Dr Oliver Mathematics
Advanced Level Paper 32: Mechanics
November 2021: Calculator
2 hours

The total number of marks available is 50.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.
(It goes with Paper 31: Statistics)

1. A particle P moves with constant acceleration $(2\mathbf{i} - 3\mathbf{j})$ m s^{-2}.
   At time $t = 0$, P is moving with velocity $4\mathbf{i}$ m s^{-1}.

   (a) Find the velocity of P at time $t = 2$ seconds. (2)

   Solution

   $$\mathbf{a}(t) = (2\mathbf{i} - 3\mathbf{j}) \Rightarrow \mathbf{v}(t) = (2\mathbf{i} - 3\mathbf{j})t + \mathbf{c},$$

   where $\mathbf{c}$ is some vector. Now,

   $$t = 0 \Rightarrow \mathbf{v}(0) = 4\mathbf{i}$$

   and so

   $$\mathbf{v}(t) = [4\mathbf{i} + (2\mathbf{i} - 3\mathbf{j})t].$$

   Finally,

   $$\mathbf{v}(t) = 4\mathbf{i} + 2(2\mathbf{i} - 3\mathbf{j}) = \underline{\underline{(8\mathbf{i} - 6\mathbf{j}) \text{ m s}^{-1}}}.$$

   At time $t = 0$, the position vector of P relative to a fixed origin O is $(\mathbf{i} + \mathbf{j})$ m.

   (b) Find the position vector of P relative to O at time $t = 3$ seconds. (2)

   Solution

   $$\mathbf{v}(t) = (2t + 4)\mathbf{i} - 3t\mathbf{j} \Rightarrow \mathbf{r}(t) = (t^2 + 4t)\mathbf{i} - \tfrac{3}{2}t^2\mathbf{j} + \mathbf{d},$$

   where $\mathbf{d}$ is some vector. Now,

   $$t = 0 \Rightarrow \mathbf{i} + \mathbf{j} = \mathbf{d}$$

   and so

   $$\mathbf{r}(t) = (t^2 + 4t + 1)\mathbf{i} - (\tfrac{3}{2}t^2 - 1)\mathbf{j}.$$

   Finally,

   $$t = 3 \Rightarrow \mathbf{r}(3) = \underline{\underline{(22\mathbf{i} - \tfrac{25}{2}\mathbf{j}) \text{ m}}}.$$

2. A small stone A of mass $3m$ is attached to one end of a string.

A small stone B of mass m is attached to the other end of the string.

Initially A is held at rest on a fixed rough plane.

The plane is inclined to the horizontal at an angle α, where $\tan\alpha = \frac{3}{4}$.

The string passes over a pulley P that is fixed at the top of the plane.

The part of the string from A to P is parallel to a line of greatest slope of the plane.

Stone B hangs freely below P, as shown in Figure 1.

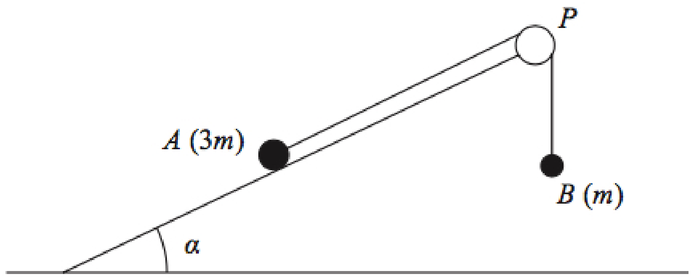

Figure 1: two small stones

The coefficient of friction between A and the plane is $\frac{1}{6}$.

Stone A is released from rest and begins to move down the plane.

The stones are modelled as particles.

The pulley is modelled as being small and smooth.

The string is modelled as being light and inextensible.

Using the model for the motion of the system before B reaches the pulley,

(a) write down an equation of motion for A, (2)

Solution

Let T N be the tension, let R N be the normal reaction, and let a ms^{-2} be the acceleration.

