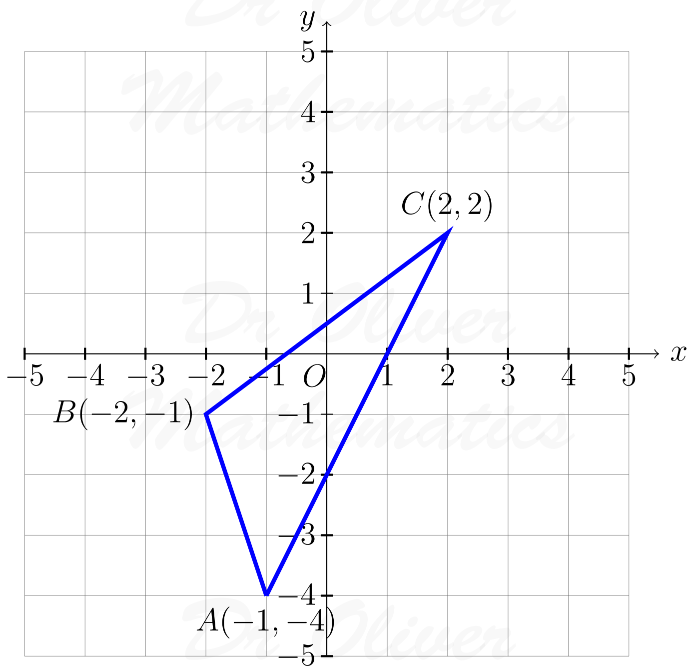
Dr Oliver Mathematics
Area by Determinant 1

In this note, we will investigate the area by determinant

1 A Problem

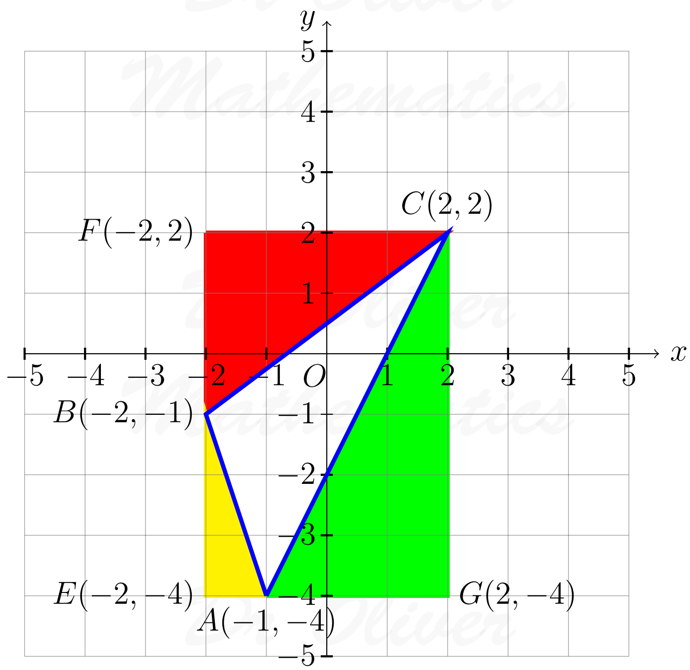


How you go about working out the area of the triangle ABC? Would you

- find the equation of the line AC, find the perpendicular through B, and find the area that way?
- find the lengths of three sides and apply Hero's rule?
  $AB = \sqrt{10}$, $AC = 3\sqrt{5}$, $BC = 5$, $s = 7.435\ldots$, and

$$\begin{aligned}\text{area of } ABC &= \sqrt{7.435\ldots(7.435\ldots - \sqrt{10})(7.435\ldots - 3\sqrt{5})(7.435\ldots - 5)} \\ &= 7\tfrac{1}{2} \text{ units}^2.\end{aligned}$$

- find the lengths of three sides, use the cosine rule to find to find one angle, and then use the sine formula for area?
  $AB = \sqrt{10}$, $AC = 3\sqrt{5}$, $BC = 5$,
- envelope the whole thing with a rectangle that is 4 units wide of 6 units tall and simply subtract the three triangles?



Well,

$$\begin{aligned}
\text{area of the rectangle } EFCG &= 4 \times 6 \\
&= 24; \\
\text{area of the red triangle} &= \tfrac{1}{2} \times 3 \times 4 \\
&= 6; \\
\text{area of the green triangle} &= \tfrac{1}{2} \times 3 \times 6 \\
&= 9; \\
\text{area of the yellow triangle} &= \tfrac{1}{2} \times 1 \times 3 \\
&= 1\tfrac{1}{2};
\end{aligned}$$

Subtract:

$$\begin{aligned}
\text{area of } ABC &= 24 - (6 + 9 + 1\tfrac{1}{2}) \\
&= 24 - 16\tfrac{1}{2} \\
&= 7\tfrac{1}{2} \text{ units}^2.
\end{aligned}$$

- or ...

