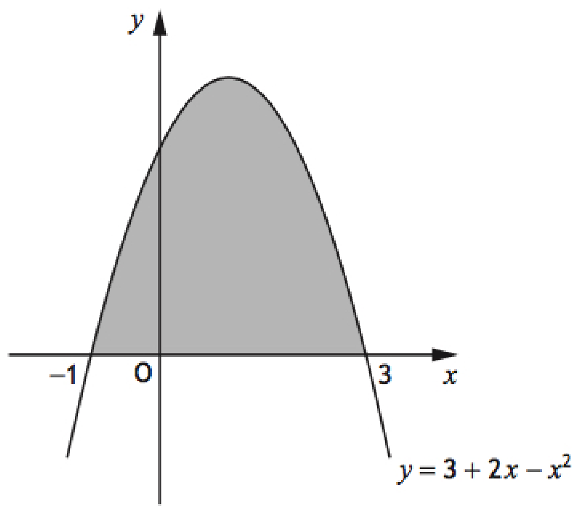
Dr Oliver Mathematics
Mathematics: Higher
2018 Paper 2: Calculator
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.

1. The diagram shows the curve with equation (4)

$$y = 3 + 2x - x^2.$$



Calculate the shaded area.

Solution

$$\begin{aligned}\text{Area} &= \int_{-1}^{3} (3 + 2x - x^2)\,\mathrm{d}x \\ &= \left[3x + x^2 - \tfrac{1}{3}x^3\right]_{x=-1}^{3} \\ &= (9 + 4 - 9) - \left(-3 + 1 + \tfrac{1}{3}\right) \\ &= \underline{\underline{5\tfrac{2}{3}}}.\end{aligned}$$

2. Vectors $\mathbf{u}$ and $\mathbf{v}$ are defined by

$$\mathbf{u} = \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} -7 \\ 8 \\ 5 \end{pmatrix}.$$

(a) Find $\mathbf{u}.\mathbf{v}$. (1)

Solution

$$\mathbf{u}.\mathbf{v} = 7 + 32 - 15 = \underline{\underline{24}}.$$

(b) Calculate the acute angle between $\mathbf{u}$ and $\mathbf{v}$. (4)

Solution

Let the acute angle between $\mathbf{u}$ and $\mathbf{v}$ be $\theta°$. Now,

$$|\mathbf{u}| = \sqrt{(-1)^2 + 4^2 + (-3)^2} = \sqrt{26}$$

and

$$|\mathbf{v}| = \sqrt{(-7)^2 + 8^2 + 5^2} = \sqrt{138}.$$

Finally,

$$\begin{aligned} & \mathbf{u}.\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \\ \Rightarrow\ & 24 = \sqrt{26} \cdot \sqrt{138} \cdot \cos\theta \\ \Rightarrow\ & \cos\theta = \frac{24}{\sqrt{26} \cdot \sqrt{138}} \\ \Rightarrow\ & \theta = 66.380\,033\,84 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{\theta = 66.4 \text{ (3 sf)}}}. \end{aligned}$$

3. A function, f, is defined on the set of real numbers by (3)

$$\mathrm{f}(x) = x^3 - 7x + 6.$$

Determine whether f is increasing or decreasing when $x = 2$.

Solution

$$\mathrm{f}(x) = x^3 - 7x + 6 \Rightarrow \mathrm{f}'(x) = 3x^2 - 7$$

and

$$f'(2) = 12 - 7 = 5 > 0;$$

hence, f is increasing.

4. Express

$$-3x^2 - 6x + 7$$

in the form

$$a(x+b)^2 + c.$$

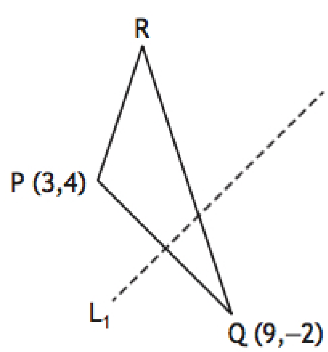
(3)

Solution

$$\begin{aligned} -3x^2 - 6x + 7 &= -3(x^2 + 2x) + 7 \\ &= -3[(x^2 + 2x + 1) - 1] + 7 \\ &= -3[(x+1)^2 - 1] + 7 \\ &= -3(x+1)^2 + 3 + 7 \\ &= -3(x+1)^2 + 10; \end{aligned}$$

hence, $a = -3$, $b = 1$, and $c = 10$.

