Dr Oliver Mathematics
Mathematics
Numerical Methods
Past Examination Questions

This booklet consists of 23 questions across a variety of examination topics.
The total number of marks available is 197.

1.
$$f(x) = 3e^x - \tfrac{1}{2}\ln x - 2, x > 0.$$

(a) Differentiate to find $f'(x)$. (3)

Solution

$$f'(x) = 3e^x - \frac{1}{2x}.$$

The curve with equation $y = f(x)$ has a turning point at P. The x-coordinate of P is α.

(b) Show that $\alpha = \frac{1}{6}e^{-\alpha}$. (2)

Solution

$$\begin{aligned} f'(\alpha) = 0 &\Rightarrow 3e^\alpha - \frac{1}{2\alpha} = 0 \\ &\Rightarrow 3e^\alpha = \frac{1}{2\alpha} \\ &\Rightarrow \alpha = \tfrac{1}{6}e^{-\alpha}. \end{aligned}$$

The iterative formula
$$x_{n+1} = \tfrac{1}{6}e^{-x_n},\ x_0 = 1$$
is used to find an approximate value for α.

(c) Calculate the values of x_1, x_2, x_3, and x_4, giving your answers to 4 decimal places. (2)

Solution

$$x_1 = 0.061\,313\,240\,2 \text{ (FCD)} = \underline{\underline{0.061\,3 \text{ (4 dp)}}}$$
$$x_2 = 0.156\,754\,763\,7 \text{ (FCD)} = \underline{\underline{0.156\,8 \text{ (4 dp)}}}$$
$$x_3 = 0.142\,485\,614\,8 \text{ (FCD)} = \underline{\underline{0.142\,5 \text{ (4 dp)}}}$$
$$x_4 = 0.144\,533\,338\,2 \text{ (FCD)} = \underline{\underline{0.144\,5 \text{ (4 dp)}}}.$$

(d) By considering the change of sign of $f'(x)$ in a suitable interval, prove that $\alpha = 0.1443$, correct to 4 decimal places. (2)

Solution
$f'(0.14425) = -6.85\ldots \times 10^{-4}$
$f'(0.14435) = 2.06\ldots \times 10^{-3}$
The function is continuous and there is a change of sign and so the root lies $0.14425 < \alpha < 0.14435$; hence,

$$\underline{\underline{\alpha = 0.1443}},$$

correct to 4 decimal places.

2.

$$f(x) = 2x^3 - x - 4.$$

(a) Show that the equation $f(x) = 0$ can be written as (3)

$$x = \sqrt{\frac{2}{x} + \frac{1}{2}}.$$

Solution

$$\begin{aligned} 2x^3 - x - 4 = 0 &\Rightarrow 2x^3 = 4 + x \\ &\Rightarrow x^2 = \frac{2}{x} + \frac{1}{2} \\ &\Rightarrow \underline{\underline{x = \sqrt{\frac{2}{x} + \frac{1}{2}}}}. \end{aligned}$$

The equation $2x^3 - x - 4 = 0$ has a root between 1.35 and 1.4.

(b) Use the iteration formula (3)

$$x_{n+1} = \sqrt{\frac{2}{x_n} + \frac{1}{2}},$$

with $x_0 = 1.35$, to find, to 2 decimal places, the values of x_1, x_2, and x_3.

Solution

$$x_1 = 1.407\,651\,051 \text{ (FCD)} = \underline{\underline{1.41 \text{ (2 dp)}}}$$

$$x_2 = 1.385\,931\,697 \text{ (FCD)} = \underline{\underline{1.39 \text{ (2 dp)}}}$$

$$x_3 = 1.393\,941\,376 \text{ (FCD)} = \underline{\underline{1.39 \text{ (2 dp)}}}.$$

The only real root of $\mathrm{f}(x) = 0$ is α.

(c) By choosing a suitable interval, prove that $\alpha = 1.392$, to 3 decimal places. (3)

Solution

$\mathrm{f}(1.3915) = -2.85\ldots \times 10^{-3}$
$\mathrm{f}(1.3925) = 7.77\ldots \times 10^{-3}$
The function is continuous and there is a change of sign and so the root lies $1.3915 < \alpha < 1.3925$; hence,

$$\underline{\underline{\alpha = 1.392}},$$

correct to 3 decimal places.

3.

$$y = (2x - 1)\tan 2x,\ 0 \leqslant x < \tfrac{\pi}{4}.$$

The curve has a minimum at the point P. The x-coordinate of P is k.

(a) Show that k satisfies the equation (6)

$$4k + \sin 4k - 2 = 0.$$

Solution

$$y = (2x - 1)\tan 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2\tan 2x + 2(2x - 1)\sec^2 2x$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 2\tan 2k + 2(2k-1)\sec^2 2k = 0\\ &\Rightarrow \frac{2\sin 2k}{\cos 2x} + \frac{2(2k-1)}{\cos^2 2k} = 0\\ &\Rightarrow 2\sin 2k \cos 2x + 2(2k-1) = 0\\ &\Rightarrow \underline{\underline{4k + \sin 4k - 2 = 0}}.\end{aligned}$$

The iterative formula

$$x_{n+1} = \tfrac{1}{4}(2 - \sin 4x_n),\ x_0 = 0.3,$$

is used to find an approximate value for k.

(b) Calculate the values of x_1, x_2, x_3, and x_4, giving your answers to 4 decimal places. (3)

Solution

$$x_1 = 0.266\,990\,228\,5 \text{ (FCD)} = \underline{\underline{0.267\,0 \text{ (4 dp)}}}$$

$$x_2 = 0.280\,945\,083\,3 \text{ (FCD)} = \underline{\underline{0.280\,9 \text{ (4 dp)}}}$$

$$x_3 = 0.274\,564\,742\,1 \text{ (FCD)} = \underline{\underline{0.274\,6 \text{ (4 dp)}}}$$

$$x_4 = 0.277\,395\,928\,8 \text{ (FCD)} = \underline{\underline{0.277\,4 \text{ (4 dp)}}}.$$

(c) Show that $k = 0.277$, correct to 3 significant figures. (2)

Solution

$f'(0.2765) = -8.71\ldots \times 10^{-5}$
$f'(0.2775) = 5.69\ldots \times 10^{-3}$
The function is continuous and there is a change of sign and so the root lies $0.2765 < \alpha < 0.2775$; hence,

$$\underline{\underline{\alpha = 0.277}},$$

correct to 3 decimal places.

