Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2007 November Paper 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Given that (4)

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix},$$

find the value of each of the constants m and n for which

$$\mathbf{A}^2 + m\mathbf{A} = n\mathbf{I},$$

where $\mathbf{I}$ is the identity matrix.

Solution

$$\begin{aligned}\mathbf{A}^2 + m\mathbf{A} &= \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} + m\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -3 \\ 9 & -2 \end{pmatrix} + \begin{pmatrix} 2m & -m \\ 3m & m \end{pmatrix} \\ &= \begin{pmatrix} 1+2m & -3-m \\ 9+3m & -2+m \end{pmatrix}\end{aligned}$$

Now, $(1, 2)$th element:

$$-3 - m = 0 \Rightarrow \underline{\underline{m = -3}}$$

and this gives that

$$\mathbf{A}^2 - 3\mathbf{A} = \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix};$$

hence, $\underline{\underline{n = -5}}$.

2. Show that (4)

$$\frac{1}{1-\cos\theta} - \frac{1}{1+\cos\theta} \equiv 2\operatorname{cosec}\theta\cot\theta.$$

Solution

Well,

$$\frac{1}{1-\cos\theta} - \frac{1}{1+\cos\theta} \equiv \frac{(1+\cos\theta)-(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

on the denominator, difference of two squares:

$$\begin{aligned} &\equiv \frac{2\cos\theta}{1-\cos^2\theta} \\ &\equiv \frac{2\cos\theta}{\sin^2\theta} \\ &\equiv 2\left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) \\ &\equiv \underline{\underline{2\operatorname{cosec}\theta\cot\theta}}, \end{aligned}$$

as required.

3. Given that

$$p = \frac{\sqrt{3}+1}{\sqrt{3}-1},$$

express in its simplest surd form,

(a) p, (3)

Solution

Now,

$\times$	$\sqrt{3}$	$+1$
$\sqrt{3}$	3	$+\sqrt{3}$
± 1	$\pm\sqrt{3}$	± 1

so we have

$$\begin{aligned}\frac{\sqrt{3}+1}{\sqrt{3}-1} &= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} \\ &= \frac{3+2\sqrt{3}+1}{3-1} \\ &= \frac{4+2\sqrt{3}}{2} \\ &= \underline{\underline{2+\sqrt{3}}}.\end{aligned}$$

(b) $p - \dfrac{1}{p}$. (2)

Solution

Well,

$$\begin{aligned}p - \frac{1}{p} &= \frac{\sqrt{3}+1}{\sqrt{3}-1} - \frac{\sqrt{3}-1}{\sqrt{3}+1} \\ &= \frac{(\sqrt{3}+1)^2 - (\sqrt{3}-1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} \\ &= \frac{(3+2\sqrt{3}+1) - (3-2\sqrt{3}+1)}{3-1} \\ &= \frac{4\sqrt{3}}{2} \\ &= \underline{\underline{2\sqrt{3}}}.\end{aligned}$$

4. A badminton team of 4 men and 4 women is to be selected from 9 men and 6 women.

(a) Find the total number of ways in which the team can be selected if there are no restrictions on the selection. (3)

Solution

The total number of ways is

$$\begin{aligned}\binom{9}{4} \times \binom{6}{4} &= 126 \times 15 \\ &= \underline{\underline{1\,890}}.\end{aligned}$$

Two of the men are twins.

(b) Find the number of ways in which the team can be selected if exactly one of the twins is in the team. (3)

Solution

"If exactly one of the twins is in the team" so that means we are left with 7 men.

The total number of ways is

$$\begin{aligned} 2 \times \binom{7}{3} \times \binom{6}{4} &= 2 \times 35 \times 15 \\ &= \underline{\underline{1\,050}}. \end{aligned}$$

5. In this question, $\mathbf{i}$ is a unit vector due east, and $\mathbf{j}$ is a unit vector due north.

A plane flies from P to Q where

$$\overrightarrow{PQ} = (960\mathbf{i} + 400\mathbf{j}) \text{ km}.$$

A constant wind is blowing with velocity

$$(-60\mathbf{i} + 60\mathbf{j}) \text{ km h}^{-1}.$$

Given that the plane takes 4 hours to travel from P to Q, find

(a) the velocity, in still air, of the plane, giving your answer in the form $(a\mathbf{i}+b\mathbf{j})$ km h^{-1}. (4)

