Dr Oliver Mathematics
Further Mathematics: Further Pure Mathematics 1 (Paper 3A)
November 2021: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. The ellipse E has equation
$$\frac{x^2}{36} + \frac{y^2}{20} = 1.$$
Find

(a) the coordinates of the foci of E, (3)

Solution
Well,
$$\begin{aligned} b^2 = a^2(1 - e^2) &\Rightarrow 20 = 36(1 - e^2) \\ &\Rightarrow \tfrac{5}{9} = 1 - e^2 \\ &\Rightarrow e^2 = \tfrac{4}{9} \\ &\Rightarrow e = \tfrac{2}{3} \end{aligned}$$
and the foci are
$$(\pm 6 \times \tfrac{2}{3}, 0) = \underline{\underline{(\pm 4, 0)}}.$$

(b) the equations of the directrices of E. (2)

Solution
$$x = \pm \frac{6}{\frac{2}{3}} \Rightarrow \underline{\underline{x = \pm 9}}.$$

2. (a) Use the substitution (5)
$$t = \tan \tfrac{1}{2}x$$
to prove the identity
$$\frac{\sin x - \cos x + 1}{\sin x + \cos x + 1} \equiv \sec x + \tan x,\ x \neq \tfrac{1}{2}n\pi,\ n \in \mathbb{Z}.$$

Solution

Now,

$$\sin x = \frac{2t}{1+t^2} \text{ and } \cos x = \frac{1-t^2}{1+t^2}$$

and

$$\begin{aligned}
\frac{\sin x - \cos x + 1}{\sin x + \cos x + 1} &\equiv \frac{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} + 1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1} \\
&\equiv \frac{2t - (1-t^2) + (1+t^2)}{2t + (1-t^2) + (1+t^2)} \\
&\equiv \frac{2t^2 + 2t}{2t - 2t^2} \\
&\equiv \frac{2t(t+1)}{2t(1-t)} \\
&\equiv \frac{t+1}{1-t}.
\end{aligned}$$

Hmm. Well, start at the other end end, $\sec x + \tan x$, and see where that takes us:

$$\begin{aligned}
\sec x + \tan x &\equiv \frac{1}{\cos x} + \frac{\sin x}{\cos x} \\
&\equiv \frac{1}{\frac{1-t^2}{1+t^2}} + \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}} \\
&\equiv \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} \\
&\equiv \frac{1+2t+t^2}{1-t^2} \\
&\equiv \frac{(1+t)^2}{(1-t)(1+t)} \\
&\equiv \frac{1+t}{1-t};
\end{aligned}$$

exactly the same! Hence,

$$\frac{\sin x - \cos x + 1}{\sin x + \cos x + 1} \equiv \underline{\underline{\sec x + \tan x}}.$$

(b) Use the substitution (5)

$$t = \tan \tfrac{1}{2}\theta$$

to determine the exact value of

$$\int_0^{\frac{1}{2}\pi} \left(\frac{5}{4 + 2\cos\theta} \right) \mathrm{d}\theta,$$

giving your answer in simplest form.

Solution

Well,

$$\theta = 0 \Rightarrow t = \tan 0 = 0,$$
$$\theta = \tfrac{1}{2}\pi \Rightarrow t = \tan \tfrac{1}{4}\pi = 1$$

and

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{2}{1+t^2} \Rightarrow \mathrm{d}\theta = \frac{2}{1+t^2}\,\mathrm{d}t.$$