4. The iterative formula

$$x_{n+1} = -\tfrac{1}{2}\mathrm{e}^{x_n},\ x_0 = -0.3,$$

is used to find an approximate value for k.

(a) Calculate the values of x_1 and x_2, giving your answers to 4 decimal places. (2)

Solution

$$x_1 = -0.370\,409\,110\,3 \text{ (FCD)} = \underline{\underline{-0.370\,4 \text{ (4 dp)}}}$$
$$x_2 = -0.345\,225\,900\,9 \text{ (FCD)} = \underline{\underline{-0.345\,2 \text{ (4 dp)}}}.$$

(b) Find the value of k to 3 decimal places. (2)

Solution

$$x_3 = -0.354\,030\,191\,9 \text{ (FCD)}$$
$$x_4 = -0.350\,926\,888\,4 \text{ (FCD)}$$
$$x_5 = -0.352\,017\,612\,6 \text{ (FCD)}$$
$$x_6 = -0.351\,633\,867\,8 \text{ (FCD)}.$$

and

$$k = \underline{\underline{-0.352 \text{ (3 dp)}}}.$$

5.

$$\mathrm{f}(x) = -x^3 + 3x^2 - 1.$$

(a) Show that the equation $\mathrm{f}(x) = 0$ can be rewritten as (2)

$$x = \sqrt{\frac{1}{3-x}}.$$

Solution

$$\begin{aligned} -x^3 + 3x^2 - 1 = 0 &\Rightarrow 3x^2 - x^3 = 1 \\ &\Rightarrow x^2(3-x) = 1 \\ &\Rightarrow x^2 = \frac{1}{3-x} \\ &\Rightarrow \underline{\underline{x = \sqrt{\frac{1}{3-x}}}}. \end{aligned}$$

(b) Starting with $x_1 = 0.6$, use the iteration **(2)**

$$x_{n+1} = \sqrt{\frac{1}{3 - x_n}}$$

to calculate the values of x_2, x_3, and x_4, giving all your answers to 4 decimal places.

> **Solution**
>
> $$x_2 = 0.645\,497\,224\,4 \text{ (FCD)} = \underline{\underline{0.645\,5 \text{ (4 dp)}}}$$
> $$x_3 = 0.651\,704\,015 \text{ (FCD)} = \underline{\underline{0.651\,7 \text{ (4 dp)}}}$$
> $$x_4 = 0.652\,564\,707\,5 \text{ (FCD)} = \underline{\underline{0.652\,6 \text{ (4 dp)}}}.$$

(c) Show that $x = 0.653$ is a root of $\mathrm{f}(x) = 0$ correct to 3 decimal places. **(3)**

> **Solution**
>
> $\mathrm{f}(0.6525) = -5.37\ldots \times 10^{-4}$
> $\mathrm{f}(0.6535) = 2.10\ldots \times 10^{-3}$
> The function is continuous and there is a change of sign and so the root lies $0.6525 < \alpha < 0.6535$; hence,
>
> $$\underline{\underline{\alpha = 0.653}},$$
>
> correct to 3 decimal places.

6.

$$\mathrm{f}(x) = \ln(x + 2) - x + 1,\ x > -2,\ x \in \mathbb{R}.$$

(a) Show that there is a root of $\mathrm{f}(x) = 0$ in the interval $2 < x < 3$. **(2)**

> **Solution**
>
> $\mathrm{f}(2) = 0.386\ldots$
> $\mathrm{f}(3) = -0.390\ldots$
> The function is continuous and there is a change of sign and so the root lies $\underline{\underline{2 < x < 3}}$.

(b) Use the iterative formula **(3)**

$$x_{n+1} = \ln(x_n + 2) + 1,\ x_0 = 2.5,$$

to calculate the values of x_1, x_2, and x_3 giving your answers to 5 decimal places.

Solution

$$x_1 = 2.504\,077\,397 \text{ (FCD)} = \underline{\underline{2.504\,08 \text{ (5 dp)}}}$$

$$x_2 = 2.504\,983\,075 \text{ (FCD)} = \underline{\underline{2.504\,98 \text{ (5 dp)}}}$$

$$x_3 = 2.505\,184\,134 \text{ (FCD)} = \underline{\underline{2.505\,18 \text{ (5 dp)}}}.$$

(c) Show that $x = 2.505$ is a root of $\mathrm{f}(x) = 0$ correct to 3 decimal places. (2)

Solution

$\mathrm{f}(2.5045) = 5.76\ldots \times 10^{-4}$
$\mathrm{f}(2.5055) = -2.01\ldots \times 10^{-4}$
The function is continuous and there is a change of sign and so the root lies $2.5045 < \alpha < 2.5055$; hence,

$$\underline{\underline{\alpha = 2.505}},$$

correct to 3 decimal places.

7.

$$\mathrm{f}(x) = 3x^3 - 2x - 6.$$

(a) Show that $\mathrm{f}(x) = 0$ has a root, α, between $x = 1.4$ and $x = 1.45$. (2)

Solution

$\mathrm{f}(1.4) = -\frac{71}{125}$
$\mathrm{f}(1.45) = \frac{1967}{8000}$
The function is continuous and there is a change of sign and so the root lies $\underline{\underline{1.4 < x < 1.45}}$.

(b) Show that the equation $\mathrm{f}(x) = 0$ can be written as (3)

$$x = \sqrt{\frac{2}{x} + \frac{2}{3}},\ x \neq 0.$$

Solution

$$
\begin{aligned}
3x^3 - 2x - 6 = 0 \Rightarrow 3x^3 &= 2x + 6 \\
\Rightarrow x^2 &= \frac{2}{3} + \frac{2}{x} \\
\Rightarrow \underline{\underline{x = \sqrt{\frac{2}{x} + \frac{2}{3}}}}.
\end{aligned}
$$

(c) Starting with $x_0 = 1.43$, use the iteration (3)

$$x_{n+1} = \sqrt{\frac{2}{x_n} + \frac{2}{3}}$$

to calculate the values of x_1, x_2, and x_3, giving your answers to 4 decimal places.