Solution

Well, the resultant velocity is

$$(960\mathbf{i} + 400\mathbf{j}) \div 4 = (240\mathbf{i} + 100\mathbf{j}).$$

Finally,

$$\begin{aligned} v_{\text{still air}} &= (240\mathbf{i} + 100\mathbf{j}) - (-60\mathbf{i} + 60\mathbf{j}) \\ &= \underline{\underline{(300\mathbf{i} + 40\mathbf{j}) \text{ km h}^{-1}}}. \end{aligned}$$

(b) the bearing, to the nearest degree, on which the plane must be directed. (2)

Solution

Well,

$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan\theta = \frac{40}{300}$$
$$\Rightarrow \tan\theta = \tfrac{2}{15}$$
$$\Rightarrow \theta = 7.594\,643\,369 \text{ (FCD)}$$
$$\Rightarrow 90 - \theta = 82.405\,356\,63 \text{ (FCD)}$$

and we have a bearing of 082° (nearest whole number).

6. A curve is such that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{\sqrt{4x+1}},$$

and $(6, 20)$ is a point on the curve.

(a) Find the equation of the curve. (4)

Solution

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{\sqrt{4x+1}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6(4x+1)^{-\frac{1}{2}}$$
$$\Rightarrow y = 3(4x+1)^{\frac{1}{2}} + c,$$

where c is some constant. Next,

$$x = 6, y = 20 \Rightarrow 20 = 3(4 \times 6 + 1)^{\frac{1}{2}} + c$$
$$\Rightarrow 20 = 3(25)^{\frac{1}{2}} + c$$
$$\Rightarrow 20 = 15 + c$$
$$\Rightarrow c = 5;$$

hence, the equation of the curve is

$$y = 3(4x+1)^{\frac{1}{2}} + 5.$$

A line with gradient $-\frac{1}{2}$ is a normal to the curve.

(b) Find the coordinates of the points at which this normal meets the coordinate axes. (4)

Solution

Now,

$$m_{\text{normal}} = -\tfrac{1}{2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 2 &\Rightarrow \frac{6}{\sqrt{4x+1}} = 2\\ &\Rightarrow 3 = \sqrt{4x+1}\\ &\Rightarrow 9 = 4x+1\\ &\Rightarrow 8 = 4x\\ &\Rightarrow x = 2\\ &\Rightarrow y = 14;\end{aligned}$$

hence, the coordinates are $(2, 14)$.

Now, the equation of the line is

$$\begin{aligned}y - 14 = -\tfrac{1}{2}(x-2) &\Rightarrow y - 14 = -\tfrac{1}{2}x + 1\\ &\Rightarrow y = -\tfrac{1}{2}x + 15.\end{aligned}$$

Now,

$$x = 0 \Rightarrow y = 15$$

and

$$y = 0 \Rightarrow x = 30;$$

hence, the coordinates are $\underline{\underline{(0, 15)}}$ and $\underline{\underline{(30, 0)}}$.

7. (a) Use the substitution $u = 2^x$ to solve the equation (5)

$$2^{2x} = 2^{x+2} + 5.$$

Solution

$$\begin{aligned}2^{2x} = 2^{x+2} + 5 &\Rightarrow (2^x)^2 = 2^2\,2^x + 5\\ &\Rightarrow u^2 = 4u + 5\\ &\Rightarrow u^2 - 4u - 5 = 0\end{aligned}$$

$$\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & -5 \end{array} \Bigg\} -5,\ +1$$

$$\begin{aligned} &\Rightarrow (u-5)(u+1) = 0 \\ &\Rightarrow u = 5 \text{ or } u = -1 \\ &\Rightarrow 2^x = 5 \text{ or } 2^x = -1 \text{ (can't happen)} \\ &\Rightarrow \underline{\underline{x = \log_2 5 \text{ or } 2.32 \text{ (3 sf)}}}. \end{aligned}$$

