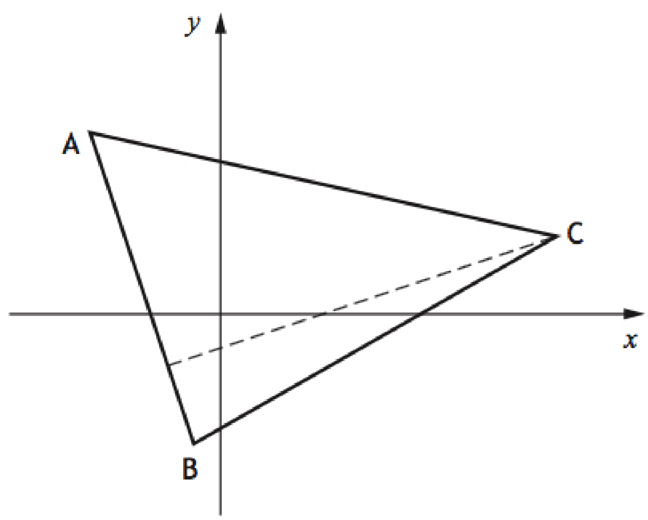
Dr Oliver Mathematics
Mathematics: Higher
2015 Paper 2: Calculator
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.

1. The vertices of triangle ABC are $A(-5, 7)$, $B(-1, -5)$, and $C(13, 3)$ as shown in the diagram.



The broken line represents the altitude from C.

(a) Show that the equation of the altitude from C is (4)

$$x - 3y = 4.$$

Solution

$$\text{Gradient of } AB = \frac{7 - (-5)}{-5 - (-1)} = \frac{12}{-4} = -3$$

and the gradient of the perpendicular is

$$-\frac{1}{-3} = \tfrac{1}{3}.$$

Finally, the equation of the altitude is

$$\begin{aligned} y - 3 = \tfrac{1}{3}(x - 13) &\Rightarrow 3(y - 3) = x - 13 \\ &\Rightarrow 3y - 9 = x - 13 \\ &\Rightarrow \underline{\underline{x - 3y = 4}}, \end{aligned}$$

as required.

(b) Find the equation of the median from B. (3)

Solution

The midpoint of AC is

$$\left(\frac{-5 + 13}{2}, \frac{7 + 3}{2}\right) = (4, 5).$$

Now, the gradient of the median from B

$$\begin{aligned} \frac{5 - (-5)}{4 - (-1)} &= \frac{10}{5} \\ &= 2 \end{aligned}$$

and the equation is

$$\begin{aligned} y + 5 = 2(x + 1) &\Rightarrow y + 5 = 2x + 2 \\ &\Rightarrow \underline{\underline{y = 2x - 3}}. \end{aligned}$$

(c) Find the coordinates of the point of intersection of the altitude from C and the median from B. (2)

Solution

$$\begin{aligned} x - 3y = 4 &\Rightarrow x - 3(2x-3) = 4 \\ &\Rightarrow x - 6x + 9 = 4 \\ &\Rightarrow -5x = -5 \\ &\Rightarrow x = 1 \\ &\Rightarrow y = -1; \end{aligned}$$

hence, the coordinates of the point of intersection is $\underline{\underline{(1, -1)}}$.

2. Functions f and g are defined on suitable domains by

$$\mathrm{f}(x) = 10 + x \text{ and } \mathrm{g}(x) = (1+x)(3-x) + 2.$$

(a) Find an expression for $\mathrm{f}(\mathrm{g}(x))$. (2)

Solution

$$\begin{aligned} \mathrm{f}(\mathrm{g}(x)) &= \mathrm{f}((1+x)(3-x)+2) \\ &= 10 + [(1+x)(3-x)+2] \\ &= \underline{\underline{(1+x)(3-x)+12}}. \end{aligned}$$

(b) Express $\mathrm{f}(\mathrm{g}(x))$ in the form $p(x+q)^2 + r$. (3)