Now,

$$\begin{aligned}
\int_0^{\frac{1}{2}\pi} \left(\frac{5}{4 + 2\cos\theta} \right) \mathrm{d}\theta &= \int_0^1 \left(\frac{5}{4 + 2\left(\frac{1-t^2}{1+t^2}\right)} \right) \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int_0^1 \left(\frac{10}{4(1+t^2) + 2(1-t^2)} \right) \mathrm{d}t \\
&= \int_0^1 \left(\frac{10}{6 + 2t^2} \right) \mathrm{d}t \\
&= \int_0^1 \left(\frac{5}{3 + t^2} \right) \mathrm{d}t \\
&= 5\int_0^1 \left(\frac{1}{3 + t^2} \right) \mathrm{d}t \\
&= 5\left[\tfrac{1}{\sqrt{3}} \arctan(\tfrac{1}{\sqrt{3}}t) \right]_{t=0}^1 \\
&= 5\left(\tfrac{1}{\sqrt{3}} \arctan(\tfrac{1}{\sqrt{3}}) - 0 \right) \\
&= 5\left(\tfrac{1}{\sqrt{3}} \times \tfrac{1}{6}\pi \right) \\
&= \underline{\underline{\frac{5\pi}{6\sqrt{3}} \text{ or } \frac{5\pi\sqrt{3}}{18}}}.
\end{aligned}$$

3. Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$ where (8)

$$\mathrm{f}(x) = \frac{x}{|x| - 2}.$$

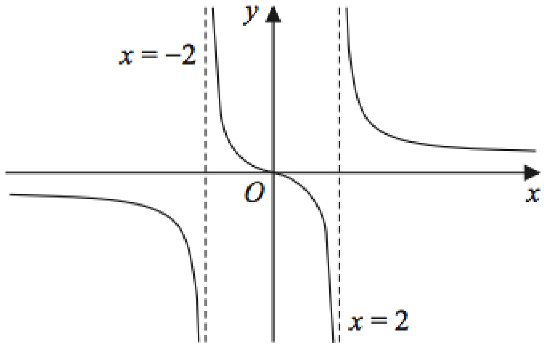


Figure 1: $\mathrm{f}(x) = \dfrac{x}{|x| - 2}$

Use algebra to determine the values of x for which

$$2x - 5 > \frac{x}{|x| - 2}.$$

Solution

$$\begin{aligned} x = 0 \Rightarrow 2x - 5 &= \frac{x}{-x - 2} \\ &\Rightarrow (2x - 5)(-x - 2) = x \\ &\Rightarrow -2x^2 + x + 10 = x \\ &\Rightarrow -2x^2 + 10 = 0 \\ &\Rightarrow -2x^2 = -10 \\ &\Rightarrow x^2 5 \\ &\Rightarrow x = \pm\sqrt{5} \end{aligned}$$