2 Area by Determinant

2.1 The formula for the area by determinant

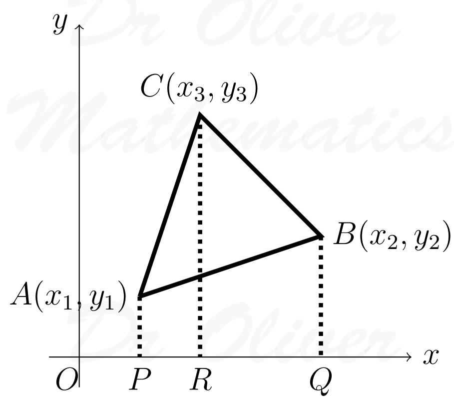
We let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Then

$$\boxed{\text{area of } ABC = \tfrac{1}{2}\left|\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right|}.$$

What?!

2.2 Proof

Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$.



Then

$$\begin{aligned}
\text{area of } ABC &= \text{area of } PACR + \text{area of } RCBQ - \text{area of } PABQ \\
&= \tfrac{1}{2}(x_3 - x_1)(y_1 + y_3) + \tfrac{1}{2}(x_2 - x_3)(y_3 + y_2) - \tfrac{1}{2}(x_2 - x_1)(y_1 + y_2) \\
&= \tfrac{1}{2}(x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3) + \tfrac{1}{2}(x_2y_3 - x_3y_3 + x_2y_2 - x_3y_2) \\
&\qquad - \tfrac{1}{2}(x_2y_1 - x_1y_1 + x_2y_2 - x_1y_2) \\
&= \tfrac{1}{2}(x_3y_1 - \cancel{x_1y_1} + \cancel{x_3y_3} - x_1y_3) + \tfrac{1}{2}(x_2y_3 - \cancel{x_3y_3} + \cancel{x_2y_2} - x_3y_2) \\
&\qquad - \tfrac{1}{2}(x_2y_1 - \cancel{x_1y_1} + \cancel{x_2y_2} - x_1y_2) \\
&= \tfrac{1}{2}(x_3y_1 - x_1y_3 + x_2y_3 - x_3y_2 - x_2y_1 + x_1y_2) \\
&= \tfrac{1}{2}\left[x_1(y_2 - y_3) - y_1(x_2 - x_3) + 1(x_2y_3 - x_3y_2)\right] \\
&= \tfrac{1}{2}\left[\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right].
\end{aligned}$$

But what if we swapped B and Cs? Then,

$$\begin{aligned}\text{area of } ABC &= \tfrac{1}{2}\left[\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}\right] \\ &= -\tfrac{1}{2}\left[\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right].\end{aligned}$$

So, it makes sense to use the absolute value:

$$\boxed{\text{area of } ABC = \tfrac{1}{2}\left|\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right|}.$$

