Dr Oliver Mathematics
Applied Mathematics: Differential Equations

The total number of marks available is 72.
You must write down all the stages in your working.

1. (a) Given the differential equation (4)

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y \cos x = 0,$$

find the general solution, expressing y explicitly in terms of x.

Solution

$$\begin{aligned} \sin x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y \cos x = 0 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} - 2y\,\frac{\cos x}{\sin x} = 0 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \left(-2\frac{\cos x}{\sin x}\right) y = 0 \end{aligned}$$

$$\begin{aligned} \text{IF} &= \mathrm{e}^{\int -2\frac{\cos x}{\sin x}\,\mathrm{d}x} \\ &= \mathrm{e}^{\int -2\frac{\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)}{\sin x}\,\mathrm{d}x} \\ &= \mathrm{e}^{-2\ln \sin x} \\ &= \mathrm{e}^{\ln \sin^{-2} x} \\ &= \sin^{-2} x \end{aligned}$$

$$\begin{aligned} &\Rightarrow \sin^{-2} x \frac{\mathrm{d}y}{\mathrm{d}x} + (-2\sin^{-3} x \cos x)y = 0 \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y \sin^{-2} x) = 0 \\ &\Rightarrow y \sin^{-2} x = c \\ &\Rightarrow \underline{\underline{y = c\sin^2 x}}. \end{aligned}$$

(b) Find the general solution of (5)

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y \cos x = 3\sin^3 x.$$

Solution

$$\begin{aligned}\sin x\frac{\mathrm{d}y}{\mathrm{d}x}-2y\cos x=3\sin^3 x &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}+\left(-2\frac{\cos x}{\sin x}\right)y=3\sin^2 x\\ &\Rightarrow \sin^{-2}x\frac{\mathrm{d}y}{\mathrm{d}x}+(-2\sin^{-3}x\cos x)y=3\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\sin^{-2}x)=3\\ &\Rightarrow y\sin^{-2}x=3x+A\\ &\Rightarrow \underline{\underline{y=(3x+A)\sin^2 x}}.\end{aligned}$$

2. Solve the differential equation (5)

$$\cos^2 x\frac{\mathrm{d}y}{\mathrm{d}x}=y,$$

given that $y>0$ and that $y=2$ when $x=0$.

Solution

$$\begin{aligned}\cos^2 x\frac{\mathrm{d}y}{\mathrm{d}x}=y &\Rightarrow \frac{1}{y}\,\mathrm{d}y=\sec^2 x\,\mathrm{d}x\\ &\Rightarrow \int\frac{1}{y}\,\mathrm{d}y=\int\sec^2 x\,\mathrm{d}x\\ &\Rightarrow \ln y=\tan x+c\end{aligned}$$

as $y>0$. Now,

$$x=0,y=2\Rightarrow \ln 2=0+c$$

and

$$\begin{aligned}\ln y=\tan x+\ln 2 &\Rightarrow \ln y-\ln 2=\tan x\\ &\Rightarrow \ln\left(\frac{y}{2}\right)=\tan x\\ &\Rightarrow \frac{y}{2}=\mathrm{e}^{\tan x}\\ &\Rightarrow \underline{\underline{y=2\mathrm{e}^{\tan x}}}.\end{aligned}$$

3. Obtain the solution of the differential equation (5)

$$x\frac{\mathrm{d}y}{\mathrm{d}x}-y=x^2\mathrm{e}^x,$$

for which $y = 2$ when $x = 1$.

Solution

$$x\frac{dy}{dx} - y = x^2e^x \Rightarrow \frac{dy}{dx} - \frac{y}{x} = xe^x$$
$$\Rightarrow \frac{dy}{dx} + \left(-\frac{1}{x}\right)y = xe^x$$

$$\begin{aligned}\text{IF} &= e^{\int -\frac{1}{x}\,dx} \\ &= e^{-\ln x} \\ &= e^{\ln x^{-1}} \\ &= x^{-1}\end{aligned}$$

$$\begin{aligned}&\Rightarrow x^{-1}\frac{dy}{dx} - x^{-2}y = e^x \\ &\Rightarrow \frac{d}{dx}(x^{-1}y) = e^x \\ &\Rightarrow x^{-1}y = \int e^x\,dx \\ &\Rightarrow x^{-1}y = e^x + c \\ &\Rightarrow y = x(e^x + c).\end{aligned}$$

Now,

$$x = 1, y = 2 \Rightarrow 2 = e + c \Rightarrow c = 2 - e$$

and

$$\underline{\underline{y = x(e^x + 2 - e)}}.$$

4. An industrial scientist finds that the differential equation

$$t\frac{dx}{dt} - 2x = 3t^2$$

models a production process.