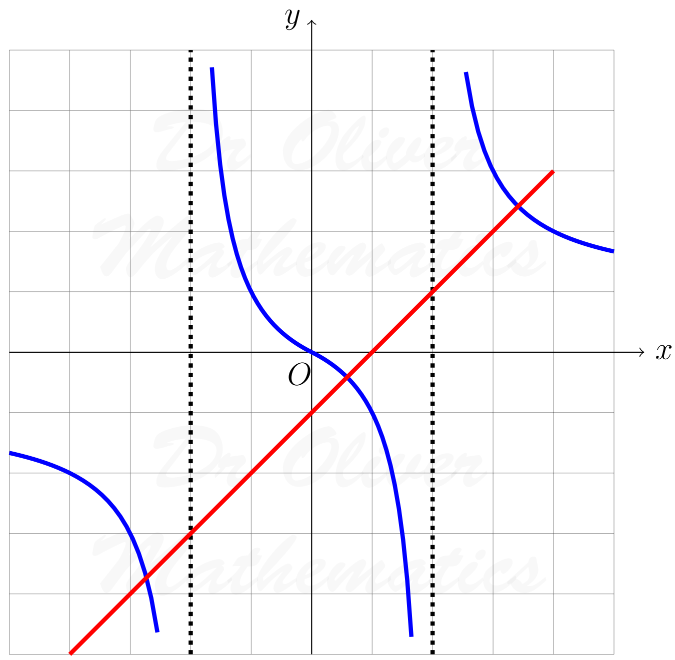
and

$$\begin{aligned} x = 0 \Rightarrow 2x - 5 &= \frac{x}{x-2} \\ &\Rightarrow (2x-5)(x-2) = x \\ &\Rightarrow 2x^2 - 9x + 10 = x \\ &\Rightarrow 2x^2 - 10x + 10 = 0 \\ &\Rightarrow 2(x^2 - 5x + 5) = 0 \\ &\Rightarrow 2[(x^2 - 5x + 6\tfrac{1}{4}) + 5 - 6\tfrac{1}{4}] = 0 \\ &\Rightarrow 2[(x - \tfrac{5}{2})^2 - \tfrac{5}{4}] = 0 \\ &\Rightarrow (x - \tfrac{5}{2})^2 = \tfrac{5}{4} \\ &\Rightarrow x - \tfrac{5}{2} = \pm\tfrac{\sqrt{5}}{2} \\ &\Rightarrow x = \tfrac{5 \pm \sqrt{5}}{2}. \end{aligned}$$

So, we have four crucial points:

$$x = -\sqrt{5},\ x = \tfrac{5-\sqrt{5}}{2},\ x = \tfrac{5+\sqrt{5}}{2},\ \text{and } x = \sqrt{5}.$$

But do we?



No! It seems that $x = \sqrt{5}$ is missing and that is because

$$2x - 5 > \frac{x}{|x| - 2}$$

is true for $x = \frac{5-\sqrt{5}}{2}$.

Hence,

$$\underline{\underline{\left\{x \in \mathbb{R} : -\sqrt{5} < x < -2\right\} \cup \left\{x \in \mathbb{R} : \frac{5-\sqrt{5}}{2} < x < 2\right\} \cup \left\{x \in \mathbb{R} : x > \frac{5-\sqrt{5}}{2}\right\}}}.$$

4. A small aircraft is landing in a field.

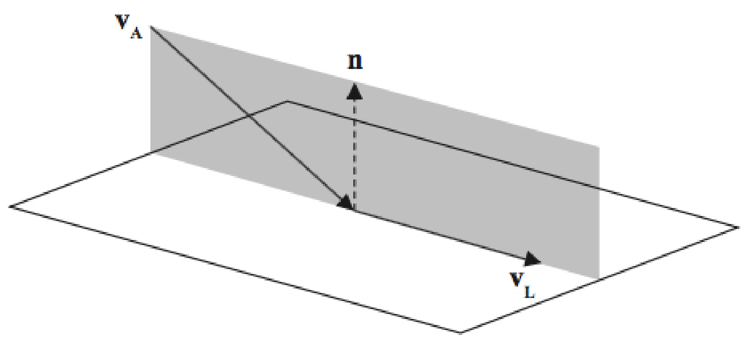


Figure 2: a small aircraft

In a model for the landing the aircraft travels in different straight lines before and after it lands, as shown in Figure 2.

The vector $\mathbf{v}_A$ is in the direction of travel of the aircraft as it approaches the field.

The vector $\mathbf{v}_L$ is in the direction of travel of the aircraft after it lands.

With respect to a fixed origin, the field is modelled as the plane with equation

$$x - 2y + 25z = 0$$

and

$$\mathbf{v}_A = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}.$$

(a) Write down a vector $\mathbf{n}$ that is a normal vector to the field. (1)

> **Solution**
> E.g.,
> $$\mathbf{n} = \underline{\underline{\begin{pmatrix} 1 \\ -2 \\ 25 \end{pmatrix}}}.$$

(b) Show that (2)

$$\mathbf{n} \times \mathbf{v}_A = \lambda \begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix},$$

where λ is a constant to be determined.

> **Solution**
> $$\begin{aligned} \mathbf{n} \times \mathbf{v}_A &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 25 \\ 3 & -2 & -1 \end{vmatrix} \\ &= (2+50)\mathbf{i} - (-1-75)\mathbf{j} + (-2+6)\mathbf{k} \\ &= 52\mathbf{i} + 76\mathbf{j} + 4\mathbf{k} \\ &= \underline{\underline{4(13\mathbf{i} + 19\mathbf{j} + \mathbf{k})}}; \end{aligned}$$
> hence, $\underline{\underline{\lambda = 4}}$.