(b) Solve the equation (4)

$$2\log_9 3 + \log_5(7y-3) = \log_2 8.$$

Solution

$$\begin{aligned} 2\log_9 3 + \log_5(7y-3) = \log_2 8 &\Rightarrow \log_9 3^2 + \log_5(7y-3) = \log_2 2^3 \\ &\Rightarrow \log_9 9 + \log_5(7y-3) = 3\log_2 2 \\ &\Rightarrow 1 + \log_5(7y-3) = 3 \\ &\Rightarrow \log_5(7y-3) = 2 \\ &\Rightarrow 7y - 3 = 5^2 \\ &\Rightarrow 7y - 3 = 25 \\ &\Rightarrow 7y = 28 \\ &\Rightarrow \underline{\underline{y = 4}}. \end{aligned}$$

8. (a) The remainder when the expression (4)

$$x^3 - 11x^2 + kx - 30$$

is divided by $(x-1)$ is 4 times the remainder when this expression is divided by $(x-2)$.

Find the value of the constant k.

Solution

We use synthetic division twice:

$$\begin{array}{c|cccc} 1 & 1 & -11 & k & -30 \\ & \downarrow & 1 & -10 & k-10 \\ \hline & 1 & -10 & k-10 & k-40 \end{array}$$

and

$$\begin{array}{c|cccc} 2 & 1 & -11 & k & -30 \\ & \downarrow & 2 & -18 & 2k-36 \\ \hline & 1 & -9 & k-18 & 2k-66 \end{array}$$

Now,

$$\begin{aligned} k - 40 = 4(2k - 66) &\Rightarrow k - 40 = 8k - 264 \\ &\Rightarrow 7k = 224 \\ &\Rightarrow \underline{\underline{k = 32}}. \end{aligned}$$

(b) Solve the equation (5)

$$x^3 - 4x^2 - 8x + 8 = 0,$$

expressing non-integer solutions in the form $a \pm \sqrt{b}$, where a and b are integers.

Solution

Let

$$f(x) = x^3 - 4x^2 - 8x + 8.$$

Now,

$$\begin{aligned} f(1) &= 1 - 4 - 8 + 8 = -3 \\ f(-1) &= -1 - 4 + 8 + 8 = 13 \\ f(2) &= 8 - 16 - 16 + 8 = -16 \\ f(-2) &= -8 - 16 + 16 + 8 = 0 \end{aligned}$$

so $(x + 2)$ is a factor.

$$\begin{array}{c|cccc} -2 & 1 & -4 & -8 & 8 \\ & \downarrow & -2 & 12 & -8 \\ \hline & 1 & -6 & 4 & 0 \end{array}$$

The quadratic factor is

$$\begin{aligned} x^2 - 6x + 4 = 0 &\Rightarrow x^2 - 6x = -4 \\ &\Rightarrow x^2 - 6x + 9 = -4 + 9 \\ &\Rightarrow (x-3)^2 = 5 \\ &\Rightarrow x - 3 = \pm\sqrt{5} \\ &\Rightarrow x = 3 \pm \sqrt{5}; \end{aligned}$$

hence,

$$\underline{\underline{x = -2,\ 3 \pm \sqrt{5}}}.$$

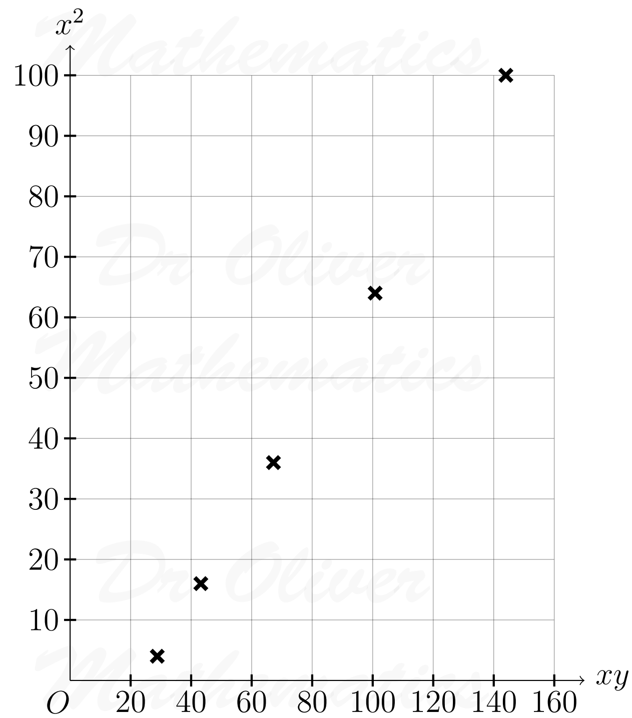
9. The table shows experimental values of two variables, x and y.

