Dr Oliver Mathematics
Applied Mathematics: Mechanics or Statistics
Section B
2005 Paper
1 hour

The total number of marks available is 32.
You must write down all the stages in your working.

1. Differentiate, and simplify as appropriate,

   (a) $\text{f}(x) = \exp(\tan \frac{1}{2}x)$, where $-\pi < x < \pi$, (3)

   Solution

   $$\begin{aligned}\text{f}(x) = \exp(\tan \tfrac{1}{2}x) &\Rightarrow \text{f}'(x) = \exp(\tan \tfrac{1}{2}x) \cdot \sec^2 \tfrac{1}{2}x \cdot \tfrac{1}{2} \\ &\Rightarrow \underline{\underline{\text{f}'(x) = \tfrac{1}{2} \sec^2 \tfrac{1}{2}x \, \exp(\tan \tfrac{1}{2}x)}}.\end{aligned}$$

   (b) $\text{g}(x) = (x^3 + 1)\ln(x^3 + 1)$, where $x > 0$. (3)

   Solution

   $$u = x^3 + 1 \Rightarrow \frac{\text{d}u}{\text{d}x} = 3x^2$$
   $$v = \ln(x^3 + 1) \Rightarrow \frac{\text{d}v}{\text{d}x} = \frac{3x^2}{x^3 + 1}$$

   $$\begin{aligned}\text{g}(x) = (x^3 + 1)\ln(x^3 + 1) &\Rightarrow \text{g}'(x) = 3x^2 \cdot \ln(x^3 + 1) + (x^3 + 1) \cdot \frac{3x^2}{(x^3 + 1)} \\ &\Rightarrow \underline{\underline{\text{g}'(x) = 3x^2 \, \ln(x^3 + 1) + 3x^2}}.\end{aligned}$$

2. Given that (3)

   $$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix},$$

   show that

   $$\mathbf{A}^2 - \mathbf{A} = k\mathbf{I}$$

   for a suitable value of k, where $\mathbf{I}$ is the 2×2 unit matrix.

Solution

$$\begin{aligned}\mathbf{A}^2 - \mathbf{A} &= \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ &= 2\mathbf{I};\end{aligned}$$

hence, $\underline{\underline{k = 2}}$.

3. A curve is defined by the parametric equations

$$x = 5t^2 - 5 \text{ and } y = 3t^3.$$

(a) Find the value of t corresponding to the point $(0, -3)$. (2)

Solution

Select y (why?):

$$\begin{aligned}3t^3 = -3 &\Rightarrow t^3 = -1 \\ &\Rightarrow \underline{\underline{t = -1}}.\end{aligned}$$

(b) Calculate the gradient of the curve at this point. (3)

Solution

$$\begin{aligned}x = 5t^2 - 5 &\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 10t \\ y = 3t^3 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 9t^2.\end{aligned}$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{9t^2}{10t} \\ &= \tfrac{9}{10}t.\end{aligned}$$

Finally,

$$t = -1 \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{9}{10}}}.$$

4. Expand and simplify (3)

$$\left(2a - \frac{3}{a}\right)^4.$$

Solution

$$\begin{aligned}
&\left(2a - \frac{3}{a}\right)^4 \\
= \quad & (2a)^4 + \binom{4}{1}(2a)^3\left(-\frac{3}{a}\right) + \binom{4}{2}(2a)^2\left(-\frac{3}{a}\right)^2 + \binom{4}{3}(2a)\left(-\frac{3}{a}\right)^3 + \left(-\frac{3}{a}\right)^4 \\
&= \underline{\underline{16a^4 - 96a^2 + 216 - \frac{216}{a^2} + \frac{81}{a^4}}}.
\end{aligned}$$

5. (a) Express (3)

$$\frac{x^2+3}{x(1+x^2)}$$

in partial fractions.

Solution

$$\begin{aligned}
\frac{x^2+3}{x(1+x^2)} &\equiv \frac{A}{x} + \frac{B+Cx}{1+x^2} \\
&\equiv \frac{A(1+x^2) + x(B+Cx)}{x(1+x^2)}
\end{aligned}$$

and so

$$x^2 + 3 \equiv A(1+x^2) + x(B+Cx).$$

$\underline{x=0}$: $3 = A$.
$\underline{x=1}$: $4 = 2A + B + C \Rightarrow B + C = -2 \quad (1)$.

$\underline{x = -1}$: $4 = 2A - B + C \Rightarrow -B + C = -2 \quad (2)$.
Add (1) + (2):

$$2C = -4 \Rightarrow C = -2$$
$$\Rightarrow B = 0.$$

Hence,

$$\frac{x^2+3}{x(1+x^2)} \equiv \underline{\underline{\frac{3}{x} - \frac{2x}{1+x^2}}}.$$

(b) Hence obtain (3)

$$\int_{\frac{1}{2}}^{1} \frac{x^2+3}{x(1+x^2)}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_{\frac{1}{2}}^{1} \frac{x^2+3}{x(1+x^2)}\,\mathrm{d}x &= \int_{\frac{1}{2}}^{1} \left(\frac{3}{x} - \frac{2x}{1+x^2}\right)\,\mathrm{d}x \\ &= \left[3\ln|x| - \ln|1+x^2|\right]_{x=\frac{1}{2}}^{1} \\ &= (3\ln 1 - \ln 2) - \left(3\ln\tfrac{1}{2} - \ln\tfrac{5}{4}\right) \\ &= -\ln 2 - \ln\tfrac{1}{8} + \ln\tfrac{5}{4} \\ &= \ln\left(\frac{\frac{5}{4}}{2\cdot\frac{1}{8}}\right) \\ &= \ln\left(\frac{\frac{5}{4}}{\frac{1}{4}}\right) \\ &= \underline{\underline{\ln 5}}.\end{aligned}$$

6. (a) Given the differential equation (4)

$$\sin x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\cos x = 0,$$

find the general solution, expressing y explicitly in terms of x.

Solution

$$\begin{aligned}
\sin x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y\cos x = 0 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} - 2y\,\frac{\cos x}{\sin x} = 0 \\
&\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \left(-2\frac{\cos x}{\sin x}\right) y = 0
\end{aligned}$$

$$\begin{aligned}
\text{IF} &= \mathrm{e}^{\int -2\frac{\cos x}{\sin x}\,\mathrm{d}x} \\
&= \mathrm{e}^{\int -2\frac{\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)}{\sin x}\,\mathrm{d}x} \\
&= \mathrm{e}^{-2\ln\sin x} \\
&= \mathrm{e}^{\ln\sin^{-2} x} \\
&= \sin^{-2} x
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \sin^{-2} x \frac{\mathrm{d}y}{\mathrm{d}x} + (-2\sin^{-3} x\cos x)y = 0 \\
&\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\sin^{-2} x) = 0 \\
&\Rightarrow y\sin^{-2} x = c \\
&\Rightarrow \underline{\underline{y = c\sin^2 x}}.
\end{aligned}$$

(b) Find the general solution of (5)

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y\cos x = 3\sin^3 x.$$

Solution

$$\begin{aligned}
\sin x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y\cos x = 3\sin^3 x &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \left(-2\frac{\cos x}{\sin x}\right) y = 3\sin^2 x \\
&\Rightarrow \sin^{-2} x \frac{\mathrm{d}y}{\mathrm{d}x} + (-2\sin^{-3} x\cos x)y = 3 \\
&\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\sin^{-2} x) = 3 \\
&\Rightarrow y\sin^{-2} x = 3x + A \\
&\Rightarrow \underline{\underline{y = (3x + A)\sin^2 x}}.
\end{aligned}$$