When the aircraft lands it remains in contact with the field and travels in the direction $\mathbf{v}_L$.

The vector $\mathbf{v}_L$ is in the same plane as both $\mathbf{v}_A$ and $\mathbf{n}$ as shown in Figure 2.

(c) Determine a vector which has the same direction as $\mathbf{v}_L$. (3)

> **Solution**
> E.g.,
> $$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 13 & 19 & 1 \\ 1 & -2 & 25 \end{vmatrix} &= (475+2)\mathbf{i} - (325-1)\mathbf{j} + (-26-19)\mathbf{k} \\ &= \underline{\underline{477\mathbf{i} - 324\mathbf{j} - 45\mathbf{k}}}. \end{aligned}$$

(d) State a limitation of the model. (1)

Solution
E..g., the plane would have some side-to-side movement.

5. The parabola C has equation

$$y^2 = 32x$$

and the hyperbola H has equation

$$\frac{x^2}{36} - \frac{y^2}{9} = 1.$$

(a) Write down the equations of the asymptotes of H. (1)

Solution

$$y = \pm\tfrac{3}{6}x \Rightarrow \underline{\underline{y = \pm\tfrac{1}{2}x}}.$$

The line l_1 is normal to C and parallel to the asymptote of H with positive gradient.

The line l_2 is normal to C and parallel to the asymptote of H with negative gradient.

(b) Determine (4)

(i) an equation for l_1,

Solution
Well, we begin with implicit differentiation:

$$\begin{aligned} y^2 = 32x &\Rightarrow 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 32 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{16}{y} \end{aligned}$$

and

$$m_{\text{normal}} = -\tfrac{1}{16}y.$$

But we know that $y = \pm\frac{1}{2}x$ are the asymptotes:

$$\begin{aligned} -\tfrac{1}{16}y = \tfrac{1}{2} &\Rightarrow y = -8 \\ &\Rightarrow 64 = 32x \\ &\Rightarrow x = 2; \end{aligned}$$

and

$$-\tfrac{1}{16}y = -\tfrac{1}{2} \Rightarrow y = 8$$
$$\Rightarrow 64 = 32x$$
$$\Rightarrow x = 2;$$

so, the two points are $(2, -8)$ and $(2, 8)$.

The equation of l_1 is

$$y - (-8) = \tfrac{1}{2}(x - 2) \Rightarrow y + 8 = \tfrac{1}{2}x - 1$$
$$\Rightarrow \underline{\underline{y = \tfrac{1}{2}x - 9}}.$$

(ii) an equation for l_2.

Solution
The equation of l_1 is

$$y - 8 = -\tfrac{1}{2}(x - 2) \Rightarrow y - 8 = -\tfrac{1}{2}x + 1$$
$$\Rightarrow \underline{\underline{y = -\tfrac{1}{2}x + 9}}.$$

The lines l_1 and l_2 meet H at the points P and Q respectively.

(c) Find the area of the triangle OPQ, where O is the origin. (4)

Solution

$$\begin{aligned}\frac{x^2}{36} - \frac{y^2}{9} = 1 &\Rightarrow \frac{x^2}{36} - \frac{(-\frac{1}{2}x + 9)^2}{9} = 1 \\ &\Rightarrow \frac{x^2}{36} - \frac{(\frac{1}{4}x^2 - 9x + 81)}{9} = 1 \\ &\Rightarrow \frac{x^2}{36} - \frac{(x^2 - 36x + 324)}{36} = 1 \\ &\Rightarrow x^2 - (x^2 - 36x + 324) = 36 \\ &\Rightarrow 36x - 324 = 36 \\ &\Rightarrow 36x = 360 \\ &\Rightarrow x = 10 \\ &\Rightarrow \frac{10^2}{36} - \frac{y^2}{9} = 1 \\ &\Rightarrow \frac{y^2}{9} = \frac{16}{9} \\ &\Rightarrow y^2 = 16 \\ &\Rightarrow y = \pm 4;\end{aligned}$$

so, $P(10, 4)$ and $Q(10, -4)$. Finally,

$$\begin{aligned}\text{area} &= \tfrac{1}{2} \times 10 \times (4 - (4)) \\ &= \tfrac{1}{2} \times 10 \times 8 \\ &= \underline{\underline{40}}.\end{aligned}$$