x	2	4	6	8	10
y	14.4	10.8	11.2	12.6	14.4

(a) Using graph paper, plot xy against x^2. (2)

Solution

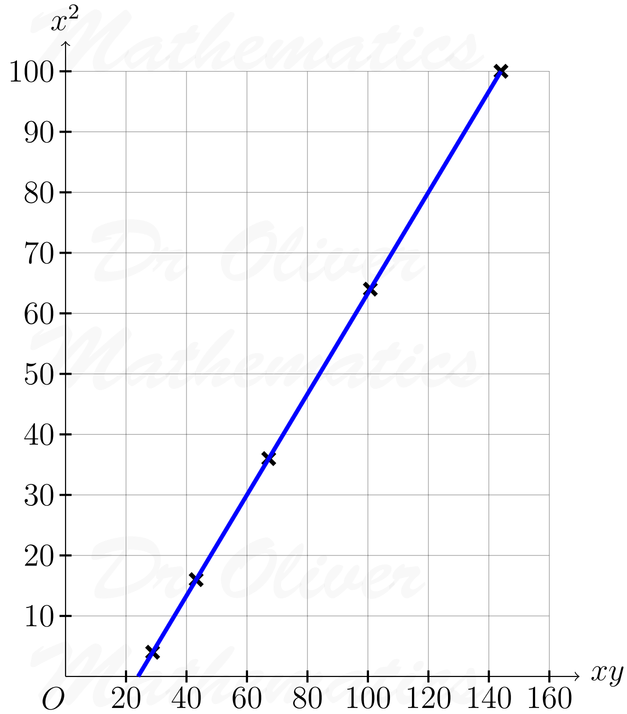
xy	28.8	43.2	67.2	100.8	144
x^2	4	16	36	64	100



(b) Use the graph of xy against x^2 to express y in terms of x. (4)

Solution

We draw in a line of best fit:



E.g., the line goes through $(12, 0)$ and $(144, 100)$. Now,

$$m = \frac{100 - 0}{144 - 12} = \tfrac{25}{33}$$

and the equation of the line is

$$x^2 - 0 = \tfrac{25}{33}(xy - 12) \Rightarrow x^2 = \tfrac{25}{33}xy - \tfrac{100}{11}$$
$$\Rightarrow x = \tfrac{25}{33}y - \tfrac{100}{11x}$$
$$\Rightarrow x + \tfrac{100}{11x} = \tfrac{25}{33}y$$
$$\Rightarrow \underline{\underline{y = \tfrac{33}{25}x + \tfrac{12}{x}}}.$$

(c) Find the value of y for which (3)

