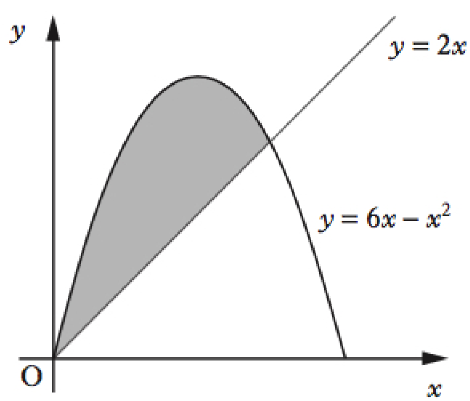


Diagram 1

(a) Calculate the area of land being developed. (5)

Solution

$$\begin{aligned} 2x = 6x - x^2 &\Rightarrow x^2 - 4x = 0 \\ &\Rightarrow x(x-4) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 4. \end{aligned}$$

Now,

$$\begin{aligned} \int_0^4 \left[(6x - x^2) - 2x\right] \mathrm{d}x &= \int_0^4 (4x - x^2)\,\mathrm{d}x \\ &= \left[2x^2 - \tfrac{1}{3}x^3\right]_{x=0}^4 \\ &= \left(32 - 21\tfrac{1}{3}\right) - (0 - 0) \\ &= 10\tfrac{2}{3}. \end{aligned}$$

Hence, the area is

$$10\tfrac{2}{3} \times 300 = \underline{\underline{3\,200 \text{ m}^2}}.$$

A road is built parallel to the railway line and is a tangent to the path as shown in Diagram 2.

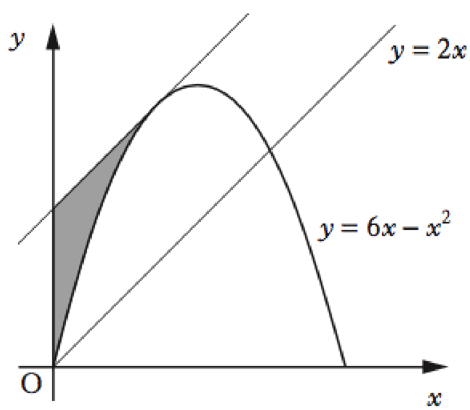


Diagram 2

It is decided that the land, represented by the shaded area in Diagram 2, will become a car park.

(b) Calculate the area of the car park. (5)

Solution

The road is $y = 2x + c$ for some c. Now,

$$y = 6x - x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6 - 2x.$$

Then,

$$\begin{aligned} 6 - 2x = 2 &\Rightarrow 2x = 4 \\ &\Rightarrow x = 2 \\ &\Rightarrow y = 6 \cdot 2 - 2^2 \\ &\Rightarrow y = 8 \end{aligned}$$

and

$$\begin{aligned} y - 8 = 2(x - 2) &\Rightarrow y - 8 = 2x - 4 \\ &\Rightarrow y = 2x + 4. \end{aligned}$$

Now,

$$\begin{aligned} \int_0^2 \left[(2x + 4) - (6x - x^2)\right] \, \mathrm{d}x &= \int_0^2 (x^2 - 4x + 4) \, \mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3 - 2x^2 + 4x\right]_{x=0}^2 \\ &= \left(2\tfrac{2}{3} - 8 + 8\right) - (0 - 0 + 0) \\ &= 2\tfrac{2}{3}. \end{aligned}$$

Hence, the area of the car park is

$$2\tfrac{2}{3} \times 300 = \underline{\underline{800 \text{ m}^2}}.$$

8. Given that the equation (5)

$$x^2 + y^2 - 2px - 4py + 3p + 2 = 0$$

represents a circle, determine the range of values of p.

Solution

$$\begin{aligned} & x^2+y^2-2px-4py+3p+2=0 \\ \Rightarrow\quad & x^2-2px+y^2-4py=-3p-2 \\ \Rightarrow\quad & (x^2-2px+p^2)+(y^2-4py+4p^2)=-3p-2+p^2+4p^2 \\ \Rightarrow\quad & (x-p)^2+(y-2p)^2=5p^2-3p-2. \end{aligned}$$

Now, because this is a circle,

$$\left.\begin{array}{ll} \text{add to:} & -3 \\ \text{multiply to:} & (+5)\times(-2)=-10 \end{array}\right\} -5,\ +2$$

$$\begin{aligned} 5p^2-3p-2>0 &\Rightarrow 5p^2-5p+2p-2>0 \\ &\Rightarrow 5p(p-1)+2(p-1)>0 \\ &\Rightarrow (5p+2)(p-1)>0 \\ &\Rightarrow \underline{\underline{p<-\tfrac{2}{5} \text{ or } p>1}}. \end{aligned}$$

9. Acceleration is defined as the rate of change of velocity.
An object is travelling in a straight line.
The velocity, v m/s, of this object, t seconds after the start of the motion, is given by

$$\mathrm{v}(t)=8\cos(2t-\tfrac{1}{2}\pi).$$

(a) Find a formula for $\mathrm{a}(t)$, the acceleration of this object, t seconds after the start of the motion. (3)

Solution

$$\begin{aligned} \mathrm{a}(t) &= 8\sin(2t-\tfrac{1}{2}\pi)\cdot(-2) \\ &= \underline{\underline{-16\sin(2t-\tfrac{1}{2}\pi)\ \mathrm{m/s^2}}}. \end{aligned}$$

(b) Determine whether the velocity of the object is increasing or decreasing when $t=10$. (2)

Solution

$$\begin{aligned} \mathrm{a}(t) &= -16\sin(20-\tfrac{1}{2}\pi) \\ &= 6.529\ldots \end{aligned}$$

and so the velocity is <u>increasing</u>.

Velocity is defined as the rate of change of displacement.

(c) Determine a formula for s(t), the displacement of the object, given that s$(t) = 4$ when $t = 0$. (3)

Solution

$$\mathrm{v}(t) = 8\cos(2t - \tfrac{1}{2}\pi) \Rightarrow \mathrm{s}(t) = 4\sin(2t - \tfrac{1}{2}\pi) + c,$$

for some constant c. Now,

$$\begin{aligned} \mathrm{s}(0) = 4 \Rightarrow 4 &= 4\sin(-\tfrac{1}{2}\pi) + c \\ \Rightarrow 4 &= -4 + c \qquad \Rightarrow c = 8, \end{aligned}$$

and we have

$$\underline{\underline{\mathrm{s}(t) = 4\sin(2t - \tfrac{1}{2}\pi) + 8}}.$$