6. Given that

$$y = (1 + \ln x)^2,\ x > 0,$$

(a) show that (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2\ln x}{x^2}.$$

Solution

Well,

$$\begin{aligned}y = (1 + \ln x)^2 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2 \times (1 + \ln x) \times \frac{1}{x} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2(1 + \ln x)}{x}\end{aligned}$$

$$u = 2(1 + \ln x) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{x}$$
$$v = x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{x \times \frac{2}{x} - 2(1 + \ln x) \times 1}{x^2}$$
$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2 - 2(1 + \ln x)}{x^2}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2\ln x}{x^2}}};$$

as required.

(b) Hence find (2)

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}.$$

Solution

Now,

$$u = 2\ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{x}$$
$$v = x^2 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x$$

and

$$\begin{aligned}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= -\frac{x^2 \times \frac{2}{x} - 2\ln x \times 2x}{(x^2)^2} \\ &= -\frac{2x - 4x\ln x}{x^4} \\ &= -\frac{x(2 - 4\ln x)}{x^4} \\ &= \underline{\underline{\frac{4\ln x - 2}{x^3}}}.\end{aligned}$$

(c) Determine the Taylor series expansion about $x = 1$ of (3)

$$(1 + \ln x)^2$$

in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$.

Give each coefficient in simplest form.

Solution

Now,

$$\begin{aligned} y(1) &= 1, \\ y'(1) &= 2, \\ y''(1) &= 0, \\ y'''(1) &= -2 \end{aligned}$$

and the Taylor series is

$$\begin{aligned} y &= 1 + 2(x-1) + 0 + \tfrac{1}{3!}(-2)(x-1)^3 + \ldots \\ &= \underline{\underline{1 + 2(x-1) - \tfrac{1}{3}(x-1)^3 + \ldots}}. \end{aligned}$$

(d) Use this series expansion to evaluate (3)

$$\lim_{x \to 1} \frac{2x - 1 - (1 + \ln x)^2}{(x-1)^3},$$

explaining your reasoning clearly.

Solution

$$\begin{aligned} \frac{2x - 1 - (1 + \ln x)^2}{(x-1)^3} &= \frac{2x - 1 - [1 + 2(x-1) - \frac{1}{3}(x-1)^3 + \ldots]}{(x-1)^3} \\ &= \frac{2x - 1 - 1 - 2(x-1) + \frac{1}{3}(x-1)^3 + \ldots}{(x-1)^3} \\ &= \frac{\frac{1}{3}(x-1)^3 + \ldots}{(x-1)^3} \\ &= \tfrac{1}{3} + \ldots \end{aligned}$$

and, finally,

$$\lim_{x \to 1} \frac{2x - 1 - (1 + \ln x)^2}{(x-1)^3} = \underline{\underline{\tfrac{1}{3}}},$$

as all remaining terms will become zero in the limit.

7. With respect to a fixed origin O, the line l has equation (7)

$$(\mathbf{r} - (12\mathbf{i} + 16\mathbf{j} - 8\mathbf{k})) \times (9\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}) = \mathbf{0}.$$

The point A lies on l such that the direction cosines of OA with respect to the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ axes are $\frac{3}{7}$, β, and γ.

Determine the coordinates of the point A.