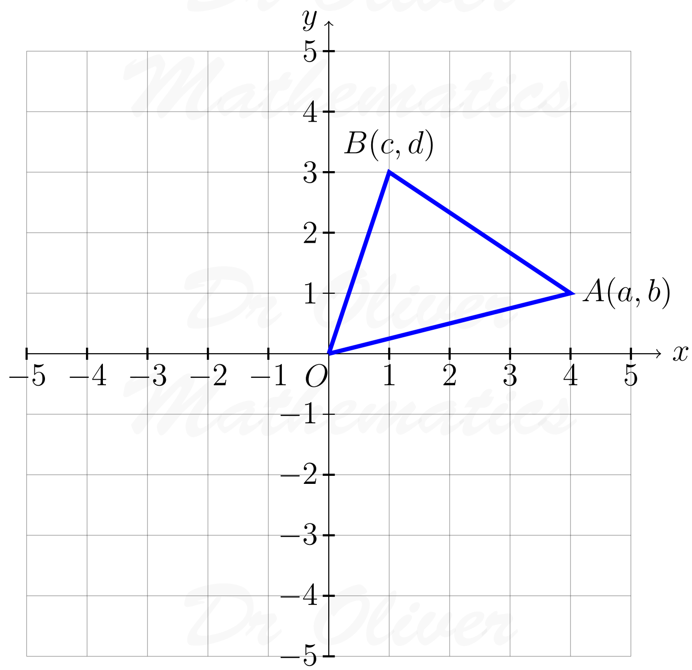
2.3 Original Example

Let $A(-1,-4)$, $B(-2,-1)$, and $C(2,2)$. Then

$$\begin{aligned}\text{area of } ABC &= \tfrac{1}{2}\left|\det\begin{pmatrix} -1 & -4 & 1 \\ -2 & -1 & 1 \\ 2 & 2 & 1 \end{pmatrix}\right| \\ &= \tfrac{1}{2}\left|-1(-1-2)+4(-2-2)+1(-4+2)\right| \\ &= \tfrac{1}{2}\left|3-16-2\right| \\ &= \tfrac{1}{2}\left|-15\right| \\ &= \underline{\underline{7\tfrac{1}{2}\text{ units}^2}}.\end{aligned}$$

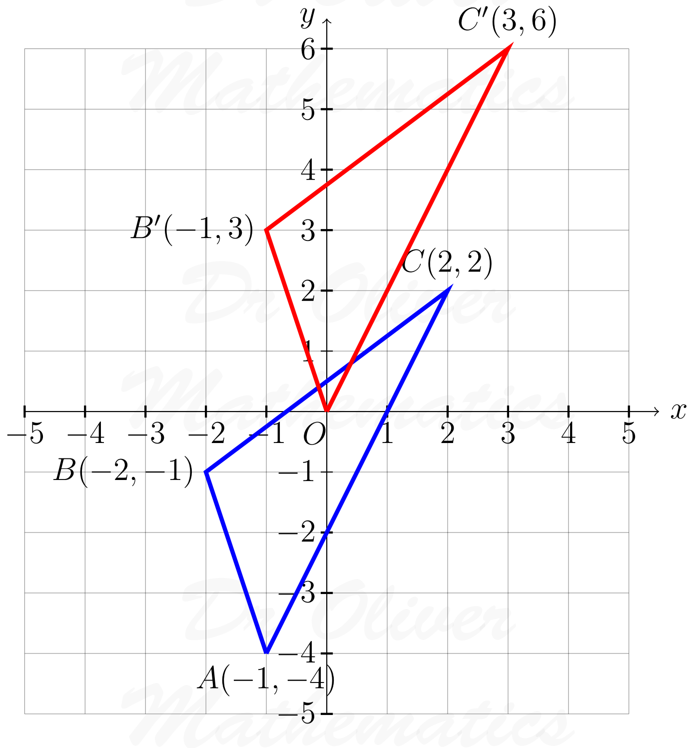
3 O is a Point on the Triangle

Let $O(0,0)$, $A(a,b)$, and $B(c,d)$. What is the area in this case?



Then

$$\begin{aligned} \text{area of } ABC &= \tfrac{1}{2}\left|\det\begin{pmatrix} 0 & 0 & 1 \\ a & b & 1 \\ c & d & 1 \end{pmatrix}\right| \\ &= \tfrac{1}{2}\left|0 - 0 + 1(ad - bc)\right| \\ &= \tfrac{1}{2}\left|ad - bc\right| \\ &= \tfrac{1}{2}\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|. \end{aligned}$$

