Dr Oliver Mathematics
Further Mathematics
Reduction Formulae
Past Examination Questions

This booklet consists of 24 questions across a variety of examination topics.
The total number of marks available is 281.

1.
$$I_n = \int_1^e x^2 (\ln x)^n \, dx, \; n \geqslant 0.$$

(a) Prove that, for $n \geqslant 1$, (4)

$$I_n = \tfrac{1}{3}e^3 - \tfrac{1}{3}nI_{n-1}.$$

(b) Find the exact value of I_3. (4)

2.
$$I_n = \int_0^a (a-x)^n \cos x \, dx, \; a > 0, \; n \geqslant 0.$$

(a) Show that, for $n \geqslant 2$, (5)

$$I_n = na^{n-1} - n(n-1)I_{n-2}.$$

(b) Hence evaluate (3)

$$\int_0^{\frac{\pi}{2}} \left(\tfrac{\pi}{2} - x\right)^2 \, dx.$$

3.
$$I_n = \int x^n e^{2x} \, dx, \; n \geqslant 0.$$

(a) Prove that, for $n \geqslant 1$, (3)

$$I_n = \tfrac{1}{2}(x^n e^{2x} - nI_{n-1}).$$

(b) Find, in terms of e, the exact value of (5)

$$\int_0^1 x^2 e^{2x} \, dx.$$

4.
$$I_n = \int_0^1 (1-x)^n \cosh x \, dx, \; n \geqslant 0.$$

(a) Prove that, for $n \geqslant 2$, (5)

$$I_n = n(n-1)I_{n-2} - n.$$

(b) Find an exact expression for I_4, giving your answer in terms of e. (3)

5. Given that

$$I_n = \int_0^4 x^n \sqrt{4-x}\,\mathrm{d}x,\ n \geqslant 0,$$

(a) show that (6)

$$I_n = \frac{8n}{2n+3} I_{n-1},\ n \geqslant 1.$$

Given that

$$\int_0^4 \sqrt{4-x}\,\mathrm{d}x = \tfrac{16}{3},$$

(b) use the result in part (a) to find the exact value of (3)

$$\int_0^4 x^2 \sqrt{4-x}\,\mathrm{d}x.$$

6. Given that $y = \sinh^{n-1} x \cosh x$,

(a) show that (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (n-1)\sinh^{n-2} x + n \sinh^n x.$$

The integral I_n is defined by

$$I_n = \int_0^{\operatorname{arsinh} 1} \sinh^n x\,\mathrm{d}x,\ n \geqslant 0.$$

(b) Using the result in part (a), or otherwise, show that (2)

$$nI_n = \sqrt{2} - (n-1)I_{n-2},\ n \geqslant 2.$$

(c) Hence find the exact value of I_4. (4)

7.

$$I_n = \int_1^5 x^n (2x-1)^{-\frac{1}{2}}\,\mathrm{d}x,\ n \geqslant 0.$$

(a) Prove that, for $n \geqslant 1$, (5)

$$(2n+1)I_n = nI_{n-1} + 3 \times 5^n - 1.$$

(b) Using the reduction formula given in part (a), find the exact value of I_2. (5)

8.
$$I_n = \int (\ln x)^n \, \mathrm{d}x, \; n \geqslant 0.$$

(a) Show that (4)
$$I_n = x(\ln x)^{n-1} - nI_{n-1}, \; n \geqslant 1.$$

(b) Hence calculate the exact value of (6)
$$\int_1^e (\ln x)^3 \, \mathrm{d}x.$$

9. Given that
$$I_n = \int_0^4 x^n \sqrt{16 - x^2} \, \mathrm{d}x \; n \geqslant 0,$$

(a) prove that, for $n \geqslant 2$, (6)
$$(n+2)I_n = 16(n-1)I_{n-2}.$$

(b) Hence, showing each step of your working, find the exact value of I_5. (5)

10.
$$I_n = \int_0^{\frac{\pi}{4}} x^n \sin 2x \, \mathrm{d}x, \; n \geqslant 0.$$

(a) Prove that, for $n \geqslant 2$, (5)
$$I_n = \tfrac{1}{4}n \left(\tfrac{\pi}{4}\right)^{n-1} - \tfrac{1}{4}n(n-1)I_{n-2}.$$

(b) Find the exact value of I_2. (4)

(c) Show that $I_4 = \frac{1}{64}(\pi^3 - 24\pi + 48)$. (2)

11.
$$I_n = \int \sin^n x \, \mathrm{d}x, \; n \geqslant 0.$$

(a) Prove that, for $n \geqslant 2$, (4)
$$I_n = \tfrac{1}{n}\left(-\sin^{n-1} x \cos x + (n-1)I_{n-2}\right).$$

Given that n is an odd number, $n \geqslant 3$,

(b) show that (4)
$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x = \frac{(n-1)(n-3)\ldots 6 \times 4 \times 2}{n(n-2)(n-4)\ldots 7 \times 5 \times 3}.$$

(c) Hence find (3)

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^2 x \, \mathrm{d}x.$$

12. Given that

$$I_n = \int_0^{\pi} e^x \sin^n x \, \mathrm{d}x, \; n \geqslant 0,$$

(a) show that, for $n \geqslant 2$, (8)

$$I_n = \frac{n(n-1)}{n^2+1} I_{n-2}.$$

(b) Find the exact value of I_4. (4)

13. Given that

$$I_n = \int_0^{8} x^n (8-x)^{\frac{1}{3}} \, \mathrm{d}x, \; n \geqslant 0,$$

(a) show that (6)

$$I_n = \frac{24n}{3n+4} I_{n-1}, \; n \geqslant 1.$$

(b) Hence find the exact value of (6)

$$\int_0^{8} x(x+5)(8-x)^{\frac{1}{3}} \, \mathrm{d}x.$$

14.

$$I_n = \int x^n \cosh x \, \mathrm{d}x, \; n \geqslant 0.$$

(a) Show that, for $n \geqslant 2$, (4)

$$I_n = x^n \sinh x - n x^{n-1} \cosh x + n(n-1) I_{n-2}.$$

(b) Hence show that (5)

$$I_4 = f(x) \sinh x + g(x) \cosh x + c,$$

where $f(x)$ and $g(x)$ are functions to be found, and c is an arbitrary constant.