Solution

Well, the position vector is

$$\overrightarrow{OA} = (12\mathbf{i} + 16\mathbf{j} - 8\mathbf{k}) + \lambda(9\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}),$$

where λ is some scalar. Now,

$$\overrightarrow{OA} = \begin{pmatrix} 12 + 9\lambda \\ 16 + 6\lambda \\ -8 + 2\lambda \end{pmatrix}$$

and so we have

$$\begin{aligned}
& \frac{12 + 9\lambda}{\sqrt{(12 + 9\lambda)^2 + (16 + 6\lambda)^2 + (-8 + 2\lambda)^2}} = \tfrac{3}{7} \\
\Rightarrow\quad & \frac{(12 + 9\lambda)^2}{(12 + 9\lambda)^2 + (16 + 6\lambda)^2 + (-8 + 2\lambda)^2} = (\tfrac{3}{7})^2 \\
\Rightarrow\quad & \frac{(12 + 9\lambda)^2}{(12 + 9\lambda)^2 + (16 + 6\lambda)^2 + (-8 + 2\lambda)^2} = \tfrac{9}{49} \\
\Rightarrow\quad & \frac{(144 + 216\lambda + 81\lambda^2)}{(144 + 216\lambda + 81\lambda^2) + (256 + 192\lambda + 36\lambda^2) + (64 - 32\lambda + 4\lambda^2)} = \tfrac{9}{49} \\
\Rightarrow\quad & \frac{(144 + 216\lambda + 81\lambda^2)}{(121\lambda^2 + 376\lambda + 464)} = \tfrac{9}{49} \\
\Rightarrow\quad & 49(144 + 216\lambda + 81\lambda^2) = 9(121\lambda^2 + 376\lambda + 464) \\
\Rightarrow\quad & 7\,056 + 10\,584\lambda + 3\,969\lambda^2 = 1\,089\lambda^2 + 3\,384\lambda + 4\,176 \\
\Rightarrow\quad & 2\,880\lambda^2 + 7\,200\lambda + 2\,880 = 0 \\
\Rightarrow\quad & 1\,440(2\lambda^2 + 5\lambda + 2) = 0
\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +5 \\ \text{multiply to:} & (+2)\times(+2) = +4\end{array}\right\} +1,\ +4$$

$$\begin{aligned}
&\Rightarrow\quad 1\,440[2\lambda^2 + 4\lambda + \lambda + 2] = 0\\
&\Rightarrow\quad 1\,440[2\lambda(\lambda + 2) + 1(\lambda + 2)] = 0\\
&\Rightarrow\quad 1\,440(2\lambda + 1)(\lambda + 2) = 0\\
&\Rightarrow\quad \lambda = -2 \text{ or } \lambda = -\tfrac{1}{2}.
\end{aligned}$$

$\underline{\lambda = -2}$:

$$\begin{aligned}
\overrightarrow{OA} &= (12\mathbf{i} + 16\mathbf{j} - 8\mathbf{k}) - 2(9\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})\\
&= -6\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}
\end{aligned}$$

and

$$\begin{aligned}
\frac{-6}{\sqrt{(-6)^2 + (4)^2 + (-12)^2}} &= \frac{-6}{\sqrt{196}}\\
&= \frac{-6}{14}\\
&= -\tfrac{3}{7}.
\end{aligned}$$

Hmm: right magnitude, wrong sign!

$\underline{\lambda = -\tfrac{1}{2}}$:

$$\begin{aligned}
\overrightarrow{OA} &= (12\mathbf{i} + 16\mathbf{j} - 8\mathbf{k}) - \tfrac{1}{2}(9\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})\\
&= 7.5\mathbf{i} + 13\mathbf{j} - 9\mathbf{k}
\end{aligned}$$

and

$$\begin{aligned}
\frac{7.5}{\sqrt{(7.5)^2 + (13)^2 + (-9)^2}} &= \frac{7.5}{\sqrt{306.25}}\\
&= \frac{7.5}{17.5}\\
&= \tfrac{3}{7}.
\end{aligned}$$

Hence, $\underline{\underline{A(7.5, 13, -9)}}$.

8. A community is concerned about the rising level of pollutant in its local pond and applies a chemical treatment to stop the increase of pollutant.

The concentration, x parts per million (ppm), of the pollutant in the pond water t days after the chemical treatment was applied, is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3 + \cosh t}{3x^2 \cosh t} - \tfrac{1}{3} x \tanh t \quad \text{(I)}.$$

When the chemical treatment was applied the concentration of pollutant was 3 ppm.