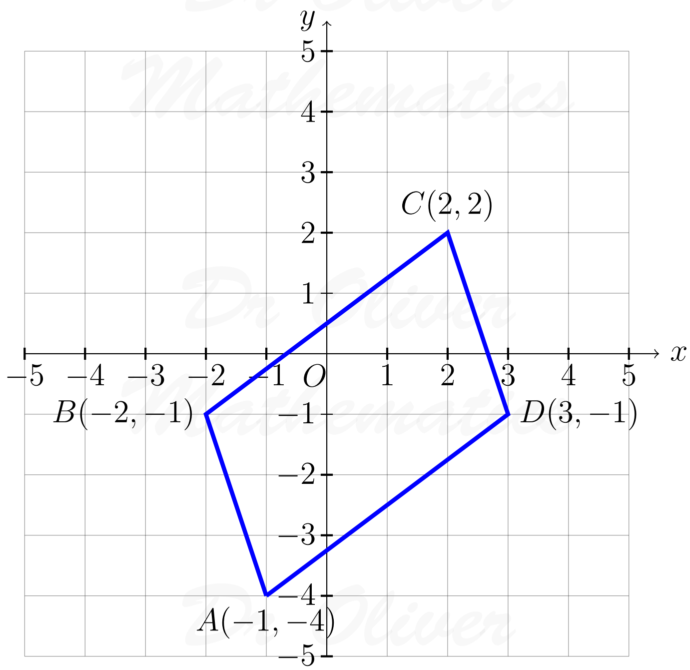


Let $O(0,0)$, $B'(-1,3)$, and $C'(3,6)$. Then

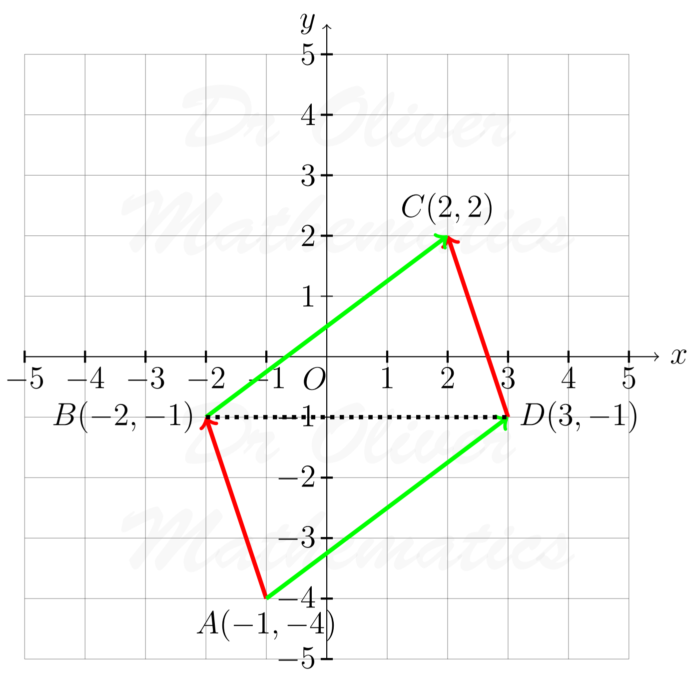
$$\begin{aligned}\text{area of } OB'C' &= \tfrac{1}{2}\left|\det\begin{pmatrix}-1 & 3\\ 3 & 6\end{pmatrix}\right|\\ &= \tfrac{1}{2}\left|-6-9\right|\\ &= \tfrac{1}{2}\left|-15\right|\\ &= \underline{\underline{7\tfrac{1}{2}\text{ units}^2}}.\end{aligned}$$

4 Area of a Parallelogram

What is the area of the parallelogram $ABCD$?



We split the parallelogram in to two pieces: triangles ABD and BCD:



Now,

- $\angle CBD = \angle BDA$ (adjacent angles),

- $\angle CDB = \angle BAD$ (adjacent angles), and
- side BD is common;

so, $\triangle BCD$ is identical to $\triangle ABD$ (AAS).

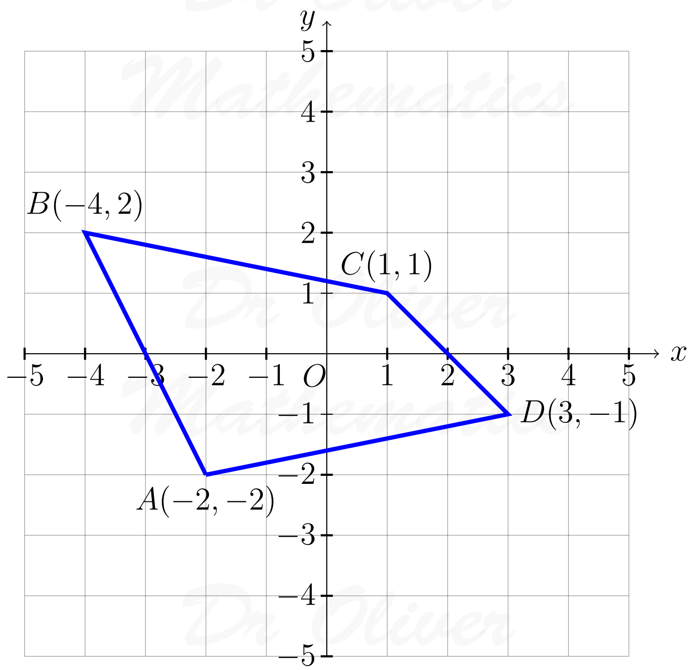
Hence,

$$\begin{aligned}
\text{area of } ABCD &= \text{area of } ACD + \text{area of } ABD \\
&= 2 \times \text{area of } ABD \\
&= 2 \times \tfrac{1}{2} \left| \det \begin{pmatrix} -1 & -4 & 1 \\ -2 & -1 & 1 \\ 3 & -1 & 1 \end{pmatrix} \right| \\
&= \left| \det \begin{pmatrix} -1 & -4 & 1 \\ -2 & -1 & 1 \\ 3 & -1 & 1 \end{pmatrix} \right| \\
&= |-1(-1+1) + 4(-2-3) + 1(2+3)| \\
&= |-15| \\
&= \underline{\underline{15 \text{ units}^2}}.
\end{aligned}$$