(c) Find the exact value of (3)

$$\int x^4 \cosh x \, \mathrm{d}x,$$

giving your answer in terms of e.

15. Given that

$$I_n = \int \frac{\sin nx}{\sin x} \, \mathrm{d}x, \; n \geqslant 1,$$

(a) prove that, for $n \geqslant 3$, (3)

$$I_n - I_{n-2} = \int 2\cos(n-1)x \, \mathrm{d}x.$$

(b) Hence, showing each step of your working, find the exact value of (7)

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sin 5x}{\sin x} \, \mathrm{d}x,$$

giving your answer in the form $\frac{1}{12}(a\pi + b\sqrt{3} + c)$, where a, b, and c are integers to be found.

16. (a) Find (2)

$$\int x e^{-\frac{1}{2}x^2} \, \mathrm{d}x.$$

Given that

$$I_n = \int_0^1 x^n e^{-\frac{1}{2}x^2} \, \mathrm{d}x,$$

(b) prove that $I_n = (n-1)I_{n-2} - e^{-\frac{1}{2}}$, $n \geqslant 2$. (5)

(c) find the value of I_5, leaving your answer in terms of e. (6)

17.

$$I_n = \int \frac{\sin nx}{\sin x} \, \mathrm{d}x, \; n > 0, \; n \in \mathbb{Z}.$$

(a) By considering $I_{n+2} - I_n$, or otherwise, show that (6)

$$I_{n+2} = \frac{2\sin(n+1)x}{n+1} + I_n.$$

(b) Hence evaluate (7)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x} \, \mathrm{d}x,$$

giving your answer in the form $p\sqrt{2} + q\sqrt{3}$, where p and q are rational numbers to be found.

18.

$$I_n = \int \frac{x^n}{\sqrt{1+x^2}} \, \mathrm{d}x.$$

(a) Show that $nI_n = x^{n-1}\sqrt{1+x^2} - (n-1)I_{n-2}$, $n \geqslant 2$. (7)

The curve C has equation
$$y^2 = \frac{x^2}{\sqrt{1+x^2}},\ y \geqslant 0.$$

The finite region, R, is bounded by C, the x-axis, and the lines with equation $x = 0$ and $x = 2$. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid so formed, giving your answer in terms of π, surds, and natural logarithms. (7)

19. Given that $I_n = \displaystyle\int \sec^n x \, \mathrm{d}x$, (14)

(a) show that
$$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2},\ n \geqslant 2.$$

(b) Hence find the exact value of
$$\int_0^{\frac{\pi}{3}} \sec^3 x \, \mathrm{d}x,$$
giving your answer in terms of natural logarithms and surds.

20.
$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x.$$

(a) Show that (8)
$$I_n = \frac{n-1}{n} I_{n-2},\ n \in \mathbb{Z},\ n \geqslant 2.$$

(b) Hence evaluate (7)
$$\int_0^{\frac{\pi}{2}} \sin^6 x (1 + \cos^2 x) \, \mathrm{d}x,$$
giving your answer as a multiple of π.

21.
$$I_n = \int_0^1 (1 - x^2)^n \, \mathrm{d}x,\ n \geqslant 0.$$

(a) Prove that $(2n+1)I_n = 2nI_{n-1}$, $n \geqslant 1$. (7)

(b) Prove, by induction, that (8)
$$I_n \leqslant \left(\frac{2n}{2n+1}\right)^n,$$
for all $n \in \mathbb{Z}^+$.

22.
$$I_n = \int_0^{\operatorname{arsinh} 1} \sinh^n x,\ \mathrm{d}x,\ n \in \mathbb{N}.$$

(a) Show that $nI_n = \sqrt{2} - (n-1)I_{n-2}$, $n \geqslant 2$. (9)

(b) Evaluate (7)

$$\int_0^{\operatorname{arsinh} 1} \sinh^5 x, \, \mathrm{d}x,$$

leaving your answer in surd form.

23.

$$I_n = \int_0^1 x^n \sqrt{1-x^2} \, \mathrm{d}x, \ n \geqslant 0.$$

(a) Find the value of I_1. (3)

(b) Show that, for $n \geqslant 2$, (9)

$$(n+2)I_n = (n-1)I_{n-2}.$$

(c) Hence find the exact value of (4)

$$\int_0^1 x^7 \sqrt{1-x^2} \, \mathrm{d}x.$$

24. (16)

$$I_n = \int_0^{\pi} \sin^{2n} x \, \mathrm{d}x, \ n \in \mathbb{N}.$$

(a) Calculate I_0 in terms of π.

(b) Show that

$$I_n = \frac{2n-1}{2n} I_{n-1}, \ n \geqslant 1.$$

(c) Find I_3 in terms of π.

The picture shows the curve with polar equation $r = a \sin^3 \theta$, $0 \leqslant \theta \leqslant \pi$, where a is a positive constant.

(d) Using your answer to part (c), or otherwise, find the exact area bounded by this curve.