Solution

$$
\begin{aligned}
x_1 &= 1.437\,104\,055 \text{ (FCD)} = \underline{\underline{1.437\,1 \text{ (4 dp)}}} \\
x_2 &= 1.434\,696\,602 \text{ (FCD)} = \underline{\underline{1.434\,7 \text{ (4 dp)}}} \\
x_3 &= 1.435\,510\,231 \text{ (FCD)} = \underline{\underline{1.435\,5 \text{ (4 dp)}}}.
\end{aligned}
$$

(d) By choosing a suitable interval, show that $\alpha = 1.435$ is correct to 3 decimal places. (3)

Solution

$f(1.4345) = -0.013\ldots$
$f(1.4355) = 3.23\ldots \times 10^{-3}$
The function is continuous and there is a change of sign and so the root lies $1.4345 < \alpha < 1.4355$; hence,

$$\underline{\underline{\alpha = 1.435}},$$

correct to 3 decimal places.

8.

$$f(x) = 3xe^x - 1.$$

The curve with equation $y = f(x)$ has a turning point P.

(a) Find the exact coordinates of P. (5)

Solution

$$\begin{aligned} f'(x) = 0 &\Rightarrow 3e^x + 3xe^x = 0 \\ &\Rightarrow 3(1+x)e^x = 0 \\ &\Rightarrow x = -1 \\ &\Rightarrow y = -3e^{-1} - 1; \end{aligned}$$

hence, the stationary point is at $\underline{\underline{(1, -3e^{-1} - 1)}}$.

The equation $f(x) = 0$ has a root between $x = 0.25$ and $x = 0.3$.

(b) Use the iterative formula (3)

$$x_{n+1} = \tfrac{1}{3}e^{-x_n},$$

with $x_0 = 0.25$ to find, to 4 decimal places, the values of x_1, x_2, and x_3.

Solution

$$x_1 = 0.259\,600\,261 \text{ (FCD)} = \underline{\underline{0.259\,6 \text{ (4 dp)}}}$$

$$x_2 = 0.257\,119\,955\,6 \text{ (FCD)} = \underline{\underline{0.257\,1 \text{ (4 dp)}}}$$

$$x_3 = 0.257\,758\,483\,2 \text{ (FCD)} = \underline{\underline{0.257\,8 \text{ (4 dp)}}}.$$

(c) By choosing a suitable interval, show that a root of $f(x) = 0$ is $x = 0.2576$ correct to 4 decimal places. (3)

Solution

$f(0.25755) = -3.79\ldots \times 10^{-4}$
$f(0.25765) = 1.09\ldots \times 10^{-4}$
The function is continuous and there is a change of sign and so the root lies $0.25755 < \alpha < 0.25765$; hence,

$$\underline{\underline{\alpha = 0.2576}},$$

correct to 4 decimal places.

9.

$$y = -x^3 + 2x^2 + 2,$$

which intersects the x-axis at the point A where $x = \alpha$. To find an approximation to α, the iterative formula

$$x_{n+1} = \frac{2}{(x_n)^2} + 2$$

is used.

(a) Taking $x_0 = 2.5$, find the values of x_1, x_2, x_3, and x_4. Give your answers to 3 decimal places where appropriate. (3)

Solution

$$x_1 = \underline{\underline{2.32 \text{ (exact)}}}$$

$$x_2 = 2\tfrac{625}{1682} \text{ (FCD)} = \underline{\underline{2.372 \text{ (3 dp)}}}$$

$$x_3 = 2.355\,593\,575 \text{ (FCD)} = \underline{\underline{2.356 \text{ (3 dp)}}}$$

$$x_4 = 2.360\,436\,923 \text{ (FCD)} = \underline{\underline{2.360 \text{ (3 dp)}}}.$$

(b) Show that $\alpha = 2.359$ correct to 3 decimal places. (3)

Solution

Let $\mathrm{f}(x) = -x^3 + 2x^2 + 2$.
$\mathrm{f}(2.3585) = 5.83\ldots \times 10^{-3}$
$\mathrm{f}(2.3595) = -1.42\ldots \times 10^{-3}$
The function is continuous and there is a change of sign and so the root lies $2.3585 < \alpha < 2.3595$; hence,

$$\underline{\underline{\alpha = 2.359}},$$

correct to 3 decimal places.

10.

$$\mathrm{f}(x) = x^3 + 2x^2 - 3x - 11.$$

(a) Show that $\mathrm{f}(x) = 0$ can be rearranged as (2)

$$x = \sqrt{\frac{3x + 11}{x + 2}},\ x \neq -2.$$

Solution

$$\begin{aligned} x^3 + 2x^2 - 3x - 11 = 0 &\Rightarrow x^3 + 2x^2 = 3x + 11 \\ &\Rightarrow x^2(x+2) = 3x + 11 \\ &\Rightarrow x^2 = \frac{3x+11}{x+2} \\ &\Rightarrow \underline{\underline{x = \sqrt{\frac{3x+11}{x+2}}}}. \end{aligned}$$

The equation $f(x) = 0$ has one positive root α. The iterative formula

$$x_{n+1} = \sqrt{\frac{3x_n + 11}{x_n + 2}}$$

is used to find a approximation to α.