Newton's Second Law:
for A,

$$\underline{\underline{3mg\sin\alpha - T - F = 3ma}}$$

(b) show that the acceleration of A is $\frac{1}{10}g$, (7)

Solution
For A,

$$\begin{aligned} \text{Parallel:} &\quad 3mg\sin\alpha - T - F = 3ma \\ \text{Perpendicular:} &\quad R = 3mg\cos\alpha \\ F = \mu R: &\quad F = \tfrac{1}{6}R \end{aligned}$$

For B,

$$T - mg = ma.$$

Now, take a deep breath:

$$\begin{aligned} & 3mg\sin\alpha - T - F = 3ma \\ \Rightarrow\quad & 3mg\sin\alpha - (ma + mg) - \tfrac{1}{6}R = 3ma \\ \Rightarrow\quad & 3mg\sin\alpha - (ma + mg) - \tfrac{1}{6}(3mg\cos\alpha) = 3ma \\ \Rightarrow\quad & \tfrac{9}{5}mg - (ma + mg) - \tfrac{1}{6}(\tfrac{12}{5}mg) = 3ma \\ \Rightarrow\quad & \tfrac{9}{5}g - a - g - \tfrac{2}{5}g = 3a \\ \Rightarrow\quad & \tfrac{2}{5}g = 4a \\ \Rightarrow\quad & \underline{\underline{a = \tfrac{1}{10}g}}, \end{aligned}$$

as required.

(c) sketch a velocity-time graph for the motion of B, from the instant when A is released from rest to the instant just before B reaches the pulley, explaining your answer. (2)

Solution

E.g., the acceleration of B is constant.

In reality, the string is not light.

(d) State how this would affect the working in part (b). (1)

Solution

E.g., the tensions in the two strings would be different.

3. A beam AB has mass m and length $2a$.

The beam rests in equilibrium with A on rough horizontal ground and with B against a smooth vertical wall.

The beam is inclined to the horizontal at an angle θ, as shown in Figure 2.

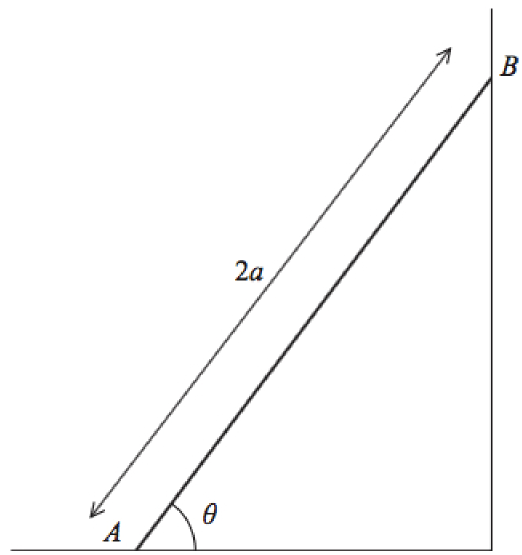

Figure 2: a beam AB has mass m and length $2a$

The coefficient of friction between the beam and the ground is μ.

The beam is modelled as a uniform rod resting in a vertical plane that is perpendicular to the wall.

Using the model,

(a) show that $\mu \geqslant \frac{1}{2}\cot\theta$. (5)

Solution

A: F N be the frictional force and R N be the normal reaction.
B: P N be the normal reaction.

$$\begin{aligned} R(\updownarrow): &\quad R = mg \\ R(\leftrightarrow): &\quad F = P \\ \text{Limiting equilibrium}: &\quad F \leqslant \mu R \\ \text{Moments about } A: &\quad (a)(mg\cos\theta) = (2P)(a\sin\theta). \end{aligned}$$

Now,

$$\begin{aligned} amg\cos\theta = 2aP\sin\theta &\Rightarrow mg\cos\theta = 2P\sin\theta \\ &\Rightarrow R\cos\theta = 2P\sin\theta \\ &\Rightarrow P = \frac{R\cos\theta}{2\sin\theta} \\ &\Rightarrow P = \tfrac{1}{2}R\cot\theta. \end{aligned}$$

Next,

$$\begin{aligned} F \leqslant \mu R \Rightarrow P &\leqslant \mu R \\ &\Rightarrow \tfrac{1}{2} R \cot\theta \leqslant \mu R \\ &\Rightarrow \underline{\underline{\mu \geqslant \tfrac{1}{2}\cot\theta}}, \end{aligned}$$

as required.

A horizontal force of magnitude kmg, where k is a constant, is now applied to the beam at A.

This force acts in a direction that is perpendicular to the wall and towards the wall.