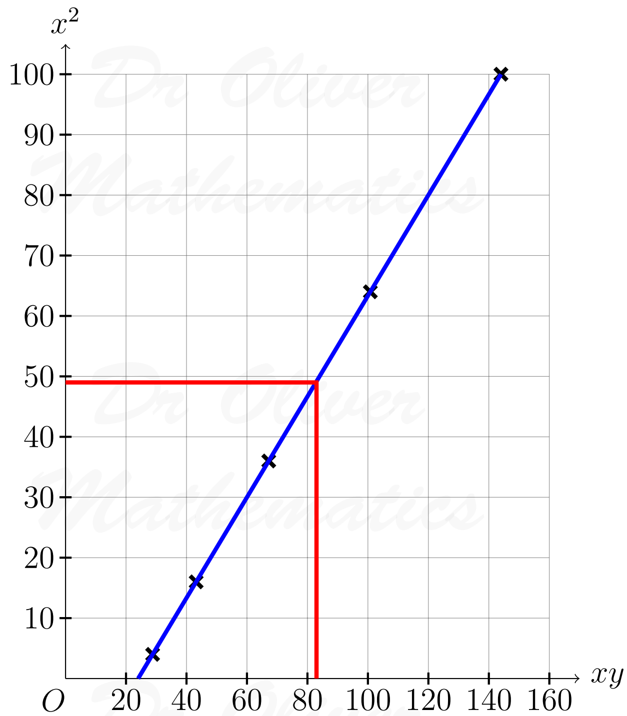
$$y = \frac{83}{x}.$$

Solution

Now,

$$y = \frac{83}{x} \Rightarrow xy = 83$$

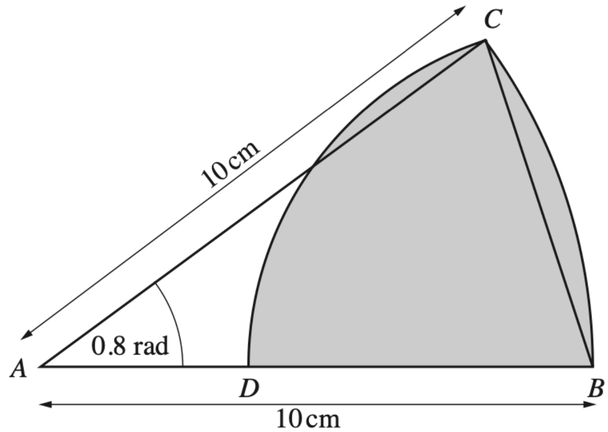
and we read-off from the line:



Finally,

$$x^2 = 49 \Rightarrow y = \frac{83}{\sqrt{49}}$$
$$\Rightarrow \underline{\underline{y = 11\tfrac{6}{7}}}$$

10. The diagram shows a sector ABC of the circle, centre A and radius 10 cm, in which angle $BAC = 0.8$ radians.



The arc CD of a circle has centre B and the point D lies on AB.

(a) Show that the length of the straight line BC is 7.79 cm, correct to 2 decimal places. (2)

Solution

$$\begin{aligned} & BC^2 = AC^2 + AE^2 - 2 \times AC \times AE \times \cos BAC \\ \Rightarrow\ & BC^2 = 10^2 + 10^2 - 2 \times 10 \times 10 \times \cos 0.8 \\ \Rightarrow\ & BC^2 = 200 - 200 \cos 0.8 \\ \Rightarrow\ & BC = 7.788\,366\,846 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{BC = 7.79 \text{ cm (2 dp)}}}, \end{aligned}$$

as required.

(b) Find the perimeter of the shaded region. (4)

Solution

Well, $\angle ABC = \frac{1}{2}(\pi - 0.8)$. Now,

$$\begin{aligned} \text{perimeter} &= \text{arc } CD + \text{arc } BC + BD \\ &= [7.788\ldots \times \tfrac{1}{2}(\pi - 0.8)] + (10 \times 0.8) + 7.788\ldots \\ &= 24.906\,958\,14 \text{ (FCD)} \\ &= \underline{\underline{24.9 \text{ cm (3 sf)}}}. \end{aligned}$$

(c) Find the area of the shaded region. (4)

Solution
Well,

$$\begin{aligned}\text{area of the sector } BCD &= \tfrac{1}{2} \times (7.788\ldots)^2 \times \tfrac{1}{2}(\pi - 0.8)\\ &= 35.509\,467\,06 \text{ (FCD)}\end{aligned}$$

and

$$\begin{aligned}\text{area of the segment on} BC &= \tfrac{1}{2} \times 10^2 \times (0.8 - \sin 0.8)\\ &= 4.132\,195\,455 \text{ (FCD)}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{area} &= 35.509\ldots + 4.132\ldots\\ &= 39.641\,662\,52 \text{ (FCD)}\\ &= \underline{\underline{39.6 \text{ cm}^2 \text{ (3 sf)}}}\end{aligned}$$

EITHER

11. A curve has the equation

$$y = xe^{2x}.$$

(a) Obtain expressions for (5)

$$\frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2}.$$

Solution
Now,

$$\begin{aligned}u = x &\Rightarrow \frac{du}{dx} = 1\\ v = e^{2x} &\Rightarrow \frac{dv}{dx} = 2e^{2x}\end{aligned}$$

and so

$$\begin{aligned}\frac{dy}{dx} &= (x)(2e^{2x}) + (1)(e^{2x})\\ &= \underline{\underline{(2x+1)e^{2x}}}.\end{aligned}$$