(a) Find the general solution of the differential equation. (5)

Solution

$$t\frac{\mathrm{d}x}{\mathrm{d}t} - 2x = 3t^2 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{2}{t}x = 3t$$
$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} + \left(-\frac{2}{t}\right)x = 3t$$

$$\begin{aligned}\text{IF} &= \mathrm{e}^{\int -\frac{2}{t}\,\mathrm{d}t}\\ &= \mathrm{e}^{-2\ln t}\\ &= \mathrm{e}^{\ln t^{-2}}\\ &= t^{-2}\end{aligned}$$

$$\begin{aligned}&\Rightarrow t^{-2}\frac{\mathrm{d}x}{\mathrm{d}t} - 2t^{-3}x = 3t^{-1}\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(t^{-2}x) = 3t^{-1}\\ &\Rightarrow t^{-2}x = \int 3t^{-1}\,\mathrm{d}t\\ &\Rightarrow t^{-2}x = 3\ln t + c\\ &\Rightarrow \underline{\underline{x = t^2(3\ln t + c)}}.\end{aligned}$$

(b) Hence find the particular solution given $x = 1$ when $t = 1$. (1)

Solution

$$x = 1, t = 1 \Rightarrow 1 = 1(0 + c)$$

and, hence,

$$\underline{\underline{x = t^2(3\ln t + 1)}}.$$

5. An industrial process is modelled by the differential equation (7)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{9t\mathrm{e}^{3t}}{y}$$

where $y > 0$ and $t \geqslant 0$.

Given that $y = 2$ when $t = 0$, find y explicitly in terms of t.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{9t\mathrm{e}^{3t}}{y} \Rightarrow y\,\mathrm{d}y = 9t\mathrm{e}^{3t}\,\mathrm{d}t$$
$$\Rightarrow \int y\,\mathrm{d}y = \int 9t\mathrm{e}^{3t}\,\mathrm{d}t$$

$$u = 9t \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = 9$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{3t} \Rightarrow v = \tfrac{1}{3}\mathrm{e}^{3t}$$

$$\Rightarrow \tfrac{1}{2}y^2 = 3t\mathrm{e}^{3t} - \int 3\mathrm{e}^{3t}\,\mathrm{d}t$$
$$\Rightarrow \tfrac{1}{2}y^2 = 3t\mathrm{e}^{3t} - \mathrm{e}^{3t} + c.$$

Now,

$$t = 0, y = 2 \Rightarrow 2 = 0 - 1 + c \Rightarrow c = 3$$

and so

$$\tfrac{1}{2}y^2 = 3t\mathrm{e}^{3t} - \mathrm{e}^{3t} + 3 \Rightarrow y^2 = 2(3t\mathrm{e}^{3t} - \mathrm{e}^{3t} + 3)$$
$$\Rightarrow \underline{\underline{y = \sqrt{2(3t\mathrm{e}^{3t} - \mathrm{e}^{3t} + 3)}}}.$$

6. At any point (x, y) on a curve C, where $x \neq 0$, the gradient of the tangent is (9)

$$4 - \frac{3y}{x}.$$

Given that the point $(1, 3)$ lies on C, obtain an equation for C in the form $y = \mathrm{f}(x)$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4 - \frac{3y}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3y}{x} = 4$$

$$\begin{aligned}\text{IF} &= e^{\int \frac{3}{x}\,dx} \\ &= e^{3\ln x} \\ &= e^{\ln x^3} \\ &= x^3\end{aligned}$$

$$\begin{aligned}&\Rightarrow x^3\frac{dy}{dx} + 3x^2y = 4x^3 \\ &\Rightarrow \frac{d}{dx}(x^3y) = 4x^3 \\ &\Rightarrow x^3y = \int 4x^3\,dx \\ &\Rightarrow x^3y = x^4 + c.\end{aligned}$$

Now,

$$x = 1, y = 3 \Rightarrow 3 = 1 + c \Rightarrow c = 2$$

and, finally,

$$x^3y = x^4 + 2 \Rightarrow \underline{\underline{y = x + \frac{2}{x^3}}}.$$

7. A turkey is taken from a refrigerator to be cooked. Its temperature is 4°C when it is placed in an oven preheated to 180°C.