Solution

$\times$	1	$+x$
3	3	$+3x$
$-x$	$-x$	$-x^2$

$$\begin{aligned} (1+x)(3-x)+12 &= (3 + 2x - x^2) + 12 \\ &= 15 + 2x - x^2 \\ &= -(x^2 - 2x) + 15 \\ &= -[(x^2 - 2x + 1) - 1] + 15 \\ &= -[(x-1)^2 - 1] + 15 \\ &= -(x-1)^2 + 1 + 15 \\ &= \underline{\underline{-(x-1)^2 + 16}}; \end{aligned}$$

hence, $\underline{\underline{p = -1}}$, $\underline{\underline{q = -1}}$, and $\underline{\underline{r = 16}}$.

Another function h is given by

$$\mathrm{h}(x) = \frac{1}{\mathrm{f}(\mathrm{g}(x))}.$$

(c) What values of x cannot be in the domain of h? (2)

Solution

$$\begin{aligned} -(x-1)^2 + 16 = 0 &\Rightarrow (x-1)^2 = 16 \\ &\Rightarrow x - 1 = -4 \text{ or } x - 1 = 4 \\ &\Rightarrow \underline{\underline{x = -3 \text{ or } x = 5}}. \end{aligned}$$

3. A version of the following problem first appeared in print in the 16th century.

A frog and a toad fall to the bottom of a well that is 50 feet deep.
Each day, the frog climbs 32 feet and then rests overnight. During the night, it slides down $\frac{2}{3}$ of its height above the floor of the well.
The toad climbs 13 feet each day before resting. Overnight, it slides down $\frac{1}{4}$ of its height above the floor of the well.

Their progress can be modelled by the recurrence relations:

- $f_{n+1} = \frac{1}{3} f_n + 32$, $f_1 = 32$,
- $t_{n+1} = \frac{3}{4} t_n + 13$, $t_1 = 13$,

where f_n and t_n are the heights reached by the frog and the toad at the end of the nth day after falling in.

(a) Calculate t_2, the height of the toad at the end of the second day. (1)

Solution

$t_2 = \frac{3}{4} t_1 + 13 = \underline{\underline{22\frac{3}{4}}}$ m.

(b) Determine whether or not either of them will eventually escape from the well. (5)

Solution

Let f and t be the limit of the frog's and the toad's sequences. Then

$$\begin{aligned} f = \tfrac{1}{3}f + 32 &\Rightarrow \tfrac{2}{3}f = 32 \\ &\Rightarrow f = 48 \end{aligned}$$

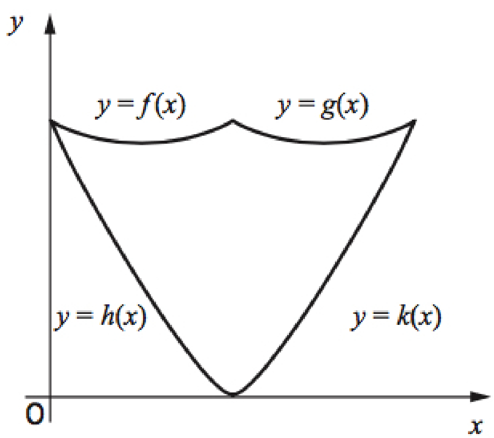
and

$$\begin{aligned} t = \tfrac{3}{4}t + 13 &\Rightarrow \tfrac{1}{4}t = 13 \\ &\Rightarrow t = 52. \end{aligned}$$

The frog does not escape but the toad does escape.

4. A wall plaque is to be made to commemorate the 150th anniversary of the publication of "*Alice's Adventures in Wonderland*".

The edges of the wall plaque can be modelled by parts of the graphs of four quadratic functions as shown in the sketch.



- $f(x) = \frac{1}{4}x^2 - \frac{1}{2}x + 3$,
- $g(x) = \frac{1}{4}x^2 - \frac{3}{2}x + 5$,
- $h(x) = \frac{3}{8}x^2 - \frac{9}{4}x + 3$
- $k(x) = \frac{3}{8}x^2 - \frac{3}{4}x$.