(b) Taking $x_1 = 0$, find, to 3 decimal places, the values of x_2, x_3, and x_4. (3)

Solution

$$x_2 = 2.345\,207\,88 \text{ (FCD)} = \underline{\underline{2.345 \text{ (3 dp)}}}$$

$$x_3 = 2.037\,324\,945 \text{ (FCD)} = \underline{\underline{2.037 \text{ (3 dp)}}}$$

$$x_4 = 2.058\,748\,112 \text{ (FCD)} = \underline{\underline{2.059 \text{ (3 dp)}}}.$$

(c) Show that $\alpha = 2.057$ correct to 3 decimal places. (3)

Solution

$f(2.0565) = -0.013\ldots$

$f(2.0575) = 4.14\ldots \times 10^{-3}$

The function is continuous and there is a change of sign and so the root lies $2.0565 < \alpha < 2.0575$; hence,

$$\underline{\underline{\alpha = 2.057}},$$

correct to 3 decimal places.

11.

$$f(x) = 4 \operatorname{cosec} x - 4x + 1,$$

where x is in radians.

(a) Show that there is a root α of $\mathrm{f}(x) = 0$ in the interval $[1.2, 1.3]$. (2)

Solution

$\mathrm{f}(1.2) = 0.491\ldots$
$\mathrm{f}(1.3) = -0.048\ldots$
The function is continuous and there is a change of sign and so the root lies $\underline{\underline{1.2 < x < 1.3}}$.

(b) Show that the equation $\mathrm{f}(x) = 0$ can be written in the form (2)

$$x = \frac{1}{\sin x} + \frac{1}{4}.$$

Solution

$$\begin{aligned} \mathrm{f}(x) = 0 &\Rightarrow 4\operatorname{cosec} x - 4x + 1 = 0 \\ &\Rightarrow 4x = \frac{4}{\sin x} + 1 \\ &\Rightarrow \underline{\underline{x = \frac{1}{\sin x} + \frac{1}{4}}}. \end{aligned}$$

(c) Use the iterative formula (3)

$$x_{n+1} = \frac{1}{\sin x_n} + \frac{1}{4},\ x_0 = 1.25,$$

to calculate the values of x_1, x_2, and x_3, giving your answers to 4 decimal places.

Solution

$$x_1 = 1.303\,757\,858 \text{ (FCD)} = \underline{\underline{1.303\,8 \text{ (4 dp)}}}$$
$$x_2 = 1.286\,745\,793 \text{ (FCD)} = \underline{\underline{1.286\,8 \text{ (4 dp)}}}$$
$$x_3 = 1.291\,744\,613 \text{ (FCD)} = \underline{\underline{1.291\,7 \text{ (4 dp)}}}.$$

(d) By considering the change of sign of $\mathrm{f}(x)$ in a suitable interval, verify that $\alpha = 1.291$ correct to 3 decimal places. (2)

Solution

$f(1.2905) = 4.45\ldots \times 10^{-4}$
$f(1.2915) = -4.75\ldots \times 10^{-3}$
The function is continuous and there is a change of sign and so the root lies $1.2905 < \alpha < 1.2915$; hence,

$$\underline{\underline{\alpha = 1.291}},$$

correct to 3 decimal places.

12.

$$f(x) = (8 - x)\ln x,\ x > 0.$$

The curve has a maximum turning point at Q.

(a) Find $f'(x)$. (3)

Solution

$$f'(x) = \underline{\underline{-\ln x + \frac{8-x}{x}}}.$$

(b) Show that the x-coordinate of Q lies between 3.5 and 3.6. (2)

Solution

$f(3.5) = 0.032\ldots$
$f(3.6) = -0.058\ldots$
The function is continuous and there is a change of sign and so the root lies $\underline{\underline{3.5 < x < 3.6}}$.

(c) Show that the x-coordinate of Q is the solution of (3)

$$x = \frac{8}{1 + \ln x}.$$

Solution

$$
\begin{aligned}
f'(x) = 0 &\Rightarrow -\ln x + \frac{8-x}{x} = 0 \\
&\Rightarrow \frac{8}{x} = 1 + \ln x \\
&\Rightarrow \frac{x}{8} = \frac{1}{1+\ln x} \\
&\Rightarrow \underline{\underline{x = \frac{8}{1+\ln x}}}.
\end{aligned}
$$

To find an approximation for the x-coordinate of Q, the iteration formula

$$x_{n+1} = \frac{8}{1 + \ln x_n}$$

is used.

(d) Taking $x_0 = 3.55$, find the values of x_1, x_2, and x_3. Give your answers to 3 decimal places. (3)

Solution

$$
\begin{aligned}
x_1 &= 3.528\,974\,374 \text{ (FCD)} = \underline{\underline{3.529 \text{ (3 dp)}}} \\
x_2 &= 3.538\,246\,011 \text{ (FCD)} = \underline{\underline{3.538 \text{ (3 dp)}}} \\
x_3 &= 3.534\,144\,722 \text{ (FCD)} = \underline{\underline{3.534 \text{ (3 dp)}}}.
\end{aligned}
$$

13.

$$f(x) = 2\sin(x^2) + x - 2,\ 0 \leqslant x < 2\pi.$$

(a) Show that $f(x) = 0$ has a root α between $x = 0.75$ and $x = 0.85$. (2)

Solution

We need radians mode: why?
$f(0.75) = -0.183\ldots$
$f(0.85) = 0.172\ldots$
The function is continuous and there is a change of sign and so the root lies $\underline{\underline{0.75 < \alpha < 0.85}}$.

The equation $f(x) = 0$ can be written as

$$x = [\arcsin(1 - 0.5x)]^{\frac{1}{2}}.$$

(b) Use the iterative formula (3)

$$x_{n+1} = [\arcsin(1 - 0.5x_n)]^{\frac{1}{2}}, \; x_0 = 0.8,$$

to find the values of x_1, x_2, and x_3, giving your answers to 5 decimal places.

Solution

$$x_1 = 0.802\,185\,208\,5 \text{ (FCD)} = \underline{\underline{0.802\,19 \text{ (5 dp)}}}$$

$$x_2 = 0.801\,333\,920\,3 \text{ (FCD)} = \underline{\underline{0.801\,33 \text{ (5 dp)}}}$$

$$x_3 = 0.801\,665\,559\,3 \text{ (FCD)} = \underline{\underline{0.801\,67 \text{ (5 dp)}}}.$$

(c) Show that $\alpha = 0.80157$ is correct to 5 decimal places. (3)

Solution

$f(0.801\,565) = -2.70\ldots \times 10^{-5}$
$f(0.801\,575) = 8.62\ldots \times 10^{-6}$
The function is continuous and there is a change of sign and so the root lies $0.801\,565 < \alpha < 0.801\,575$; hence,

$$\underline{\underline{\alpha = 0.801\,57}},$$

correct to 5 decimal places.

