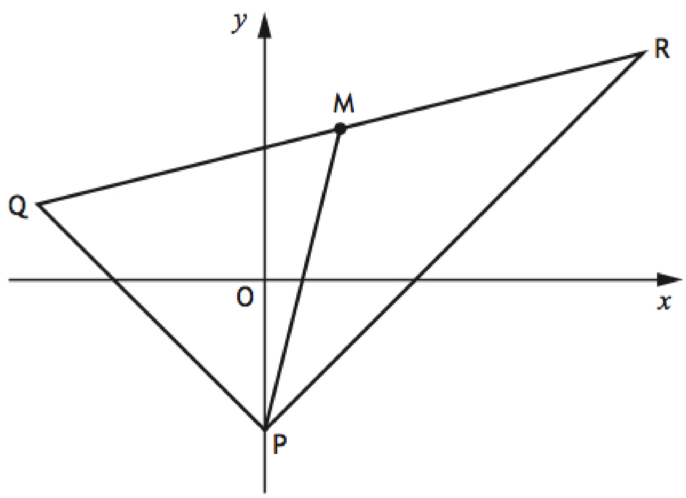
Dr Oliver Mathematics
Mathematics: Higher
2016 Paper 2: Calculator
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.

1. PQR is a triangle with vertices $P(0, -4)$, $Q(-6, 2)$, and $R(10, 6)$.



(a) (i) State the coordinates of M, the midpoint of QR. (1)

Solution

$$\left(\frac{-6+10}{2}, \frac{2+6}{2}\right) = \underline{\underline{(2, 4)}}.$$

(ii) Hence find the equation of PM, the median through P. (2)

Solution

$$\begin{aligned}\text{Gradient of } PM &= \frac{4-(-4)}{2-0} \\ &= \frac{8}{2} \\ &= 4\end{aligned}$$

and the equation is

$$y - (-4) = 4(x - 0) \Rightarrow \underline{\underline{y = 4x - 4}}.$$

(b) Find the equation of the line, L, passing through M and perpendicular to PR. (3)

Solution

$$\left(\frac{0+10}{2}, \frac{-4+6}{2}\right) = (5, 1).$$

$$\begin{aligned}\text{Gradient of } PR &= \frac{6-(-4)}{10-0} \\ &= \frac{10}{10} \\ &= 1\end{aligned}$$

and the gradient of the perpendicular is $m = -1$. Hence, the equation is

$$\begin{aligned}y - 4 = -(x-2) &\Rightarrow y - 4 = -x + 2 \\ &\Rightarrow \underline{\underline{y = -x + 6}}.\end{aligned}$$

(c) Show that line L passes through the midpoint of PR. (3)

Solution

$$x = 5 \Rightarrow y = -5 + 6 = 1$$

and, hence, the line L passes through the midpoint of PR.

2. Find the range of values for p such that (3)

$$x^2 - 2x + 3 - p = 0$$

has no real roots.

Solution

$a = 1$, $b = -2$, and $c = 3 - p$:

$$\begin{aligned}\text{`}b^2 - 4ac = 0\text{'} &\Rightarrow (-2)^2 - 4 \times 1 \times (3-p) < 0 \\ &\Rightarrow 4 - 4(3-p) < 0 \\ &\Rightarrow 1 < 3 - p \\ &\Rightarrow \underline{\underline{p < 2}}.\end{aligned}$$

3. (a) (i) Show that $(x+1)$ is a factor of (2)

$$2x^3 - 9x^2 + 3x + 14.$$

Solution

$$\begin{array}{r|rrrr} -1 & 2 & -9 & 3 & 14 \\ & \downarrow & -2 & 11 & -14 \\ \hline & 2 & -11 & 14 & 0 \end{array}$$

Hence, because there is no remainder, $(x+1)$ is a factor of $2x^3 - 9x^2 + 3x + 14$.

(ii) Hence solve the equation (3)

