# Dr Oliver Mathematics
# Cramer's Rule

In this note, we present Cramer's Rule. We will do $2 \times 2$ matrices, $3 \times 3$ matrices, $n \times n$ matrices, and then some examples.

## 1 $2 \times 2$ matrices

Suppose

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2. \end{aligned}$$

What are $x$ and $y$? We will convert it in to matrix form:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

We will denote

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

Now, if the determinant does not equal zero,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

we can proceed as follows.

Determinant:

$$\det \mathbf{A} = a_1 b_2 - a_2 b_1$$

Matrix of minors:

$$\begin{pmatrix} b_2 & a_2 \\ b_1 & a_1 \end{pmatrix}$$

Matrix of cofactors:

$$\begin{pmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{pmatrix}$$

<u>Transpose:</u>

$$\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$$

<u>Inverse:</u>

$$\frac{1}{a_1 b_2 - a_2 b_1} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$$

Now,

$$\begin{aligned}
\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a_1 b_2 - a_2 b_1} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \begin{pmatrix} b_2 c_1 - b_1 c_2 \\ -a_2 c_1 + a_1 c_2 \end{pmatrix} \\
&\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \begin{pmatrix} c_1 b_2 - c_2 b_1 \\ a_1 c_2 - a_2 c_1 \end{pmatrix} \\
&\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \\ \\ \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \end{pmatrix}
\end{aligned}$$

Hence

$$\boxed{x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.}$$

## 2 $3 \times 3$ matrices

Suppose

$$\begin{aligned}
a_1 x + b_1 y + c_1 z &= d_1 \\
a_2 x + b_2 y + c_2 z &= d_2 \\
a_3 x + b_3 y + c_3 z &= d_3.
\end{aligned}$$

What are $x$, $y$, and $z$?

We will convert it in to matrix form:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Now, if the determinant does not equal zero,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0,$$

we can proceed as follows.

Determinant:

$$\det \mathbf{A} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$

Matrix of minors:

$$\begin{pmatrix} b_2c_3 - b_3c_2 & a_2c_3 - a_3c_2 & a_2b_3 - a_3b_2 \\ b_1c_3 - b_3c_1 & a_1c_3 - a_3c_1 & a_1b_3 - a_3b_1 \\ b_1c_2 - b_2c_1 & a_1c_2 - a_2c_1 & a_1b_2 - a_2b_1 \end{pmatrix}$$

Matrix of cofactors:

$$\begin{pmatrix} b_2c_3 - b_3c_2 & -(a_2c_3 - a_3c_2) & a_2b_3 - a_3b_2 \\ -(b_1c_3 - b_3c_1) & a_1c_3 - a_3c_1 & -(a_1b_3 - a_3b_1) \\ b_1c_2 - b_2c_1 & -(a_1c_2 - a_2c_1) & a_1b_2 - a_2b_1 \end{pmatrix}$$

Transpose:

$$\begin{pmatrix} b_2c_3 - b_3c_2 & -(b_1c_3 - b_3c_1) & b_1c_2 - b_2c_1 \\ -(a_2c_3 - a_3c_2) & a_1c_3 - a_3c_1 & -(a_1c_2 - a_2c_1) \\ a_2b_3 - a_3b_2 & -(a_1b_3 - a_3b_1) & a_1b_2 - a_2b_1 \end{pmatrix}$$

Inverse:

$$\frac{1}{\det \mathbf{A}} \begin{pmatrix} b_2c_3 - b_3c_2 & -(b_1c_3 - b_3c_1) & b_1c_2 - b_2c_1 \\ -(a_2c_3 - a_3c_2) & a_1c_3 - a_3c_1 & -(a_1c_2 - a_2c_1) \\ a_2b_3 - a_3b_2 & -(a_1b_3 - a_3b_1) & a_1b_2 - a_2b_1 \end{pmatrix}$$

Now,

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{1}{\det \mathbf{A}} \begin{pmatrix} b_2c_3 - b_3c_2 & -(b_1c_3 - b_3c_1) & b_1c_2 - b_2c_1 \\ -(a_2c_3 - a_3c_2) & a_1c_3 - a_3c_1 & -(a_1c_2 - a_2c_1) \\ a_2b_3 - a_3b_2 & -(a_1b_3 - a_3b_1) & a_1b_2 - a_2b_1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \\
&= \frac{1}{\det \mathbf{A}} \begin{pmatrix} d_1(b_2c_3 - b_3c_2) - d_2(b_1c_3 - b_3c_1) + d_3(b_1c_2 - b_2c_1) \\ -d_1(a_2c_3 - a_3c_2) + d_2(a_1c_3 - a_3c_1) - d_3(a_1c_2 - a_2c_1) \\ d_1(a_2b_3 - a_3b_2) - d_2(a_1b_3 - a_3b_1) + d_3(a_1b_2 - a_2b_1) \end{pmatrix} \\
&= \frac{1}{\det \mathbf{A}} \begin{pmatrix} d_1(b_2c_3 - b_3c_2) - b_1c_3d_2 + b_3c_1d_2 + b_1c_2d_3 - b_2c_1d_3 \\ -a_2c_3d_1 + a_3c_2d_1 + a_1c_3d_2 - a_3c_1d_2 - a_1c_2d_3 - a_2c_1d_3 \\ a_2b_3d_1 - a_3b_2d_1 - a_1b_3d_2 + a_3b_1d_2 + a_1b_2d_3 - a_2b_1d_3 \end{pmatrix} \\
&= \frac{1}{\det \mathbf{A}} \begin{pmatrix} d_1(b_2c_3 - b_3c_2) - b_1(c_3d_2 - c_2d_3) + c_1(b_3d_2 - b_2d_3) \\ a_1(c_3d_2 - c_2d_3) - d_1(a_2c_3 - a_3c_2) + c_1(a_2d_3 - a_3d_2) \\ a_1(b_2d_3 - b_3d_2) - b_1(a_2d_3 - a_3d_2) + d_1(a_2b_3 - a_3b_2) \end{pmatrix}.
\end{aligned}$$