Next,

$$u = 2x + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
$$v = \mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2\mathrm{e}^{2x}$$

and so

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= (2x+1)(2\mathrm{e}^{2x}) + (2)(\mathrm{e}^{2x}) \\ &= (4x+4)\mathrm{e}^{2x} \\ &= \underline{\underline{4(x+1)\mathrm{e}^{2x}}}.\end{aligned}$$

(b) Show that the y-coordinate of the stationary point of the curve is $-\frac{1}{2\mathrm{e}}$. (3)

Solution

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow (2x+1)\mathrm{e}^{2x} = 0$$

$\mathrm{e}^x > 0$:

$$\begin{aligned}&\Rightarrow 2x + 1 = 0 \\ &\Rightarrow x = -\tfrac{1}{2} \\ &\Rightarrow y = (-\tfrac{1}{2})\mathrm{e}^{2(-\frac{1}{2})} \\ &\Rightarrow y = -\tfrac{1}{2}\mathrm{e}^{-1} \\ &\Rightarrow \underline{\underline{y = -\tfrac{1}{2\mathrm{e}}}},\end{aligned}$$

as required.

(c) Determine the nature of this stationary point. (2)

Solution

Now,

$$x = -\tfrac{1}{2} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.735\ldots > 0$$

and it is a <u>minimum</u>.

OR

12. (a) Show that (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\ln x}{x^2}\right) = \frac{1 - 2\ln x}{x^3}.$$

Solution

Now,

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$v = x^2 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\ln x}{x^2}\right) &= \frac{(x^2)\left(\frac{1}{x}\right) - (\ln x)(2x)}{(x^2)^2} \\ &= \frac{x - 2x\ln x}{x^4} \\ &= \frac{x(1 - 2\ln x)}{x^4} \\ &= \underline{\underline{\frac{1 - 2\ln x}{x^3}}},\end{aligned}$$

as required.

(b) Show that the y-coordinate of the stationary point of the curve (3)

$$y = \frac{\ln x}{x^2}$$

is $\frac{1}{2\mathrm{e}}$.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \frac{1-2\ln x}{x^3} = 0\\ &\Rightarrow 1 - 2\ln x = 0\\ &\Rightarrow 2\ln x = 1\\ &\Rightarrow \ln x = \tfrac{1}{2}\\ &\Rightarrow x = \mathrm{e}^{\frac{1}{2}}\\ &\Rightarrow y = \frac{\ln \mathrm{e}^{\frac{1}{2}}}{(\mathrm{e}^{\frac{1}{2}})^2}\\ &\Rightarrow y = \frac{\frac{1}{2}}{\mathrm{e}}\\ &\Rightarrow \underline{\underline{y = \tfrac{1}{2\mathrm{e}}}},\end{aligned}$$

as required.

(c) Use the result from part (a) to find (4)

$$\int \left(\frac{\ln x}{x^3}\right) \mathrm{d}x.$$

Solution

Now,

$$\begin{aligned} & \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\ln x}{x^2}\right) = \frac{1-2\ln x}{x^3}\\ \Rightarrow\quad & \frac{\ln x}{x^2} = \int \left(\frac{1-2\ln x}{x^3}\right) \mathrm{d}x\\ \Rightarrow\quad & \frac{\ln x}{x^2} = \int \frac{1}{x^3}\,\mathrm{d}x - 2\int \left(\frac{\ln x}{x^3}\right) \mathrm{d}x\\ \Rightarrow\quad & \frac{\ln x}{x^2} = \int x^{-3}\,\mathrm{d}x - 2\int \left(\frac{\ln x}{x^3}\right) \mathrm{d}x\\ \Rightarrow\quad & \frac{\ln x}{x^2} = -\tfrac{1}{2}x^{-2} + c - 2\int \left(\frac{\ln x}{x^3}\right) \mathrm{d}x\\ \Rightarrow\quad & 2\int \left(\frac{\ln x}{x^3}\right) \mathrm{d}x = -\tfrac{1}{2}x^{-2} - \frac{\ln x}{x^2} + c\\ \Rightarrow\quad & \int \left(\frac{\ln x}{x^3}\right) \mathrm{d}x = \underline{\underline{-\tfrac{1}{4}x^{-2} - \frac{\ln x}{2x^2} + c.}}\end{aligned}$$