(a) Find the x-coordinate of the point of intersection of the graphs with equations $y = f(x)$ and $y = g(x)$. (2)

Solution

$$\tfrac{1}{4}x^2 - \tfrac{1}{2}x + 3 = \tfrac{1}{4}x^2 - \tfrac{3}{2}x + 5 \Rightarrow \underline{\underline{x = 2}}.$$

The graphs of the functions $\mathrm{f}(x)$ and $\mathrm{h}(x)$ intersect on the y-axis.
The plaque has a vertical line of symmetry.

(b) Calculate the area of the wall plaque. (7)

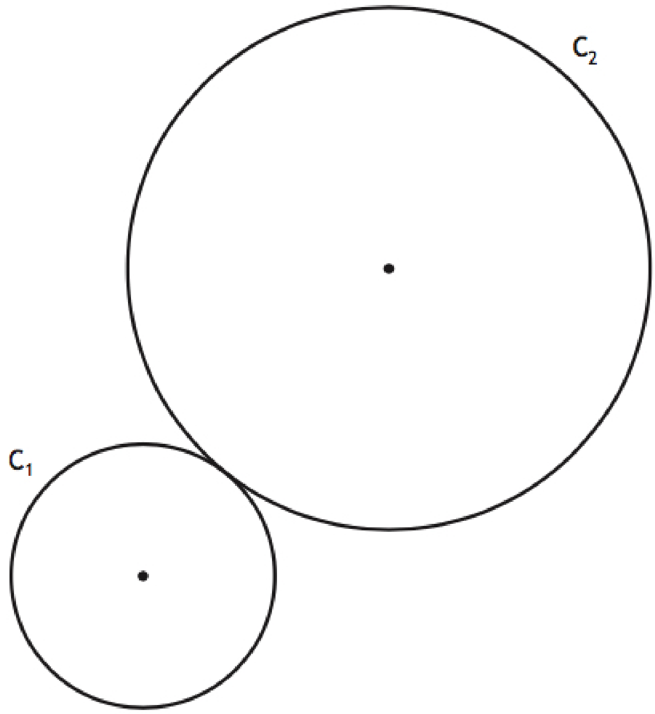
Solution

$$\begin{aligned}\text{Area} &= 2\int_0^2 \left[(\tfrac{1}{4}x^2 - \tfrac{1}{2}x + 3) - (\tfrac{3}{8}x^2 - \tfrac{9}{4}x + 3)\right] \mathrm{d}x \\ &= 2\int_0^2 (-\tfrac{1}{8}x^2 + \tfrac{7}{4}x)\,\mathrm{d}x \\ &= 2\left[-\tfrac{1}{24}x^3 + \tfrac{7}{8}x^2\right]_{x=0}^2 \\ &= 2\left[\left(-\tfrac{1}{3} + 3\tfrac{1}{2}\right) - (0+0)\right] \\ &= \underline{\underline{6\tfrac{1}{3}}}.\end{aligned}$$

5. Circle C_1 has equation

$$x^2 + y^2 + 6x + 10y + 9 = 0.$$

The centre of circle C_2 is $(9, 11)$.



Circles C_1 and C_2 touch externally.

(a) Determine the radius of C_2. (4)

Solution

$$\begin{aligned} & x^2 + y^2 + 6x + 10y + 9 = 0 \\ \Rightarrow\ & x^2 + 6x + y^2 + 10y = -9 \\ \Rightarrow\ & (x^2 + 6x + 9) + (y^2 + 10y + 25) = -9 + 9 + 25 \\ \Rightarrow\ & (x + 3)^2 + (y + 5)^2 = 25; \end{aligned}$$

hence, the centre of C_1 is $(-3, -5)$ and the radius is 5. Now,

$$\begin{aligned} C_1C_2 &= \sqrt{(9 - (-3))^2 + (11 - (-5))^2} \\ &= \sqrt{144 + 256} \\ &= 20. \end{aligned}$$

Finally, the radius of C_2 is

$$20 - 5 = \underline{\underline{15}}.$$

A third circle, C_3, is drawn such that:

- both C_1 and C_2 touch C_3 internally,
- the centres of C_1, C_2, and C_3 are collinear.