Hence

$$\boxed{x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \text{ and } z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}}$$

## 3 $n \times n$ matrices

Suppose

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n.
\end{aligned}$$

Then

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} & \dots & a_{1n} \\ a_{21} & b_2 & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & b_n & a_{n3} & \dots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}}$$

and so on, up to

$$x_n = \frac{\begin{vmatrix} a_{11} & \dots & a_{1,n-1} & b_1 \\ a_{21} & \dots & a_{2,n-1} & b_2 \\ \vdots & & & \\ a_{n1} & \dots & a_{n,n-1} & b_n \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}}$$

# 4 Examples

1. Solve

$$\begin{aligned} 2x + 3y &= 12 \\ 5x - 2y &= 11. \end{aligned}$$

**Solution**

The determinants of the three relevant matrices are

$$\begin{vmatrix} 12 & 3 \\ 11 & -2 \end{vmatrix}, \begin{vmatrix} 2 & 12 \\ 5 & 11 \end{vmatrix}, \text{ and } \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}.$$

$$\begin{aligned} x &= \frac{\begin{vmatrix} 12 & 3 \\ 11 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}} \\ &= \frac{12 \times (-2) - 11 \times 3}{2 \times (-2) - 3 \times 5} \\ &= \frac{-57}{-19} \\ &= 3 \end{aligned}$$

and

$$\begin{aligned} y &= \frac{\begin{vmatrix} 2 & 12 \\ 5 & 11 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}} \\ &= \frac{2 \times 11 - 12 \times 5}{2 \times (-2) - 3 \times 5} \\ &= \frac{-38}{-19} \\ &= 2; \end{aligned}$$

hence,

$$\underline{\underline{x = 3 \text{ and } y = 2}}.$$

2. Solve

$$\begin{aligned} 4x - 3y &= 16 \\ 3x + 4y &= -13. \end{aligned}$$

**Solution**

The determinants of the three relevant matrices are

$$\begin{vmatrix} 16 & -3 \\ -13 & 4 \end{vmatrix}, \begin{vmatrix} 4 & 16 \\ 3 & -13 \end{vmatrix}, \text{ and } \begin{vmatrix} 4 & -3 \\ 3 & 4 \end{vmatrix}.$$

$$\begin{aligned} x &= \frac{\begin{vmatrix} 16 & -3 \\ -13 & 4 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ 3 & 4 \end{vmatrix}} \\ &= \frac{16 \times 4 - (-13) \times (-3)}{4 \times 4 - 3 \times (-3)} \\ &= \frac{25}{25} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} y &= \frac{\begin{vmatrix} 4 & 16 \\ 3 & -13 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ 3 & 4 \end{vmatrix}} \\ &= \frac{4 \times (-13) - 16 \times 3}{4 \times 4 - 3 \times (-3)} \\ &= \frac{-100}{25} \\ &= -4; \end{aligned}$$

hence,

$$\underline{\underline{x = 1 \text{ and } y = -4}}.$$

3. Solve

$$\begin{aligned} 2x + y + z &= 4 \\ 3x - z &= 7 \\ x + 4y - 3z &= 9. \end{aligned}$$

**Solution**

The determinants of the four relevant matrices are

$$\begin{vmatrix} 4 & 1 & 1 \\ 7 & 0 & -1 \\ 9 & 4 & -3 \end{vmatrix}, \begin{vmatrix} 2 & 4 & 1 \\ 3 & 7 & -1 \\ 1 & 9 & -3 \end{vmatrix}, \begin{vmatrix} 2 & 1 & 4 \\ 3 & 0 & 7 \\ 1 & 4 & 9 \end{vmatrix}, \text{ and } \begin{vmatrix} 2 & 1 & 1 \\ 3 & 0 & -1 \\ 1 & 4 & -3 \end{vmatrix}.$$

$$\begin{aligned} x &= \frac{\begin{vmatrix} 4 & 1 & 1 \\ 7 & 0 & -1 \\ 9 & 4 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 1 \\ 3 & 0 & -1 \\ 1 & 4 & -3 \end{vmatrix}} \\ &= \frac{4(0+4) - 1(-21+9) + 1(28-0)}{2(0+4) - 1(-9+1) + 1(12-0)} \\ &= \frac{56}{28} \\ &= 2, \end{aligned}$$

$$\begin{aligned} y &= \frac{\begin{vmatrix} 2 & 4 & 1 \\ 3 & 7 & -1 \\ 1 & 9 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 1 \\ 3 & 0 & -1 \\ 1 & 4 & -3 \end{vmatrix}} \\ &= \frac{2(-21+9) - 4(-9+1) + 1(27-7)}{2(0+4) - 1(-9+1) + 1(12-0)} \\ &= \frac{28}{28} \\ &= 1, \end{aligned}$$

$$
\begin{aligned}
z &= \frac{\begin{vmatrix} 2 & 1 & 4 \\ 3 & 0 & 7 \\ 1 & 4 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 1 \\ 3 & 0 & -1 \\ 1 & 4 & -3 \end{vmatrix}} \\
&= \frac{2(0-28) - 1(27-7) + 4(12-0)}{2(0+4) - 1(-9+1) + 1(12-0)} \\
&= -\frac{28}{28} \\
&= -1;
\end{aligned}
$$

hence,

$$\underline{\underline{x = 2,\ y = 1,\ \text{and } z = -1.}}$$