14.

$$f(x) = x^2 - 3x + 2\cos(\tfrac{1}{2}x), \; 0 \leqslant x \leqslant \pi.$$

(a) Show that the equation $f(x) = 0$ has a solution in the interval $0.8 < x < 0.9$ (2)

Solution

$f(0.8) = 0.082\ldots$
$f(0.9) = -0.089\ldots$
The function is continuous and there is a change of sign and so the root lies $\underline{\underline{0.8 < x < 0.9}}$.

The curve with equation $y = \mathrm{f}(x)$ has a minimum point P.

(b) Show that the x-coordinate of P is the solution of the equation (4)

$$x = \frac{3 + \sin(\frac{1}{2}x)}{2}.$$

Solution

$$\mathrm{f}'(x) = 2x - 3 - \sin(\tfrac{1}{2}x)$$

and

$$\begin{aligned} \mathrm{f}'(x) = 0 &\Rightarrow 2x - 3 - \sin(\tfrac{1}{2}x) = 0 \\ &\Rightarrow 2x = 3 + \sin(\tfrac{1}{2}x) \\ &\Rightarrow \underline{\underline{x = \frac{3 + \sin(\frac{1}{2}x)}{2}}}. \end{aligned}$$

(c) Using the iteration formula (3)

$$x_{n+1} = \frac{3 + \sin(\frac{1}{2}x_n)}{2},\ x_0 = 2,$$

find the values of x_1, x_2, and x_3 , giving your answers to 3 decimal places.

Solution

$$x_1 = 1.920\,735\,492 \text{ (FCD)} = \underline{\underline{1.921 \text{ (3 dp)}}}$$

$$x_2 = 1.909\,701\,211 \text{ (FCD)} = \underline{\underline{1.910 \text{ (3 dp)}}}$$

$$x_3 = 1.908\,113\,72 \text{ (FCD)} = \underline{\underline{1.908 \text{ (3 dp)}}}.$$

(d) By choosing a suitable interval, show that the x-coordinate of P is 1.9078 correct to 4 decimal places. (3)

Solution

$\mathrm{f}'(1.907\,75) = -1.63\ldots \times 10^{-4}$
$\mathrm{f}'(1.907\,85) = 7.66\ldots \times 10^{-6}$
The function is continuous and there is a change of sign and so the root lies

$1.907\,75 < x < 1.907\,85$; hence,

$$\underline{\underline{x = 1.907\,8}},$$

correct to 4 decimal places.

15.

$$f(x) = x^3 + 3x^2 + 4x - 12.$$

(a) Show that the equation $f(x) = 0$ can be written as (3)

$$x = \sqrt{\frac{4(3-x)}{3+x}},\ x \neq -3.$$

Solution

$$\begin{aligned} x^3 + 3x^2 + 4x - 12 = 0 &\Rightarrow x^3 + 3x^2 = 12 - 4x \\ &\Rightarrow x^2(x+3) = 4(3-x) \\ &\Rightarrow x^2 = \frac{4(3-x)}{3+x} \\ &\Rightarrow \underline{\underline{x = \sqrt{\frac{4(3-x)}{3+x}}}}. \end{aligned}$$

The equation $x^3 + 3x^2 + 4x - 12 = 0$ has a single root which is between 1 and 2.

(b) Use the iteration formula (3)

$$x_{n+1} = \sqrt{\frac{4(3-x_n)}{3+x_n}},\ n \geqslant 0,$$

with $x_0 = 1$ to find, to 2 decimal places, the value of x_1, x_2, and x_3.

Solution

$$x_1 = 1.414\,213\,562 \text{ (FCD)} = \underline{\underline{1.41 \text{ (2 dp)}}}$$

$$x_2 = 1.198\,741\,87 \text{ (FCD)} = \underline{\underline{1.20 \text{ (2 dp)}}}$$

$$x_3 = 1.309\,961\,056 \text{ (FCD)} = \underline{\underline{1.31 \text{ (2 dp)}}}.$$

The root of $f(x) = 0$ is α.

(c) By choosing a suitable interval, prove that $\alpha = 1.272$ to 3 decimal places. (3)

Solution

$f(1.271\,5) = -8.21\ldots \times 10^{-3}$
$f(1.272\,5) = 8.27\ldots \times 10^{-3}$
The function is continuous and there is a change of sign and so the root lies $1.271\,5 < \alpha < 1.272\,5$; hence,

$$\underline{\underline{\alpha = 1.272}},$$

correct to 3 decimal places.

16.

$$g(x) = e^{x-1} + x - 6.$$

(a) Show that the equation $g(x) = 0$ can be written as (3)

$$x = \ln(6 - x) + 1,\ x < 6.$$

Solution

$$\begin{aligned} g(x) = 0 &\Rightarrow e^{x-1} + x - 6 = 0 \\ &\Rightarrow e^{x-1} = 6 - x \\ &\Rightarrow x - 1 = \ln(6 - x) \\ &\Rightarrow \underline{\underline{x = \ln(6 - x) + 1}}. \end{aligned}$$

The root of $g(x) = 0$ is α. The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1,\ x_0 = 2,$$

is used to find an approximate value for α.

(b) Calculate the values of x_1, x_2, and x_3 to 4 decimal places. (3)

Solution

$$x_1 = 2.386\,294\,361\ (\text{FCD}) = \underline{\underline{2.386\,3\ (4\text{ dp})}}$$

$$x_2 = 2.284\,733\,739\ (\text{FCD}) = \underline{\underline{2.284\,7\ (4\text{ dp})}}$$

$$x_3 = 2.312\,450\,348\ (\text{FCD}) = \underline{\underline{2.312\,5\ (4\text{ dp})}}.$$

(c) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places. (3)

Solution

$f(2.306\,5) = -2.75\ldots \times 10^{-4}$
$f(2.307\,5) = 4.41\ldots \times 10^{-3}$
The function is continuous and there is a change of sign and so the root lies $2.306\,5 < \alpha < 2.307\,5$; hence,

$$\underline{\underline{\alpha = 2.307}},$$

correct to 3 decimal places.