(b) Determine the equation of C_3. (4)

Solution

$$\tfrac{3}{4} \times 12 = 9 \text{ and } \tfrac{3}{4} \times 16 = 12.$$

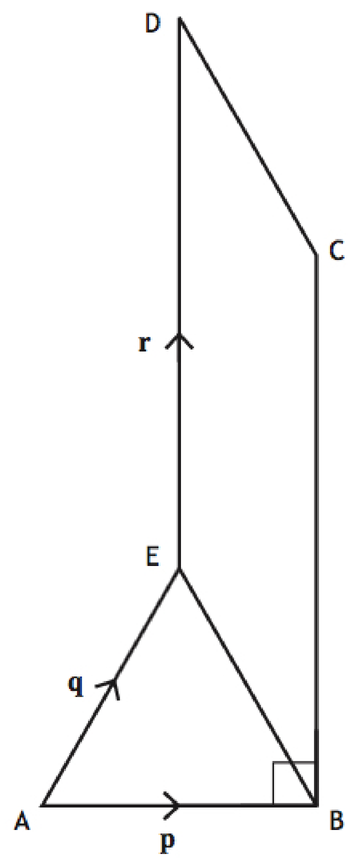
Now,

$$\begin{pmatrix} -3 \\ -5 \end{pmatrix} + \begin{pmatrix} 12 \\ 16 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix};$$

the centre of C_3 is $(6, 7)$. Clearly, the radius of C_3 is 20 which means that the equation of C_3 is

$$\underline{\underline{(x-6)^2 + (y-7)^2 = 400}}.$$

6. Vectors $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are represented on the diagram as shown.



- $BCDE$ is a parallelogram,
- ABE is an equilateral triangle,
- $|\mathbf{p}| = 3$,
- Angle $ABC = 90°$.

(a) Evaluate (3)

$$\mathbf{p}.(\mathbf{q} + \mathbf{r}).$$

Solution

$$\begin{aligned} \mathbf{p}.(\mathbf{q} + \mathbf{r}) &= \mathbf{p}.\mathbf{q} + \mathbf{p}.\mathbf{r} \\ &= |\mathbf{p}||\mathbf{q}| \cos BAC + 0 \\ &= 3 \times 3 \times \tfrac{1}{2} \\ &= \underline{\underline{4\tfrac{1}{2}}}. \end{aligned}$$

(b) Express $\overrightarrow{EC}$ in terms of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$. (1)

Solution

$$\begin{aligned} \overrightarrow{EC} &= \overrightarrow{EA} + \overrightarrow{AB} + \overrightarrow{BC} \\ &= \underline{\underline{-\mathbf{q} + \mathbf{p} + \mathbf{r}}}. \end{aligned}$$

(c) Given that (3)

$$\overrightarrow{AE}.\overrightarrow{EC} = 9\sqrt{3} - \tfrac{9}{2},$$

find $|\mathbf{r}|$.

Solution

$$\begin{aligned} \overrightarrow{AE}.\overrightarrow{EC} &= 9\sqrt{3} - \tfrac{9}{2} \\ \Rightarrow \quad \mathbf{q}.(-\mathbf{q} + \mathbf{p} + \mathbf{r}) &= 9\sqrt{3} - \tfrac{9}{2} \\ \Rightarrow \quad -\mathbf{q}.\mathbf{q} + \mathbf{q}.\mathbf{p} + \mathbf{q}.\mathbf{r} &= 9\sqrt{3} - \tfrac{9}{2} \\ \Rightarrow \quad -9 + 4\tfrac{1}{2} + |\mathbf{q}||\mathbf{r}|\cos 30^\circ &= 9\sqrt{3} - \tfrac{9}{2} \\ \Rightarrow \quad 3|\mathbf{r}|\cos 30^\circ &= 9\sqrt{3} \\ \Rightarrow \quad |\mathbf{r}| &= \frac{3\sqrt{3}}{\cos 30^\circ} \\ \Rightarrow \quad |\mathbf{r}| &= \underline{\underline{6}}. \end{aligned}$$