5. PQR is a triangle with $P(3, 4)$ and $Q(9, -2)$.



(a) Find the equation of L_1, the perpendicular bisector of PQ. (3)

Solution

The midpoint of PQ is

$$\left(\frac{3+9}{2}, \frac{4+(-2)}{2}\right) = (6, 1).$$

Now, the gradient of PQ is

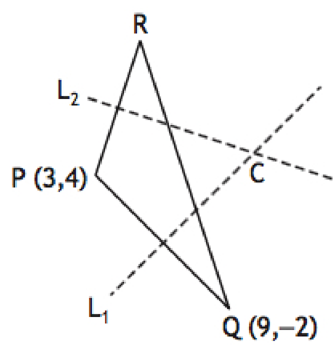
$$\frac{4-(-2)}{3-9} = -1$$

and so the perpendicular bisector is 1. Finally,

$$y - 1 = x - 6 \Rightarrow \underline{\underline{y = x - 5}}.$$

The equation of L_2, the perpendicular bisector of PR is

$$3y + x = 25.$$



(b) Calculate the coordinates of C, the point of intersection of L_1 and L_2. (2)

Solution

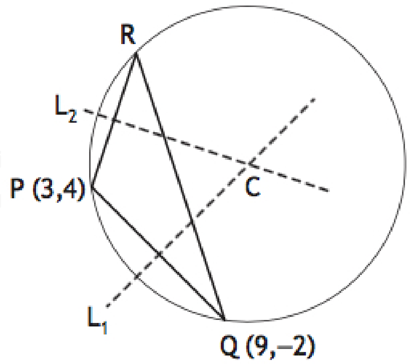
$$x = y + 5 \quad (1)$$
$$x = 25 - 3y \quad (2)$$

and do (1) − (2):

$$0 = 4y - 20 \Rightarrow 4y = 20$$
$$\Rightarrow y = 5$$
$$\Rightarrow x = 10;$$

hence, $\underline{\underline{C(10, 5)}}$.

C is the centre of the circle which passes through the vertices of triangle PQR.



(c) Determine the equation of this circle. (2)

Solution

$$\begin{aligned} CP^2 &= (10-3)^2 + (5-4)^2 \\ &= 49 + 1 \\ &= 50 \end{aligned}$$

and, hence, the equation of this circle is

$$\underline{\underline{(x-10)^2 + (y-5)^2 = 50}}.$$

6. Functions, f and g, are given by

$$\mathrm{f}(x) = 3 + \cos x \text{ and } \mathrm{g}(x) = 2x,\ x \in \mathbb{R}.$$

(a) Find expressions for

(i) $\mathrm{f}(\mathrm{g}(x))$ and (2)

Solution

$$\mathrm{f}(\mathrm{g}(x)) = \mathrm{f}(2x) = \underline{\underline{3 + \cos 2x}}.$$

(ii) $\mathrm{g}(\mathrm{f}(x))$. (1)

Solution

$$\mathrm{g}(\mathrm{f}(x)) = \mathrm{g}(3 + \cos x) = \underline{\underline{2(3 + \cos x)}}.$$

(b) Determine the value(s) of x for which (6)

$$\mathrm{f}(\mathrm{g}(x)) = \mathrm{g}(\mathrm{f}(x))$$

where $0 \leqslant x < 2\pi$.

Solution

$$\begin{aligned} 3 + \cos 2x = 2(3 + \cos x) &\Rightarrow 3 + (2\cos^2 x - 1) = 6 + 2\cos x \\ &\Rightarrow 2\cos^2 x - 2\cos x - 4 = 0 \\ &\Rightarrow 2(\cos^2 x - \cos x - 2) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -2 \\ \text{multiply to:} & +1 \end{array}\right\} -2,\ +1$$

$$\Rightarrow 2(\cos x - 2)(\cos x + 1) = 0$$

$\cos x = 2$ does not have solution in the real numbers

$$\Rightarrow \cos x = -1$$

$$\Rightarrow \underline{\underline{x = \pi}}.$$

7. (a) (i) Show that $(x - 2)$ is a factor of (2)

$$2x^3 - 3x^2 - 3x + 2.$$

Solution

2	2	−3	−3	2
	↓	2	2	−2
	2	1	−1	0

Hence, because there is no remainder, $(x-2)$ is a <u>factor</u> of $2x^3-3x^2-3x+2$.

(ii) Hence, factorise (2)

$$2x^3 - 3x^2 - 3x + 2$$

fully.

Solution

$$2x^3 - 3x^2 - 3x + 2 = (x-2)(2x^2 + x - 1)$$

$$\left.\begin{array}{ll} \text{add to:} & +1 \\ \text{multiply to:} & (+2) \times (-1) = -2 \end{array}\right\} +2, -1$$

$$\begin{aligned} &= (x-2)(2x^2 + 2x - x - 1) \\ &= (x-2)[2x(x+1) - (x+1)] \\ &= \underline{\underline{(x-2)(2x-1)(x+1)}}. \end{aligned}$$

The fifth term, u_5, of a sequence is

$$u_5 = 2a - 3.$$

The terms of the sequence satisfy the recurrence relation

$$u_{n+1} = au_n - 1.$$

(b) Show that (1)

$$u_7 = 2a^3 - 3a^2 - a - 1.$$

Solution

$$\begin{aligned} u_6 &= au_5 - 1 \\ &= a(2a-3) - 1 \\ &= 2a^2 - 3a - 1 \\ u_7 &= au_6 - 1 \\ &= a(2a^2 - 3a - 1) - 1 \\ &= \underline{\underline{2a^3 - 3a^2 - a - 1}}, \end{aligned}$$

as required.