17.

$$f(x) = 25x^2e^{2x} - 16,\ x \in \mathbb{R}.$$

(a) Show that the equation $f(x) = 0$ can be written as (1)

$$x = \pm\tfrac{4}{5}e^{-x}.$$

Solution

$$\begin{aligned} 25x^2e^{2x} - 16 = 0 &\Rightarrow 25x^2e^{2x} = 16 \\ &\Rightarrow (xe^x)^2 = \tfrac{16}{25} \\ &\Rightarrow xe^x = \pm\tfrac{4}{5} \\ &\Rightarrow \underline{\underline{x = \pm\tfrac{4}{5}e^{-x}}}. \end{aligned}$$

The equation $f(x) = 0$ has a root α, where $\alpha = 0.5$ to 1 decimal place.

(b) Starting with $x_0 = 0.5$, use the iteration formula (3)

$$x_{n+1} = \tfrac{4}{5}e^{-x_n}$$

to calculate the values of x_1, x_2, and x_3, giving your answers to 3 decimal places.

Solution

$$x_1 = 0.485\,224\,527\,8 \text{ (FCD)} = \underline{\underline{0.485 \text{ (3 dp)}}}$$

$$x_2 = 0.492\,447\,176\,9 \text{ (FCD)} = \underline{\underline{0.492 \text{ (3 dp)}}}$$

$$x_3 = 0.488\,903\,217\,5 \text{ (FCD)} = \underline{\underline{0.489 \text{ (3 dp)}}}.$$

(c) Give an accurate estimate for α to 2 decimal places, and justify your answer. (2)

Solution

$\mathrm{f}(0.485) = -0.48\ldots$
$\mathrm{f}(0.495) = 0.48\ldots$

The function is continuous and there is a change of sign and so the root lies $0.485 < \alpha < 0.495$; hence,

$$\underline{\underline{\alpha = 0.49}},$$

correct to 2 decimal places.

18.

$$\mathrm{f}(x) = (x^2 + 3x + 1)\mathrm{e}^{x^2}.$$

The curve has a minimum turning point at the point P.

(a) Find $\mathrm{f}'(x)$. (3)

Solution

$$\begin{aligned} \mathrm{f}'(x) &= (2x+3)\mathrm{e}^{x^2} + 2x(x^2+3x+1)\mathrm{e}^{x^2} \\ &= \underline{\underline{(2x^3+6x^2+4x+3)\mathrm{e}^{x^2}}}. \end{aligned}$$

(b) Show that the x-coordinate of P is the solution of (3)

$$x = -\frac{3(2x^2+1)}{2(x^2+2)}.$$

Solution

$$\begin{aligned} \mathrm{f}'(x) = 0 &\Rightarrow (2x^3+6x^2+4x+3)\mathrm{e}^{x^2} = 0 \\ &\Rightarrow 2x^3+6x^2+4x+3 = 0 \\ &\Rightarrow 2x(x^2+2) = -3(2x^2+1) \\ &\Rightarrow \underline{\underline{x = -\frac{3(2x^2+1)}{2(x^2+2)}}}. \end{aligned}$$

(c) Use the iteration formula (3)

$$x_{n+1} = -\frac{3(2x_n^2+1)}{2(x_n^2+2)},\ x_0 = -2.4,$$

to calculate the values of x_1, x_2, and x_3, giving your answers to 3 decimal places.

Solution

$$x_1 = -2.420\,103\,093 \text{ (FCD)} = \underline{\underline{-2.420 \text{ (3 dp)}}}$$
$$x_2 = -2.427\,254\,95 \text{ (FCD)} = \underline{\underline{-2.427 \text{ (3 dp)}}}$$
$$x_3 = -2.429\,771\,016 \text{ (FCD)} = \underline{\underline{-2.430 \text{ (3 dp)}}}.$$

The x-coordinate of P is α.

(d) By choosing a suitable interval, prove that $\alpha = -2.43$ correct to 2 decimal places. (2)

Solution

$\mathrm{f}'(-2.435) = -15.25\ldots$
$\mathrm{f}'(-2.425) = 22.45\ldots$
The function is continuous and there is a change of sign and so the root lies $-2.435 < \alpha < -2.425$; hence,

$$\underline{\underline{\alpha = -2.43}},$$

correct to 2 decimal places.

19.

$$y = 2\cos(\tfrac{1}{2}x^2) + x^3 - 3x - 2.$$

The curve crosses the x-axis at the point Q and has a minimum turning point at R.

(a) Show that the x-coordinate of Q lies between 2.1 and 2.2. (2)

Solution

Let $\mathrm{f}(x) = 2\cos(\tfrac{1}{2}x^2) + x^3 - 3x - 2$.
$\mathrm{f}(2.1) = -0.224\ldots$
$\mathrm{f}(2.2) = 0.546\ldots$
The function is continuous and there is a change of sign and so the root lies $\underline{\underline{2.1 < x < 2.2}}$.