7. (a) Find (2)

$$\int (3\cos 2x + 1)\,\mathrm{d}x.$$

Solution

$$\int (3\cos 2x + 1)\,\mathrm{d}x = \underline{\underline{\tfrac{3}{2}\sin 2x + x + c}}.$$

(b) Show that (2)

$$3\cos 2x + 1 \equiv 4\cos^2 x - 2\sin^2 x.$$

Solution

$$\begin{aligned} 3\cos 2x + 1 &\equiv 3(\cos^2 x - \sin^2 x) + (\sin^2 x + \cos^2 x) \\ &\equiv \underline{\underline{4\cos^2 x - 2\sin^2 x}}, \end{aligned}$$

as required.

(c) Hence, or otherwise, find (2)

$$\int (\sin^2 x - 2\cos^2 x)\,\mathrm{d}x.$$

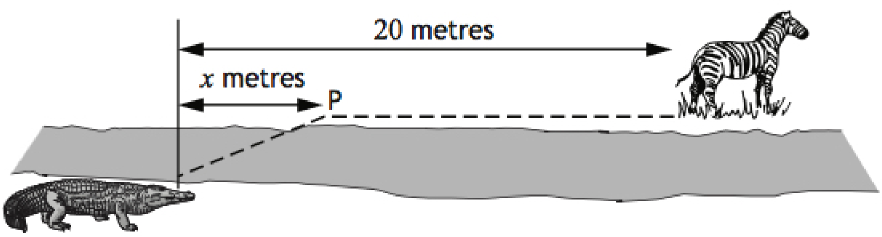
Solution

$$\begin{aligned}\int (\sin^2 x - 2\cos^2 x)\,\mathrm{d}x &= -\tfrac{1}{2}\int (4\cos^2 x - 2\sin^2 x)\,\mathrm{d}x \\ &= -\tfrac{1}{2}\int (3\cos 2x + 1)\,\mathrm{d}x \\ &= -\tfrac{1}{2}(\tfrac{3}{2}\sin 2x + x + c) \\ &= \underline{\underline{-\tfrac{3}{4}\sin 2x - \tfrac{1}{2}x + d}}.\end{aligned}$$

8. A crocodile is stalking prey located 20 metres further upstream on the opposite bank of a river.

Crocodiles travel at different speeds on land and in water.

The time taken for the crocodile to reach its prey can be minimised if it swims to a particular point, P, x metres upstream on the other side of the river as shown in the diagram.



The time taken, T, measured in tenths of a second, is given by

$$T(x) = 5\sqrt{36 + x^2} + 4(20 - x).$$

(a) (i) Calculate the time taken if the crocodile does not travel on land. (1)

Solution
I remember this question: for days on end, it caused an outpouring on social media (you know the kind of thing: *it's unfair! we've not been taught that! will the exam board be lowering the pass mark because of this question?* and

on and on and on . . .)

$$
\begin{aligned}
T(20) &= 5\sqrt{36 + 20^2} + 4(20 - 20) \\
&= 5\sqrt{436} \\
&= (10\sqrt{109})\ 0.1 \text{ s (FCD)} \\
&= 10.440\,306\,51 \text{ s (FCD)} \\
&= \underline{\underline{10.4 \text{ s (3 sf)}}}.
\end{aligned}
$$

(ii) Calculate the time taken if the crocodile swims the shortest distance possible. (1)

Solution

$$
\begin{aligned}
T(0) &= 5\sqrt{36 + 0} + 4(20 - 0) \\
&= (30 + 80)\ 0.1 \text{ s (FCD)} \\
&= \underline{\underline{11 \text{ s (3 sf)}}}.
\end{aligned}
$$

Between these two extremes there is one value of x which minimises the time taken.