For this sequence, it is known that

- $u_7 = u_5$ and
- a limit exists.

(c) (i) Determine the value of a. (3)

Solution

$$\begin{aligned} u_7 = u_5 &\Rightarrow 2a^3 - 3a^2 - a - 1 = 2a - 3 \\ &\Rightarrow 2a^3 - 3a^2 - 3a + 2 = 0 \\ &\Rightarrow (a-2)(2a-1)(a+1) = 0 \\ &\Rightarrow a = 2, a = \tfrac{1}{2}, \text{ or } a = -1. \end{aligned}$$

Because $|a| < 1$ (why?), $\underline{\underline{a = \tfrac{1}{2}}}$.

(ii) Calculate the limit. (1)

Solution
The sequence is

$$u_{n+1} = \tfrac{1}{2}u_n - 1$$

and let the limit be u. Then,

$$\begin{aligned} u = \tfrac{1}{2}u - 1 &\Rightarrow \tfrac{1}{2}u = -1 \\ &\Rightarrow \underline{\underline{u = -2}}. \end{aligned}$$

8. (a) Express (4)

$$2\cos x^\circ - \sin x^\circ$$

in the form

$$k\cos(x-a)^\circ,\ k > 0,\ 0 < a < 360.$$

Solution

$$\begin{aligned} 2\cos x^\circ - \sin x^\circ &\equiv k\cos(x-a)^\circ \\ &\equiv k(\cos x^\circ \cos a^\circ + \sin x^\circ \sin a^\circ) \\ &\equiv k\cos x^\circ \cos a^\circ + k\sin x^\circ \sin a^\circ \end{aligned}$$

which means that

$$k\cos a^\circ = 2 \text{ and } k\sin a^\circ = -1.$$

Now,

$$\begin{aligned} k &= \sqrt{k^2} \\ &= \sqrt{k^2(\cos^2 a^\circ + \sin^2 a^\circ)} \\ &= \sqrt{(k\cos a^\circ)^2 + (k\sin a^\circ)^2} \\ &= \sqrt{2^2 + (-1)^2} \\ &= \sqrt{5} \end{aligned}$$

and

$$\begin{aligned} \tan a^\circ = \frac{k\sin a^\circ}{k\cos a^\circ} &\Rightarrow \tan a^\circ = -\tfrac{1}{2} \\ &\Rightarrow a = 333.434\,948\,8 \text{ (FCD)}. \end{aligned}$$

Hence,

$$2\cos x^\circ - \sin x^\circ = \underline{\underline{\sqrt{5}\cos(x - 333.434\ldots)^\circ}}.$$

(b) Hence, or otherwise, find

(i) the minimum value of (1)

$$6\cos x^\circ - 3\sin x^\circ,$$

Solution

$$6\cos x^\circ - 3\sin x^\circ = 3(\sqrt{5}\cos(x - 333.434\ldots)^\circ)$$

and so it has a minimum value at $\underline{\underline{-3\sqrt{5}}}$.

(ii) the value of x for which it occurs where $0 \leqslant x < 360$. (2)

Solution

$$\begin{aligned} \cos(x - 333.434\ldots)^\circ = -1 &\Rightarrow x - 333.434\ldots = -180 \\ &\Rightarrow x = 153.434\,948\,8 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 153 \text{ (3 sf)}}}. \end{aligned}$$

9. A sector with a particular fixed area has radius x cm. (6)

The perimeter, P cm, of the sector is given by

$$P = 2x + \frac{128}{x}.$$

Find the minimum value of P.

Solution

$$P = 2x + \frac{128}{x} \Rightarrow P = 2x + 128x^{-1}$$
$$\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}x} = 2 - 128x^{-2}.$$

Now,

$$\frac{\mathrm{d}P}{\mathrm{d}x} = 0 \Rightarrow 2 - 128x^{-2} = 0$$
$$\Rightarrow 2 = \frac{128}{x^2}$$
$$\Rightarrow 1 = \frac{64}{x^2}$$
$$\Rightarrow x^2 = 64$$
$$\Rightarrow x = 8$$

because $x > 0$. Next,

$$\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = 256x^{-3} > 0$$

for all values of $x > 0$ so it is a minimum. Hence,

$$x = 8 \Rightarrow P = 16 + 16 = \underline{\underline{32 \text{ cm}}}.$$

10. The equation (4)

$$x^2 + (m-3)x + m = 0$$

has two real and distinct roots.