(b) Show that the x-coordinate of R is a solution of the equation (4)

$$x = \sqrt{1 + \tfrac{2}{3}x\sin(\tfrac{1}{2}x^2)}.$$

Solution

$$\mathrm{f}'(x) = -2x\sin(\tfrac{1}{2}x^2) + 3x^2 - 3$$

and

$$\begin{aligned}\mathrm{f}'(x) = 0 &\Rightarrow -2x\sin(\tfrac{1}{2}x^2) + 3x^2 - 3 = 0\\ &\Rightarrow 3x^2 = 3 + 2x\sin(\tfrac{1}{2}x^2)\\ &\Rightarrow x^2 = 1 + \tfrac{2}{3}x\sin(\tfrac{1}{2}x^2)\\ &\Rightarrow \underline{\underline{x = \sqrt{1 + \tfrac{2}{3}x\sin(\tfrac{1}{2}x^2)}}}.\end{aligned}$$

Using the iterative formula

$$x_{n+1} = \sqrt{1 + \tfrac{2}{3}x_n\sin(\tfrac{1}{2}x_n^2)},\ x_0 = 1.3,$$

(c) find the values of x_1 and x_2 to 3 decimal places. (2)

Solution

$$x_1 = 1.283\,838\,631\text{ (FCD)} = \underline{\underline{1.284\text{ (3 dp)}}}$$

$$x_2 = 1.276\,002\,755\text{ (FCD)} = \underline{\underline{1.276\text{ (3 dp)}}}.$$

20. A curve C has equation

$$y = \mathrm{e}^{4x} + x^4 + 8x + 5.$$

(a) Show that the x-coordinate of any turning point of C satisfies the equation (3)

$$x^3 = -2 - \mathrm{e}^{4x}.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4\mathrm{e}^{4x} + 4x^3 + 8$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 4\mathrm{e}^{4x} + 4x^3 + 8 = 0$$
$$\Rightarrow 4x^3 = -8 - 4\mathrm{e}^{4x}$$
$$\Rightarrow \underline{\underline{x^3 = -2 - \mathrm{e}^{4x}}}.$$

The iteration formula

$$x_{n+1} = (-2 - \mathrm{e}^{4x_n})^{\frac{1}{3}},\ x_0 = -1,$$

can be used to find an approximate value for this root.

(b) Calculate the values of x_1 and x_2, giving your answers to 5 decimal places. (2)

Solution

$$x_1 = -1.263\,755\,412 \text{ (FCD)} = \underline{\underline{-1.263\,76 \text{ (5 dp)}}}$$
$$x_2 = -1.261\,258\,763 \text{ (FCD)} = \underline{\underline{-1.261\,26 \text{ (5 dp)}}}.$$

(c) Hence deduce the coordinates, to 2 decimal places, of the turning point of the curve C. (2)

Solution

$\mathrm{f}'(-1.265) = -0.071\ldots$
$\mathrm{f}'(-1.255) = 0.119\ldots$
The function is continuous and there is a change of sign and so the root lies $-1.265 < \alpha < -1.255$; hence,

$$\underline{\underline{\alpha = -1.26}},$$

correct to 2 decimal places.

21. Figure 1 is a sketch showed part of the curve with equation $y = 2^{x+1} - 3$ and part of the line with equation $y = 17 - x$.

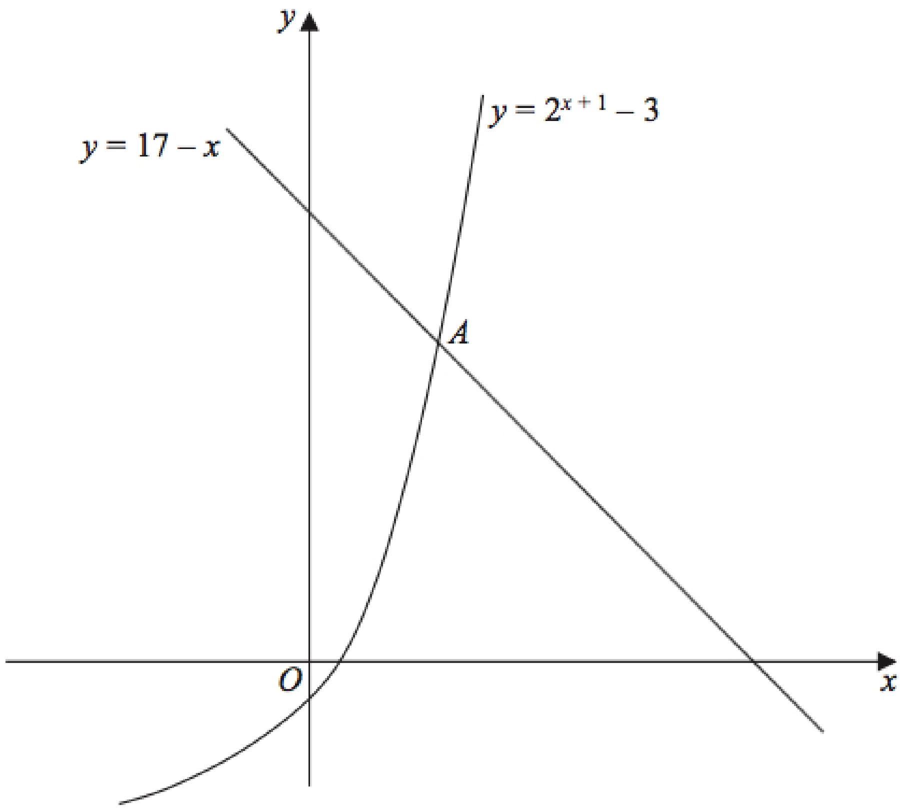

Figure 1: $y = 2^{x+1} - 3$ and $y = 17 - x$

The curve and the line intersect at the point A.

(a) Show that the $x$coordinate of A satisfies the equation (3)

$$x = \frac{\ln(20 - x)}{\ln 2} - 1.$$

Solution

$$\begin{aligned} 2^{x+1} - 3 = 17 - x &\Rightarrow 2^{x+1} = 20 - x \\ &\Rightarrow \ln 2^{x+1} = \ln(20 - x) \\ &\Rightarrow (x + 1) \ln 2 = \ln(20 - x) \\ &\Rightarrow x + 1 = \frac{\ln(20 - x)}{\ln 2} \\ &\Rightarrow x = \frac{\ln(20 - x)}{\ln 2} - 1. \end{aligned}$$

(b) Use the iterative formula (3)

$$x_{n+1} = \frac{\ln(20 - x_n)}{\ln 2} - 1, \; x_0 = 3,$$

to calculate the values of x_1, x_2, and x_3, giving your answers to 3 decimal places.