(b) Find this value of x and hence calculate the minimum possible time. (8)

Solution

$$
\begin{aligned}
T(x) = 5\sqrt{36 + x^2} + 4(20 - x) &\Rightarrow T(x) = 5(36 + x^2)^{\frac{1}{2}} + 80 - 4x \\
&\Rightarrow T'(x) = 5x(36 + x^2)^{-\frac{1}{2}} - 4
\end{aligned}
$$

and

$$
\begin{aligned}
T'(x) = 0 &\Rightarrow 5x(36 + x^2)^{-\frac{1}{2}} - 4 = 0 \\
&\Rightarrow 5x(36 + x^2)^{-\frac{1}{2}} = 4 \\
&\Rightarrow (36 + x^2)^{-\frac{1}{2}} = \frac{4}{5x} \\
&\Rightarrow (36 + x^2)^{\frac{1}{2}} = \frac{5x}{4} \\
&\Rightarrow 36 + x^2 = \tfrac{25}{16}x^2 \\
&\Rightarrow 36 = \tfrac{9}{16}x^2 \\
&\Rightarrow x^2 = 64 \\
&\Rightarrow x = 8.
\end{aligned}
$$

Finally,

$$\begin{aligned} T(8) &= 5\sqrt{36+8^2} + 4(20-8) \\ &= 50 + 48 \\ &= 98\ 0.1 \text{ s (FCD)} \\ &= \underline{\underline{9.8 \text{ s}}}. \end{aligned}$$

9. The blades of a wind turbine are turning at a steady rate. (8)
The height, h metres, of the tip of one of the blades above the ground at time, t seconds, is given by the formula

$$h = 36\sin(1.5t) - 15\cos(1.5t) + 65.$$

Express

$$36\sin(1.5t) - 15\cos(1.5t)$$

in the form

$$k\sin(1.5t - a),$$

where $k > 0$ and $0 < a < \frac{1}{2}\pi$, and hence find the two values of t for which the tip of this blade is at a height of 100 metres above the ground during the first turn.

Solution

$$\begin{aligned} k\sin(1.5t - a) &\equiv k[\sin(1.5t)\cos a - \sin a\cos(1.5t)] \\ &\equiv k\sin(1.5t)\cos a - k\sin a\cos(1.5t) \end{aligned}$$

and so

$$k\cos a = 36 \text{ and } k\sin a = 15.$$

Now,

$$\begin{aligned} k &= \sqrt{k^2} \\ &= \sqrt{(k\sin a)^2 + (k\cos a)^2} \\ &= \sqrt{15^2 + 36^2} \\ &= 39 \end{aligned}$$

and

$$\begin{aligned} \tan a = \frac{k\sin a}{k\cos a} &\Rightarrow \tan a = \tfrac{15}{36} \\ &\Rightarrow a = 0.394\,791\,119\,7 \text{ (FCD)}; \end{aligned}$$

hence,

$$36\sin(1.5t) - 15\cos(1.5t) = \underline{\underline{39\sin(1.5t - 0.394\ldots)}}.$$

Next,

$$\begin{aligned}
& 36\sin(1.5t) - 15\cos(1.5t) + 65 = 100 \\
\Rightarrow\quad & 39\sin(1.5t - 0.394\ldots) + 65 = 100 \\
\Rightarrow\quad & 39\sin(1.5t - 0.394\ldots) = 35 \\
\Rightarrow\quad & \sin(1.5t - 0.394\ldots) = \tfrac{35}{39} \\
\Rightarrow\quad & 1.5t - 0.394\ldots = 1.113\,922\,323,\ 2.027\,670\,33 \text{ (FCD)} \\
\Rightarrow\quad & 1.5t = 1.508\,713\,443,\ 2.422\,461\,45 \text{ (FCD)} \\
\Rightarrow\quad & t = 1.005\,808\,962,\ 1.614\,974\,3 \text{ (FCD)} \\
\Rightarrow\quad & \underline{\underline{t = 1.01,\ 1.61 \text{ s (3 sf)}}}.
\end{aligned}$$