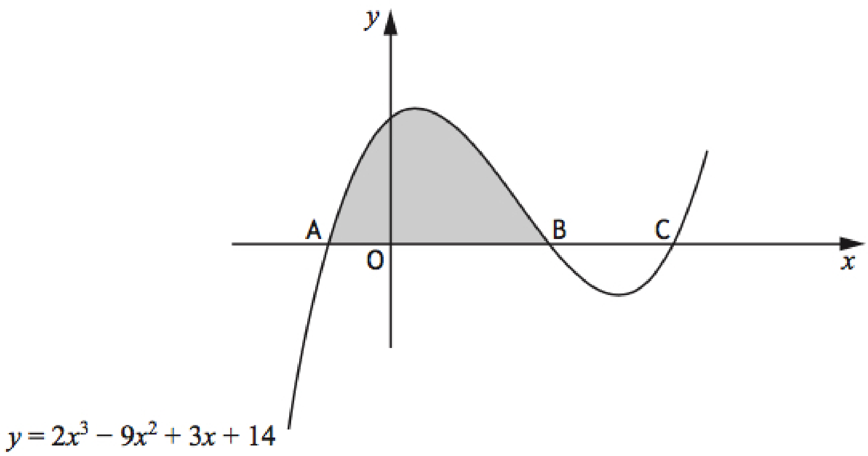
$$2x^3 - 9x^2 + 3x + 14.$$

Solution

$$2x^3 - 9x^2 + 3x + 14 = 0 \Rightarrow (x+1)(2x^2 - 11x + 14) = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -11 \\ \text{multiply to:} & (+2) \times (+14) = +28 \end{array}\right\} -7, -4$$

$$\begin{aligned} &\Rightarrow (x+1)(2x^2 - 7x - 4x + 14) = 0 \\ &\Rightarrow (x+1)[x(2x-7) - 2(2x-7)] = 0 \\ &\Rightarrow (x+1)(x-2)(2x-7) = 0 \\ &\Rightarrow x = -1,\ x = 2, \text{ or } x = 3\tfrac{1}{2}. \end{aligned}$$



(b) (i) Write down the coordinates of the points A and B. (1)

Solution
$\underline{\underline{A(-1,0)}}$ and $\underline{\underline{B(2,0)}}$.

(ii) Hence calculate the shaded area in the diagram. (4)

Solution

$$\begin{aligned}\text{Shaded area} &= \int_{-1}^{2} (2x^3 - 9x^2 + 3x + 14)\,\mathrm{d}x \\ &= \left[\tfrac{1}{2}x^4 - 3x^3 + \tfrac{3}{2}x^2 + 14x\right]_{x=-1}^{2} \\ &= (8 - 24 + 6 + 28) - \left(\tfrac{1}{2} + 3 + \tfrac{3}{2} - 14\right) \\ &= \underline{\underline{27}}.\end{aligned}$$

4. Circles C_1 and C_2 have equations

$$(x+5)^2 + (y-6)^2 = 9$$

and

$$x^2 + y^2 - 6x - 16 = 0$$

respectively.

(a) Write down the centres and radii of C_1 and C_2. (4)

Solution

C_1 has centre $\underline{\underline{(-5, 6)}}$ and radius 3. Now,

$$\begin{aligned} x^2 + y^2 - 6x - 16 = 0 &\Rightarrow x^2 - 6x + y^2 = 16 \\ &\Rightarrow (x^2 - 6x + 9) + y^2 = 16 + 9 \\ &\Rightarrow (x - 3)^2 + y^2 = 25; \end{aligned}$$

C_2 has centre $\underline{\underline{(3, 0)}}$ and radius 5.

(b) Show that C_1 and C_2 do not intersect. (3)

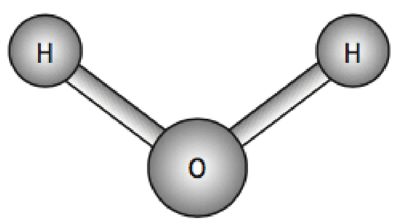
Solution

The distances between the centres is

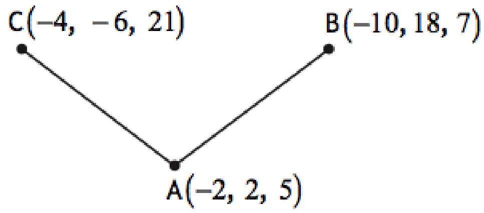
$$\begin{aligned} \sqrt{(-5-3)^2 + (6-0)^2} &= \sqrt{64 + 36} \\ &= \sqrt{100} \\ &= 10 \\ &> 3 + 5; \end{aligned}$$

hence, C_1 and C_2 $\underline{\underline{\text{do not intersect}}}$.

5. The picture shows a model of a water molecule.



Relative to suitable coordinate axes, the oxygen atom is positioned at point $A(-2, 2, 5)$. The two hydrogen atoms are positioned at points $B(-10, 18, 7)$ and $C(-4, -6, 21)$ as shown in the diagram below.