Solution

$$x_1 = 3.087\,462\,841 \text{ (FCD)} = \underline{\underline{3.087 \text{ (3 dp)}}}$$

$$x_2 = 3.080\,021\,199 \text{ (FCD)} = \underline{\underline{3.080 \text{ (3 dp)}}}$$

$$x_3 = 3.080\,655\,856 \text{ (FCD)} = \underline{\underline{3.081 \text{ (3 dp)}}}.$$

(c) Use your answer to part (b) to deduce the coordinates of the point A, giving your answers to one decimal place. (2)

Solution

Let $\mathrm{f}(x) = 2^{x+1} + x - 20$.
$\mathrm{f}(3.05) = -0.385\ldots$
$\mathrm{f}(3.15) = 0.903\ldots$
The function is continuous and there is a change of sign and so the root lies $3.05 < x < 3.15$; hence,

$$\underline{\underline{(3.1, 13.9)}},$$

correct to 1 decimal places.

22.

$$\mathrm{f}(x) = 4\mathrm{e}^{2x} - 25 \text{ and } \mathrm{g}(x) = 2x + 43.$$

The equation $\mathrm{f}(x) = \mathrm{g}(x)$ has a positive root at $x = \alpha$.

(a) Show that α is a solution of (2)

$$x = \tfrac{1}{2}\ln(\tfrac{1}{2}x + 17).$$

Solution

$$\begin{aligned} 4e^{2x} - 25 = 2x + 43 &\Rightarrow 4e^{2x} = 2x + 68 \\ &\Rightarrow e^{2x} = \tfrac{1}{2}x + 17 \\ &\Rightarrow 2x = \ln(\tfrac{1}{2}x + 17) \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}\ln(\tfrac{1}{2}x + 17)}}. \end{aligned}$$

The iteration formula

$$x_{n+1} = \tfrac{1}{2}\ln(\tfrac{1}{2}x_n + 17)$$

can be used to find an approximation for α.

(b) Taking $x_0 = 1.4$, find x_1 and x_2. Give each answer to 4 decimal places. (2)

Solution

$$x_1 = 1.436\,782\,32 \text{ (FCD)} = \underline{\underline{1.436\,8 \text{ (4 dp)}}}$$

$$x_2 = 1.437\,301\,574 \text{ (FCD)} = \underline{\underline{1.437\,3 \text{ (4 dp)}}}.$$

(c) By choosing a suitable interval, show that $\alpha = 1.437$ to 3 decimal places. (2)

Solution

Let $h(x) = 4e^{2x} - 2x - 68$.

$h(1.436\,5) = -0.112\ldots$

$h(1.437\,5) = 0.026\ldots$

The function is continuous and there is a change of sign and so the root lies $1.436\,5 < \alpha < 1.437\,5$; hence,

$$\underline{\underline{\alpha = 1.437}},$$

correct to 3 decimal places.

23. Figure 2 shows a sketch of part of the curve C with equation

$$y = 2\ln(2x + 5) - \tfrac{3}{2}x,\ x > -2.5.$$

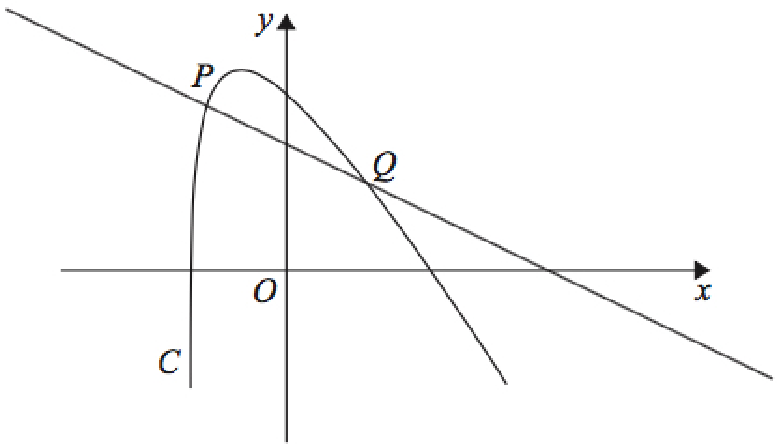

Figure 2: $y = 2\ln(2x+5) - \frac{3}{2}x$

The point P with x-coordinate -2 lies on C. The normal to C at P cuts the curve again at the point Q.

(a) Find an equation of the normal to C at P. Write your answer in the form $ax+by=c$, where a, b, and c are integers. (5)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{2x+5} - \frac{3}{2}$$

and

$$x = -2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{5}{2}.$$

Then

$$\begin{aligned} y - 3 = -\tfrac{2}{5}(x+2) &\Rightarrow 5y - 15 = -2(x+2) \\ &\Rightarrow 5y - 15 = -2x - 4 \\ &\Rightarrow \underline{\underline{2x + 5y = 11}}. \end{aligned}$$

(b) Show that the x-coordinate of Q is a solution of the equation (3)

$$x = \tfrac{20}{11}\ln(2x+5) - 2.$$

Solution

$$\begin{aligned} & 2\ln(2x+5) - \tfrac{3}{2}x = \frac{11-2x}{5} \\ \Rightarrow\quad & 2\ln(2x+5) - \tfrac{3}{2}x = \tfrac{11}{5} - \tfrac{2}{5}x \\ \Rightarrow\quad & \tfrac{11}{10}x = 2\ln(2x+5) - \tfrac{11}{5} \\ \Rightarrow\quad & \underline{\underline{x = \tfrac{20}{11}\ln(2x+5) - 2}}. \end{aligned}$$

The iteration formula

$$x_{n+1} = \tfrac{20}{11}\ln(2x_n + 5) - 2$$

can be used to find an approximation for the x-coordinate of Q.

(c) Taking $x_1 = 2$, find the values of x_2 and x_3, giving each answer to 4 decimal places. (2)

Solution

$$x_2 = 1.994\,953\,777 \text{ (FCD)} = \underline{\underline{1.995\,0 \text{ (4 dp)}}}$$

$$x_3 = 1.992\,913\,755 \text{ (FCD)} = \underline{\underline{1.992\,9 \text{ (4 dp)}}}.$$