# Dr Oliver Mathematics
# Induction

In this note, we explore induction.

There are three steps to doing induction:

(a) Show that a propositional form $P(x)$ is true for some basis case.

(b) Assume that $P(n)$ is true for some n, and show that this implies that $P(n+1)$ is true.

(c) Then, by the principle of induction, the propositional form $P(x)$ is true for all $n$ greater or equal to the basis case.

In the first example, we start with a famous case.

**Example 1**
Prove that

$$1 + 2 + \ldots + n = \tfrac{1}{2}n(n+1).$$

**Solution**
Base case: Let us see what the case looks like taking the LHS and RHS separately.

$$\begin{aligned} 1 &= 1, \\ \tfrac{1}{2} \times 1 \times 2 &= 1, \end{aligned}$$

and we agree. So $n = 1$ is true.

Induction: Suppose the solution is true for $n = k$, i.e.,

$$1 + 2 + \ldots + k = \tfrac{1}{2}k(k+1).$$

Then

$$\begin{aligned} 1 + 2 + \ldots + k + (k+1) &= \tfrac{1}{2}k(k+1) + (k+1) \text{ (by the inductive hypothesis)} \\ &= \tfrac{1}{2}(k+1)[k+2], \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required. ■.

**Example 2**
Prove that

$$1^2 + 2^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

**Solution**

Base case: Let us see what the case looks like taking the LHS and RHS separately.

$$1^2 = 1,$$
$$\tfrac{1}{6} \times 1 \times 2 \times 3 = 1,$$

and we agree. So $n = 1$ is true.

Induction: Suppose the solution is true for $n = k$, i.e.,

$$1^2 + 2^2 + \ldots + k^2 = \tfrac{1}{6}k(k+1)(2k+1).$$

Then

$$\begin{aligned} 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 &= \tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2 \text{ (by the inductive hypothesis)} \\ &= \tfrac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \\ &= \tfrac{1}{3}(k+1)(2k^2 + 7k + 6) \\ &= \tfrac{1}{3}(k+1)(k+2)(2k+3) \\ &= \tfrac{1}{3}(k+1)(k+2)[2(k+1)+1], \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required. ■.

Here are two examples for you to try: if you like that, go the the "A Level Further Mathematics" page and click on "Induction Questions" link.

1. Prove that
$$1^3 + 2^3 + \ldots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

2. Prove by induction that $11^n - 6$ is divisible by 5 for every positive integer $n$.