(a) (a) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in component form. (2)

Solution

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} -10 \\ 18 \\ 7 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} -8 \\ 16 \\ 2 \end{pmatrix}}}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} \\ &= -\begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} -4 \\ -6 \\ 21 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} -2 \\ -8 \\ 16 \end{pmatrix}}}.\end{aligned}$$

(b) Hence, or otherwise, find the size of angle BAC. (4)

Solution

$$\begin{aligned}
&\overrightarrow{AB}.\overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}| \cos\ BAC \\
\Rightarrow\ & 16 - 128 + 32 = \sqrt{(-8)^2 + 16^2 + 2^2}\sqrt{(-2)^2 + (-8)^2 + 16^2} \cos\ BAC \\
\Rightarrow\ & -80 = 18 \cdot 18 \cdot \cos\ BAC \\
\Rightarrow\ & \cos\ BAC = -\tfrac{20}{81} \\
\Rightarrow\ & \angle BAC = 104.294\,948\,6 \text{ (FCD)} \\
\Rightarrow\ & \underline{\underline{\angle BAC = 104 \text{ (3 sf)}}}.
\end{aligned}$$

6. Scientists are studying the growth of a strain of bacteria. The number of bacteria present is given by the formula

$$B(t) = 200\mathrm{e}^{0.107t},$$

where t represents the number of hours since the study began.

(a) State the number of bacteria present at the start of the study. (1)

Solution

$\underline{\underline{200}}$.

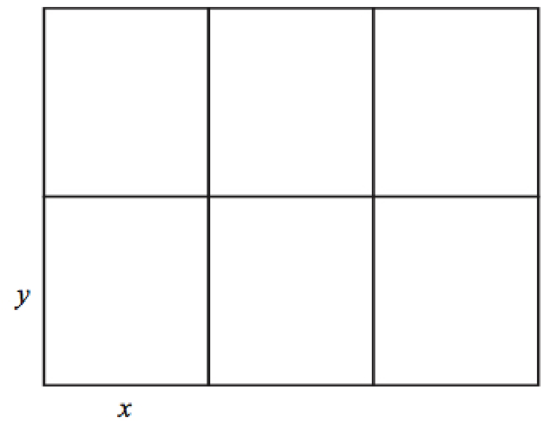
(b) Calculate the time taken for the number of bacteria to double. (4)

Solution

$$\begin{aligned}
400 = 200\mathrm{e}^{0.107t} &\Rightarrow 2 = \mathrm{e}^{0.107t} \\
&\Rightarrow 0.107t = \ln 2 \\
&\Rightarrow t = \frac{\ln 2}{0.107} \\
&\Rightarrow t = 6.478\,011\,033 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{t = 6.48 \text{ hours (3 sf)}}}.
\end{aligned}$$

7. A council is setting aside an area of land to create six fenced plots where local residents can grow their own food.

Each plot will be a rectangle measuring x metres by y metres as shown in the diagram.



The area of land being set aside is 108 m^2.

(a) Show that the total length of fencing, L metres, is given by (3)

$$L(x) = 9x + \frac{144}{x}.$$

Solution

Well, let A be the area of land. Now,

$$A = 6xy \Rightarrow y = \frac{108}{6x} = \frac{18}{x}$$

and

$$L = 9x + 8y \Rightarrow L = 9x + 8\left(\frac{18}{x}\right)$$
$$\Rightarrow \underline{\underline{L = 9x + \frac{144}{x}}},$$

as required.

(b) Find the value of x that minimises the length of fencing required. (6)

Solution

$$L(x) = 9x + \frac{144}{x} \Rightarrow L(x) = 9x + 144x^{-1}$$
$$\Rightarrow L'(x) = 9 - 144x^{-2}.$$

Now,

$$\begin{aligned} L'(x) = 0 &\Rightarrow 9 - 144x^{-2} = 0 \\ &\Rightarrow 9 = \frac{144}{x^2} \\ &\Rightarrow x^2 = 16 \\ &\Rightarrow \underline{\underline{x = 4 \text{ (because the lengths are positive)}}}. \end{aligned}$$

Why? Well,

$$L'(x) = 9 - 144x^{-2} \Rightarrow L''(x) = 288x^{-3}$$

and

$$L''(4) = 4\tfrac{1}{2} > 0$$

so $x = 4$ is a minimum.

8. (a) Express (4)

$$5 \cos x - 2 \sin x$$

in the form $k \cos(x + a)$, where $k > 0$ and $0 < a < 2\pi$.

Solution

$$\begin{aligned} k \cos(x + a) &\equiv k \cos x \cos a - k \sin x \sin a \\ &\equiv 5 \cos x - 2 \sin x \end{aligned}$$

and so

$$k \sin a = 2,\ k \cos a = 5.$$

Now,

$$\begin{aligned} k &= \sqrt{k^2} \\ &= \sqrt{(k \sin a)^2 + (k \cos a)^2} \\ &= \sqrt{2^2 + 5^2} \\ &= \sqrt{29} \end{aligned}$$

and

$$\begin{aligned} \tan a = \frac{k \sin a}{k \cos a} \Rightarrow \tan a &= \tfrac{2}{5} \\ &= 0.380\,506\,377\,1 \text{ (FCD)}. \end{aligned}$$