   Its temperature, T°C, after a time of x hours in the oven satisfies the equation

$$\frac{dT}{dx} = k(180 - T).$$

   (a) Show that (4)

$$T = 180 - 176e^{-kx}.$$

Solution

$$
\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}x} = k(180 - T) &\Rightarrow \frac{1}{(180 - T)}\,\mathrm{d}T = k\,\mathrm{d}x \\
&\Rightarrow \int \frac{1}{(180 - T)}\,\mathrm{d}T = \int k\,\mathrm{d}x \\
&\Rightarrow -\ln(180 - T) = kx + c \\
&\Rightarrow \ln(180 - T) = -kx - c \\
&\Rightarrow 180 - T = \mathrm{e}^{-kx-c} \\
&\Rightarrow T = 180 - \mathrm{e}^{-kx}\mathrm{e}^{-c} \\
&\Rightarrow T = 180 - A\mathrm{e}^{-kx}
\end{aligned}
$$

for some constant A. Now,

$$x = 0, T = 4 \Rightarrow 4 = 180 - A \Rightarrow T = 176$$

and

$$\underline{\underline{T = 180 - 176\mathrm{e}^{-kx}}}.$$

After an hour in the oven the temperature of the turkey is 30°C.

(b) Calculate the value of k correct to 2 decimal places. (2)

Solution

$$
\begin{aligned}
x = 1, T = 30 &\Rightarrow 30 = 180 - 176\mathrm{e}^{-k} \\
&\Rightarrow 176\mathrm{e}^{-k} = 150 \\
&\Rightarrow \mathrm{e}^{-k} = \tfrac{75}{88} \\
&\Rightarrow -k = \ln\tfrac{75}{88} \\
&\Rightarrow k = -\ln\tfrac{75}{88} \\
&\Rightarrow k = 0.159\,848\,700\,9 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{k = 0.16 \text{ (2 dp)}}}.
\end{aligned}
$$

The turkey will be cooked when it reaches a temperature of 80°C.

(c) After how long (to the nearest minute) will the turkey be cooked? (3)

Solution

$$
\begin{aligned}
80 = 180 - 176\mathrm{e}^{-kx} &\Rightarrow 176\mathrm{e}^{-kx} = 100 \\
&\Rightarrow \mathrm{e}^{-kx} = \tfrac{100}{176} \\
&\Rightarrow -kx = \ln \tfrac{100}{176} \\
&\Rightarrow x = -\tfrac{1}{k} \ln \tfrac{100}{176} \\
&\Rightarrow x = 3.536\,555\,41 \text{ hours (FCD)} \\
&\Rightarrow x = \underline{\underline{3 \text{ hours } 32 \text{ mins (nearest minute)}}}.
\end{aligned}
$$

8. Find the general solution of the differential equation (6)

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6,\ x \neq 0.$$

Solution

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 6x$$

$$
\begin{aligned}
\mathrm{IF} &= \mathrm{e}^{\int 2x\,\mathrm{d}x} \\
&= \mathrm{e}^{x^2}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \mathrm{e}^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\mathrm{e}^{x^2}y = 6x\,\mathrm{e}^{x^2} \\
&\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x^2}y) = 6x\,\mathrm{e}^{x^2} \\
&\Rightarrow \mathrm{e}^{x^2}y = \int 6x\,\mathrm{e}^{x^2}\,\mathrm{d}x \\
&\Rightarrow \mathrm{e}^{x^2}y = 3\,\mathrm{e}^{x^2} + c \\
&\Rightarrow \underline{\underline{y = 3 + c\,\mathrm{e}^{-x^2}}}.
\end{aligned}
$$

9. A flu-like virus starts to spread through the 20 000 inhabitants of Dumbarton. The situation can be modelled by the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(20\,000 - N)}{10\,000},$$

where N is the number of people infected after t days and $0 < N < 20\,000$.

(a) How many people are infected when the infection is spreading most rapidly? (1)

Solution

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(20\,000 - N)}{10\,000} &\Rightarrow \frac{\mathrm{d}N}{\mathrm{d}t} = \tfrac{1}{10\,000}(20\,000N - N^2)\\ &\Rightarrow \frac{\mathrm{d}^2N}{\mathrm{d}t^2} = \tfrac{1}{10\,000}(20\,000 - 2N)\\ &\Rightarrow \frac{\mathrm{d}^2N}{\mathrm{d}t^2} = \tfrac{1}{5\,000}(10\,000 - N)\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2N}{\mathrm{d}t^2} = 0 &\Rightarrow \tfrac{1}{5\,000}(10\,000 - N)\\ &\Rightarrow \underline{\underline{N = 10\,000}}.\end{aligned}$$

(b) Express (5)

$$\frac{10\,000}{N(20\,000 - N)}$$

in partial fractions and show that

$$\ln\left(\frac{N}{20\,000 - N}\right) = 2t + c,$$

for some constant c.