Determine the range of values for m.

Solution

$a = 1$, $b = m - 3$, and $c = m$:

$$\begin{aligned}(m-3)^2 - 4 \cdot 1 \cdot m > 0 &\Rightarrow (m^2 - 6m + 9) - 4m > 0\\ &\Rightarrow m^2 - 10m + 9 > 0\\ &\Rightarrow (m-1)(m-9) > 0\\ &\Rightarrow \underline{\underline{m < 1 \text{ or } m > 9}}.\end{aligned}$$

11. A supermarket has been investigating how long customers have to wait at the checkout.

During any half hour period, the percentage, $P\%$, of customers who wait for less than t minutes, can be modelled by

$$P = 100(1 - \mathrm{e}^{kt}),$$

where k is a constant.

(a) If 50% of customers wait for less than 3 minutes, determine the value of k. (4)

Solution

$$\begin{aligned}50 = 100(1 - \mathrm{e}^{3k}) &\Rightarrow \tfrac{1}{2} = 1 - \mathrm{e}^{3k}\\ &\Rightarrow \mathrm{e}^{3k} = \tfrac{1}{2}\\ &\Rightarrow 3k = \ln \tfrac{1}{2}\\ &\Rightarrow k = \tfrac{1}{3} \ln \tfrac{1}{2}\\ &\Rightarrow k = -0.231\,049\,060\,2 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{k = -0.231 \text{ (3 sf)}}}.\end{aligned}$$

(b) Calculate the percentage of customers who wait for 5 minutes or longer. (2)

Solution

$$\begin{aligned}\text{P(who wait for 5 minutes or less)} &= 100(1 - \mathrm{e}^{5 \times -0.231\ldots})\\ &= 68.501\,973\,75 \text{ (FCD)}\end{aligned}$$

and hence

$$\begin{aligned}\text{P(who wait for 5 minutes or more)} &= 100 - 68.501\ldots\\ &= \underline{\underline{31.5\% \text{ (3 sf)}}}.\end{aligned}$$

12. Circle C_1 has equation

$$(x - 13)^2 + (y + 4)^2 = 100.$$

Circle C_2 has equation

$$x^2 + y^2 + 14x - 22y + c = 0.$$

(a) (i) Write down the coordinates of the centre of C_1. (1)

Solution

$\underline{\underline{(13, -4)}}$.

The centre of C_1 lies on the circumference of C_2.

(ii) Show that $c = -455$. (1)

Solution

$$\begin{aligned} x = 13, y = -4 &\Rightarrow 13^2 + (-4)^2 + 14(13) - 22(-4) + c = 0 \\ &\Rightarrow 169 + 16 + 182 + 88 + c = 0 \\ &\Rightarrow \underline{\underline{c = -455}}, \end{aligned}$$

as required.

The line joining the centres of the circles intersects C_1 at P.

(b) (i) Determine the ratio in which P divides the line joining the centres of the circles. (2)

Solution

$$\begin{aligned} & x^2 + y^2 + 14x - 22y + c = 0 \\ \Rightarrow\ & x^2 + 14x + y^2 - 22y = 455 \\ \Rightarrow\ & (x^2 + 14x + 49) + (y^2 - 22y + 121) = 455 + 49 + 121 \\ \Rightarrow\ & (x + 7)^2 + (y - 11)^2 = 625. \end{aligned}$$

So, the centre of C_2 is $(-7, 11)$ and the radius is 25. Finally, the ratio in which P divides the line joining the centres of the circles is

$$10 : 15 = \underline{\underline{2 : 3}}.$$

(ii) Hence, or otherwise, determine the coordinates of P. (2)

Solution

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OC_1} + \overrightarrow{C_1P} \\ &= \begin{pmatrix} 13 \\ -4 \end{pmatrix} + \tfrac{2}{5}\overrightarrow{C_1C_2} \\ &= \begin{pmatrix} 13 \\ -4 \end{pmatrix} + \tfrac{2}{5}\begin{pmatrix} -20 \\ 15 \end{pmatrix} \\ &= \begin{pmatrix} 13 \\ -4 \end{pmatrix} + \begin{pmatrix} -8 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 2 \end{pmatrix};\end{aligned}$$

hence, $\underline{\underline{P(5, 2)}}$.

P is the centre of a third circle, C_3.
C_2 touches C_3 internally.

(c) Determine the equation of C_3. (1)

Solution

$$\begin{aligned}(x-5)^2 + (y-2)^2 &= (15+25)^2 \\ &= 40^2 \\ &= 1\,600;\end{aligned}$$

hence,

$$\underline{\underline{(x-5)^2 + (y-2)^2 = 1\,600}}.$$