(a) Use the iteration formula (4)

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{(y_{n+1} - y_n)}{h},$$

once to estimate the concentration of the pollutant in the pond water 6 hours after the chemical treatment was applied.

Solution

Well, 6 hours equals 0.25 days. Now,

$$\begin{aligned} t = 0, x = 3 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{3 + \cosh 0}{3 \times 3^2 \cosh 0} - \tfrac{1}{3}(3) \tanh 0 \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} &= \tfrac{4}{27} \end{aligned}$$

so

$$\begin{aligned} x_1 &\approx 3 + (0.25 \times \tfrac{4}{27}) \\ &= 3.\dot{0}3\dot{7} \\ &= \underline{\underline{3.04 \text{ ppm (3 sf)}}}. \end{aligned}$$

(b) Show that the transformation $u = x^3$ transforms the differential equation (I) into the differential equation (3)

$$\frac{\mathrm{d}u}{\mathrm{d}t} + u \tanh t = 1 + \frac{3}{\cosh t} \quad \text{(II)}.$$

Solution

Well,

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{\mathrm{d}u}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 3x^2 \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 3x^2 \left(\frac{3 + \cosh t}{3x^2 \cosh t} - \tfrac{1}{3} x \tanh t \right) \\ &= \frac{3 + \cosh t}{\cosh t} - x^3 \tanh t \\ &= \frac{3}{\cosh t} + 1 - u \tanh t\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} + u \tanh t &= \left(\frac{3}{\cosh t} + 1 - u \tanh t \right) + u \tanh t \\ &= \underline{\underline{1 + \frac{3}{\cosh t}}},\end{aligned}$$

as required.

(c) Determine the general solution of equation (II). (4)

Solution

Now,

$$\begin{aligned}\text{IF} &= \mathrm{e}^{\int \tanh t \, \mathrm{d}t} \\ &= \mathrm{e}^{\ln \cosh t} \\ &= \cosh t\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} + u \tanh t = 1 + \frac{3}{\cosh t} &\Rightarrow \cosh t \frac{\mathrm{d}u}{\mathrm{d}t} + u \sinh t = \cosh t + 3 \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}u}(u \cosh t) = \cosh t + 3 \\ &\Rightarrow u \cosh t = \int (\cosh t + 3) \, \mathrm{d}t \\ &\Rightarrow u \cosh t = \sinh t + 3t + c \\ &\Rightarrow \underline{\underline{u = \tanh t + \frac{3t + c}{\cosh t}}},\end{aligned}$$

for some constant, c.

(d) Hence find an equation for the concentration of pollutant in the pond water t days after the chemical treatment was applied. (3)

Solution

Now, $t = 0$ and $x = 3$ makes $u = 3^3 = 27$:

$$27 \cosh 0 = \sinh 0 + 0 + c \Rightarrow c = 27$$

and

$$u = \tanh t + \frac{3t + 27}{\cosh t} \Rightarrow x^3 = \tanh t + \frac{3t + 27}{\cosh t}$$
$$\Rightarrow \underline{\underline{x = \sqrt[3]{\tanh t + \frac{3t + 27}{\cosh t}}}}.$$

(e) Find the percentage error of the estimate found in part (a) compared to the value predicted by the model, stating if it is an overestimate or an underestimate. (3)

Solution

Well,

$$t = 0.25 \Rightarrow x = \sqrt[3]{\tanh 0.25 + \frac{3 \times 0.25 + 27}{\cosh 0.25}}$$
$$\Rightarrow x = 3.005\,536\,235 \text{ (FCD)}$$

and

$$\text{percentage error} = \left(\frac{3.005\ldots - 3.\dot{0}3\dot{7}}{3.005\ldots}\right) \times 100\%$$
$$= -1.048\,092\,563 \text{ (FCD)};$$

Hence, it is an $\underline{\underline{\text{overestimate}}}$ by $\underline{\underline{1.05\% \text{ (3 sf)}}}$.