Solution

$$\begin{aligned}\frac{10\,000}{N(20\,000 - N)} &\equiv \frac{A}{N} + \frac{B}{20\,000 - N}\\ &\equiv \frac{A(20\,000 - N) + BN}{N(20\,000 - N)}\end{aligned}$$

which means

$$10\,000 \equiv A(20\,000 - N) + BN.$$

$\underline{N = 0}$: $10\,000 = 20\,000A \Rightarrow A = \frac{1}{2}$.
$\underline{N = 20\,000}$: $10\,000 = 20\,000B \Rightarrow B = \frac{1}{2}$.
Hence,

$$\frac{10\,000}{N(20\,000 - N)} \equiv \underline{\underline{\frac{\frac{1}{2}}{N} + \frac{\frac{1}{2}}{20\,000 - N}}}$$

and

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(20\,000-N)}{10\,000} &\Rightarrow \frac{10\,000}{N(20\,000-N)}\,\mathrm{d}N = \mathrm{d}t \\ &\Rightarrow \left(\frac{\frac{1}{2}}{N} + \frac{\frac{1}{2}}{20\,000-N}\right)\mathrm{d}N = \mathrm{d}t \\ &\Rightarrow \int \left(\frac{\frac{1}{2}}{N} + \frac{\frac{1}{2}}{20\,000-N}\right)\mathrm{d}N = \int \mathrm{d}t \\ &\Rightarrow \tfrac{1}{2}\ln N - \tfrac{1}{2}\ln(20\,000-N) = t + a \\ &\Rightarrow \tfrac{1}{2}\ln\left(\frac{N}{20\,000-N}\right) = t + a \\ &\Rightarrow \underline{\underline{\ln\left(\frac{N}{20\,000-N}\right) = 2t + c,}}\end{aligned}$$

where $c = 2a$.

Initially there were 100 people infected.

(c) Show that (4)

$$N = \frac{20\,000\,\mathrm{e}^{2t}}{199 + \mathrm{e}^{2t}}.$$

Solution

$$\begin{aligned}t = 0, N = 100 &\Rightarrow \ln\left(\frac{100}{20\,000-100}\right) = 0 + c \\ &\Rightarrow c = \ln\tfrac{1}{199}\end{aligned}$$

and

$$\begin{aligned}
\ln\left(\frac{N}{20\,000-N}\right) = 2t + \ln\tfrac{1}{199} &\Rightarrow \frac{N}{20\,000-N} = \mathrm{e}^{2t+\ln\frac{1}{199}} \\
&\Rightarrow \frac{N}{20\,000-N} = \mathrm{e}^{2t}\,\mathrm{e}^{\ln\frac{1}{199}} \\
&\Rightarrow \frac{N}{20\,000-N} = \tfrac{1}{199}\mathrm{e}^{2t} \\
&\Rightarrow N = \tfrac{1}{199}\mathrm{e}^{2t}(20\,000-N) \\
&\Rightarrow N = \tfrac{20\,000}{199}\mathrm{e}^{2t} - \tfrac{1}{199}\mathrm{e}^{2t}N \\
&\Rightarrow N + \tfrac{1}{199}\mathrm{e}^{2t}N = \tfrac{20\,000}{199}\mathrm{e}^{2t} \\
&\Rightarrow \tfrac{1}{199}N\left(199+\mathrm{e}^{2t}\right) = \tfrac{20\,000}{199}\mathrm{e}^{2t} \\
&\Rightarrow N\left(199+\mathrm{e}^{2t}\right) = 20\,000\mathrm{e}^{2t} \\
&\Rightarrow \underline{\underline{N = \frac{20\,000\mathrm{e}^{2t}}{199+\mathrm{e}^{2t}}}},
\end{aligned}$$

as required.

10. Find the general solution, in the form $y = \mathrm{f}(x)$, of the differential equation (6)

$$\frac{1}{\cos x}\,\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \tan x,\ 0 < x < \pi.$$

Solution

$$\begin{aligned}
\frac{1}{\cos x}\,\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \tan x &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x\tan x = \cos x\tan x \\
&\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = \sin x
\end{aligned}$$

$$
\begin{aligned}
\text{IF} &= e^{\int \sin x \, dx} \\
&= e^{-\cos x}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow e^{-\cos x}\frac{dy}{dx} + ye^{-\cos x}\sin x = e^{-\cos x}\sin x \\
&\Rightarrow \frac{d}{dx}(e^{-\cos x}y) = \sin x \, e^{-\cos x} \\
&\Rightarrow e^{-\cos x}y = \int \sin x \, e^{-\cos x} \, dx \\
&\Rightarrow e^{-\cos x}y = e^{-\cos x} + c \\
&\Rightarrow \underline{\underline{y = 1 + c\,e^{\cos x}}}.
\end{aligned}
$$