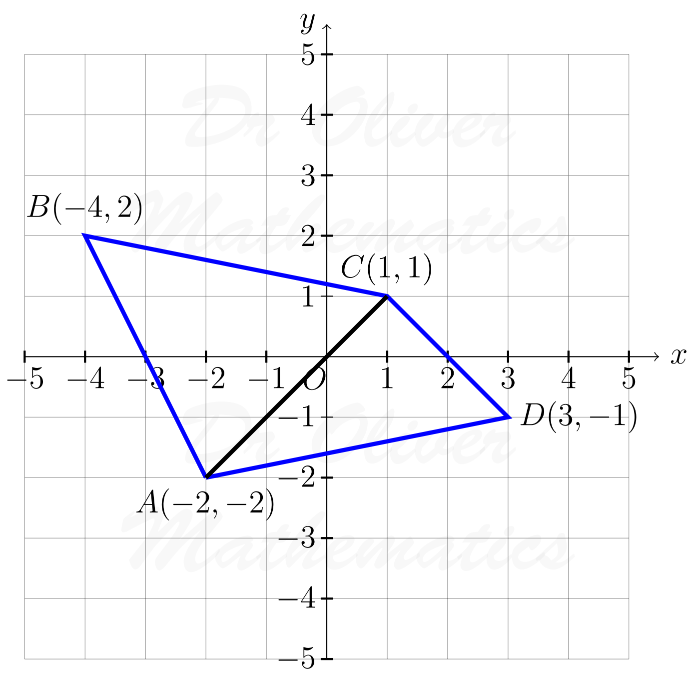
We could have split $ABCD$ into ABC and ACD: it makes no difference.

5 Area of a Quadrilateral

How about extending the result to 4-sided shape, $ABCD$?



We split it up into two areas:



Now,

$$\begin{aligned}
\text{area of } ABC &= \tfrac{1}{2}\left|\det\begin{pmatrix} -2 & -2 & 1 \\ -4 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right| \\
&= \tfrac{1}{2}\left|-2(2-1)+2(-4-1)+(-4-2)\right| \\
&= \tfrac{1}{2}\left|-2-10-6\right| \\
&= \tfrac{1}{2}\left|-18\right| \\
&= 18
\end{aligned}$$

and

$$\begin{aligned}
\text{area of } ADC &= \tfrac{1}{2}\left|\det\begin{pmatrix} -2 & -2 & 1 \\ 3 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right| \\
&= \tfrac{1}{2}\left|-2(-1-1)+2(3-1)+(3+1)\right| \\
&= \tfrac{1}{2}\left|4+4+4\right| \\
&= \tfrac{1}{2}\left|12\right| \\
&= 6;
\end{aligned}$$

thus,

$$\begin{aligned}
\text{area of } ABCD &= \text{area of } ABC + \text{area of } ADC \\
&= 18+6 \\
&= \underline{\underline{24 \text{ units}^2}}.
\end{aligned}$$

6 Test for Collinear Points

Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be distinct points. Then

$$\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = 0$$

if and only if the points are collinear.

What?!

Well,

$$\begin{aligned}
\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = 0 &\Leftrightarrow \text{area of } ABC = 0 \\
&\Leftrightarrow \text{one-dimensional length} \\
&\Leftrightarrow \text{the points are collinear.}
\end{aligned}$$

6.1 Collinear?

$A(-3,7)$, $B(1,-1)$, and $C(5,-10)$?

Well,

$$\det\begin{pmatrix} -3 & 7 & 1 \\ 1 & -1 & 1 \\ 5 & -10 & 1 \end{pmatrix} = -3(-1+10) - 7(1-5) + 1(-10+5)$$
$$= -27 + 28 - 5$$
$$= -4$$
$$\neq 0,$$

so the <u>three points are not collinear</u>.

6.2 Collinear?

$A(1,5.5)$, $B(-3,3.5)$, and $C(6,8)$?

Well,

$$\det\begin{pmatrix} 1 & 5.5 & 1 \\ -3 & 3.5 & 1 \\ 6 & 8 & 1 \end{pmatrix} = 1(3.5-8) - 5.5(-3-6) + 1(-24-21)$$
$$= -4.5 + 49.5 - 45$$
$$= 0,$$

so the <u>three points are collinear</u>.