Given that $\tan\theta = \frac{5}{4}$, $\mu = \frac{1}{2}$, and the beam is now in limiting equilibrium,

(b) use the model to find the value of k. (5)

Solution

$$\begin{aligned} R(\updownarrow): &\quad R = mg \\ R(\leftrightarrow): &\quad kmg - F = P \\ \text{Limiting equilibrium}: &\quad F = \tfrac{1}{2}R \\ \text{Moments about } A: &\quad (a)(mg\cos\theta) = (2P)(a\sin\theta). \end{aligned}$$

Now, another deep breath:

$$\begin{aligned} (a)(mg\cos\theta) = (2P)(a\sin\theta) &\Rightarrow mg\cos\theta = 2P\sin\theta \\ &\Rightarrow mg = 2P\tan\theta \\ &\Rightarrow mg = \tfrac{5}{2}P \\ &\Rightarrow mg = \tfrac{5}{2}(kmg - F) \\ &\Rightarrow mg = \tfrac{5}{2}(kmg - \tfrac{1}{2}R) \\ &\Rightarrow mg = \tfrac{5}{2}(kmg - \tfrac{1}{2}mg) \\ &\Rightarrow 1 = \tfrac{5}{2}(k - \tfrac{1}{2}) \\ &\Rightarrow \tfrac{2}{5} = k - \tfrac{1}{2} \\ &\Rightarrow \underline{\underline{k = \tfrac{9}{10}}}. \end{aligned}$$

4. A small stone is projected with speed 65 $\mathrm{m\,s^{-1}}$ from a point O at the top of a vertical cliff.

Point O is 70 m vertically above the point N.

Point N is on horizontal ground.

The stone is projected at an angle α above the horizontal, where $\tan\alpha = \frac{5}{12}$.

The stone hits the ground at the point A, as shown in Figure 3.

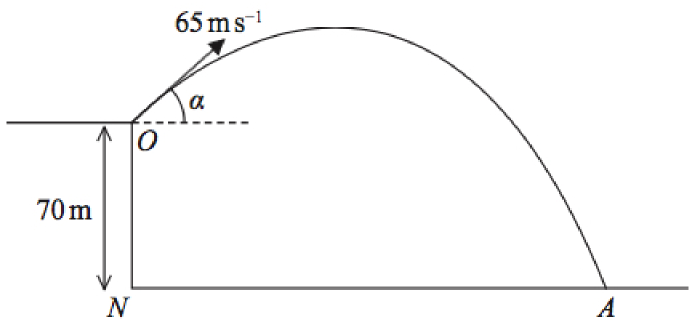

Figure 3: small stone is projected with speed 65 ms^{-1}

The stone is modelled as a particle moving freely under gravity.

The acceleration due to gravity is modelled as having magnitude 10 m s^{-2}.

Using the model,

(a) find the time taken for the stone to travel from O to A, (4)

Solution

Vertically: $s = -70$, $u = 65\sin\alpha$, $v = ?$, $a = -10$, and $t = ?$:
we use $s = ut + \frac{1}{2}at^2$:

$$-70 = 65\sin\alpha t + \tfrac{1}{2} \times (-10) \times t^2 \Rightarrow -70 = 65(\tfrac{5}{13})t - 5t^2$$
$$\Rightarrow -70 = 25t - 5t^2$$
$$\Rightarrow 5t^2 - 25t - 70 = 0$$
$$\Rightarrow 5(t^2 - 5t - 14) = 0$$

add to: -5; multiply to: -14 $\Big\} -7, +2$

$$\Rightarrow 5(t-7)(t+2) = 0$$
$$\Rightarrow t = 7 \text{ or } t = -2;$$

but $t > 0$ and so $\underline{\underline{t = 7}}$.

(b) find the speed of the stone at the instant just before it hits the ground at A. (5)

Solution

Horizontally:

$$u = 65\cos\alpha = 65 \times \tfrac{12}{13} = 60.$$

Vertically: $s = ?$, $u = 65\sin\alpha$, $v = ?$, $a = -10$, and $t = 7$:
we use $v = u + at$:

$$\begin{aligned} v &= 65\sin\alpha - 7g \\ &= 25 - 70 \\ &= -45. \end{aligned}$$

Finally,

$$\begin{aligned} \text{speed} &= \sqrt{60^2 + (-45)^2} \\ &= \underline{\underline{75 \text{ m s}^{-1}}}. \end{aligned}$$

One limitation of the model is that it ignores air resistance.