Hence,

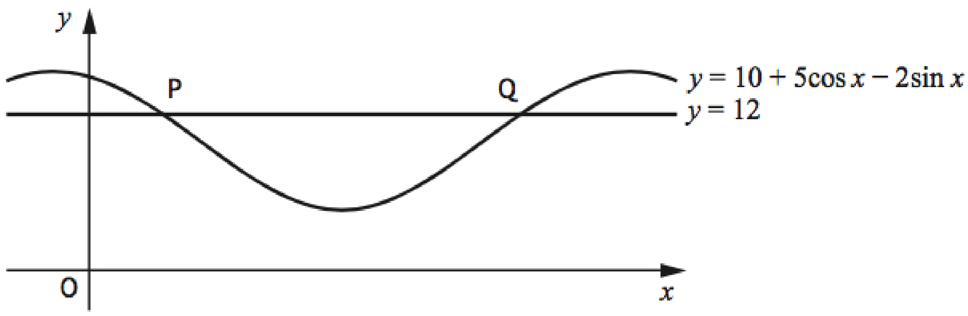
$$5 \cos x - 2 \sin x = \underline{\underline{\sqrt{29} \cos(x + 0.380\ldots)}}.$$

The diagram shows a sketch of part of the graph of

$$y = 10 + 5\cos x - 2\sin x$$

and the line with equation $y = 12$.

The line cuts the curve at the points P and Q.



(b) Find the x-coordinates of P and Q. (4)

Solution

$$\begin{aligned} 10 + 5\cos x - 2\sin x = 12 &\Rightarrow 5\cos x - 2\sin x = 2 \\ &\Rightarrow \sqrt{29}\cos(x + 0.380\ldots) = 2 \\ &\Rightarrow \cos(x + 0.380\ldots) = \tfrac{2}{\sqrt{29}} \\ &\Rightarrow x + 0.380\ldots = 1.190\ldots,\ 5.092\ldots \\ &\Rightarrow x = 0.809\,783\,572\,6,\ \tfrac{3}{2}\pi \\ &\Rightarrow \underline{\underline{x = 0.810\ (3\text{ sf}),\ \tfrac{3}{2}\pi}}. \end{aligned}$$

9. For a function f, defined on a suitable domain, it is known that: (4)

- $f'(x) = \dfrac{2x + 1}{\sqrt{x}}$ and
- $f(9) = 40$.

Express $f(x)$ in terms of x.

Solution

$$\mathrm{f}'(x) = \frac{2x+1}{\sqrt{x}} \Rightarrow \mathrm{f}'(x) = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$
$$\Rightarrow \mathrm{f}(x) = \tfrac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c.$$

Now,

$$\begin{aligned} \mathrm{f}(9) = 40 &\Rightarrow \tfrac{4}{3}(9^{\frac{3}{2}}) + 2(9^{\frac{1}{2}}) + c = 40 \\ &\Rightarrow \tfrac{4}{3}(27) + 2(3) + c = 40 \\ &\Rightarrow 36 + 6 + c = 40 \\ &\Rightarrow c = -2 \end{aligned}$$

and, hence,

$$\underline{\underline{\mathrm{f}(x) = \tfrac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 2}}.$$

10. (a) Given that (2)

$$y = (x^2+7)^{\frac{1}{2}},$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$y = (x^2+7)^{\frac{1}{2}} \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = x(x^2+7)^{-\frac{1}{2}}}}.$$

(b) Hence find (1)

$$\int \frac{4x}{\sqrt{x^2+7}}\,\mathrm{d}x.$$

Solution

$$\begin{aligned} \int \frac{4x}{\sqrt{x^2+7}}\,\mathrm{d}x &= 4\int x(x^2+7)^{-\frac{1}{2}}\,\mathrm{d}x \\ &= \underline{\underline{4(x^2+7)^{\frac{1}{2}} + c}}. \end{aligned}$$

11. (a) Show that (4)

$$\sin 2x \tan x \equiv 1 - \cos 2x,$$

where $\frac{1}{2}\pi < x < \frac{3}{2}\pi$.

Solution

$$\begin{aligned}\sin 2x \tan x &\equiv (2 \sin x \cos x)\left(\frac{\sin x}{\cos x}\right) \\ &\equiv 2 \sin x^2 \\ &\equiv 1 - (1 - 2 \sin x^2) \\ &\equiv \underline{\underline{1 - \cos 2x}},\end{aligned}$$

as required.

(b) Given that (2)

$$f(x) = \sin 2x \tan x,$$

find $f'(x)$.

Solution

$$\begin{aligned}f(x) = \sin 2x \tan x &\Rightarrow f(x) = 1 - \cos 2x \\ &\Rightarrow \underline{\underline{f'(x) = 2 \sin 2x}}.\end{aligned}$$