(c) State one other limitation of the model that could affect the reliability of your answers. (1)

Solution

E.g., we have use an approximation for g, wind effect, spin of the stone, etc.

5. At time t seconds, a particle P has velocity $\mathbf{v}$ m s^{-1}, where

$$\mathbf{v} = 3t^{\frac{1}{2}}\mathbf{i} - 2t\mathbf{j},\ t > 0.$$

(a) Find the acceleration of P at time t seconds, where $t > 0$. (2)

Solution

$$\mathbf{v}(t) = 3t^{\frac{1}{2}}\mathbf{i} - 2t\mathbf{j} \Rightarrow \underline{\underline{\mathbf{a}(t) = (\tfrac{3}{2}t^{-\frac{1}{2}}\mathbf{i} - 2\mathbf{j})\ \mathrm{m\,s^{-2}}}}.$$

(b) Find the value of t at the instant when P is moving in the direction of $\mathbf{i} - \mathbf{j}$. (3)

Solution

Well, we solve:

$$\begin{aligned} 3t^{\frac{1}{2}} = 2t &\Rightarrow 3t^{\frac{1}{2}} - 2t = 0 \\ &\Rightarrow t^{\frac{1}{2}}(3 - 2t^{\frac{1}{2}}) = 0 \\ &\Rightarrow t^{\frac{1}{2}} = 0 \text{ or } t^{\frac{1}{2}} = \tfrac{3}{2} \\ &\Rightarrow t = 0 \text{ or } t = 2\tfrac{1}{4}; \end{aligned}$$

since $t > 0$, $\underline{\underline{t = 2\frac{1}{4}}}$ s.

At time t seconds, where $t > 0$, the position vector of P, relative to a fixed origin O, is $\mathbf{r}$ metres.

When $t = 1$, $\mathbf{r} = -\mathbf{j}$.

(c) Find an expression for $\mathbf{r}$ in terms of t. (3)

Solution

$$\mathbf{v}(t) = 3t^{\frac{1}{2}}\mathbf{i} - 2t\mathbf{j} \Rightarrow \mathbf{r}(t) = 2t^{\frac{3}{2}}\mathbf{i} - t^2\mathbf{j} + \mathbf{c},$$

for some vector $\mathbf{c}$. Now,

$$\begin{aligned} t = 1, \mathbf{r} = -\mathbf{j} &\Rightarrow -\mathbf{j} = 2\mathbf{i} - \mathbf{j} + \mathbf{c} \\ &\Rightarrow \mathbf{c} = -2\mathbf{i} \end{aligned}$$

and so

$$\underline{\underline{\mathbf{r}(t) = (2t^{\frac{3}{2}} - 2)\mathbf{i} - t^2\mathbf{j}}}.$$

(d) Find the exact distance of P from O at the instant when P is moving with speed $10\ \mathrm{m\,s^{-1}}$. (6)

Solution

Now,

$$\sqrt{\left(3t^{\frac{1}{2}}\right)^2 + (-2t)^2} = 10 \Rightarrow \left(3t^{\frac{1}{2}}\right)^2 + (-2t)^2 = 10^2$$
$$\Rightarrow 9t + 4t^2 = 100$$
$$\Rightarrow 4t^2 + 9t - 100 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & +9 \\ \text{multiply to:} & (+4) \times (-100) = -400 \end{array}\right\} +25,\ -16$$

$$\Rightarrow 4t^2 + 25t - 16t - 100 = 0$$
$$\Rightarrow t(4t + 25) - 4(4t + 25) = 0$$
$$\Rightarrow (t - 4)(4t + 25) = 0$$
$$\Rightarrow t = 4 \text{ or } t = -6\tfrac{1}{4};$$

since $t > 0$, $t = 4$ and so

$$t = 4 \Rightarrow \mathbf{r}(t) = 14\mathbf{i} - 16\mathbf{j}.$$

Finally,

$$\begin{aligned} \text{distance} &= \sqrt{14^2 + (16)^2} \\ &= \sqrt{452} \\ &= \underline{\underline{2\sqrt{113}}}. \end{aligned}$$