Dr Oliver Mathematics
Mathematics: Advanced Higher
2022 Paper 2: Calculator
2 hours

The total number of marks available is 65.
You must write down all the stages in your working.

1. Express (3)

$$\frac{3x^2 - 3x + 5}{x(x^2 + 5)}$$

in partial fractions.

Solution

$$\begin{aligned}\frac{3x^2 - 3x + 5}{x(x^2 + 5)} &\equiv \frac{A}{x} + \frac{Bx + C}{x^2 + 5} \\ &\equiv \frac{A(x^2 + 5) + x(Bx + C)}{x(x^2 + 5)}\end{aligned}$$

which means

$$3x^2 - 3x + 5 \equiv A(x^2 + 5) + x(Bx + C).$$

$\underline{x = 0}$: $5 = 5A \Rightarrow A = 1$.
$\underline{x = 1}$: $3 - 3 + 5 = 6 + B + C \Rightarrow B + C = -1 \quad (1)$.
$\underline{x = -1}$: $3 + 3 + 5 = 6 + B - C \Rightarrow B - C = 5 \quad (2)$.
Do (1) + (2):

$$\begin{aligned}2B = 4 &\Rightarrow B = 2 \\ &\Rightarrow C = -3.\end{aligned}$$

Hence,

$$\frac{3x^2 - 3x + 5}{x(x^2 + 5)} \equiv \underline{\underline{\frac{1}{x} + \frac{2x - 3}{x^2 + 5}}}.$$

2. Find the exact value of (2)

$$\int_0^3 \frac{4}{2x + 1}\,\mathrm{d}x.$$

Solution

$$\int_0^3 \frac{4}{2x+1}\,\mathrm{d}x = [2\ln|2x+1|]_{x=0}^{3}$$
$$= 2\ln 7 - 2\ln 1$$
$$= \underline{\underline{2\ln 7}}.$$

3. Use the Euclidean algorithm to find integers a and b such that (3)

$$634a + 87b = 1.$$

Solution

$$634 = 7 \times 87 + 25$$
$$87 = 3 \times 25 + 12$$
$$25 = 2 \times 12 + 1$$

and

$$1 = 25 - 2 \times 12$$
$$= 25 - 2(87 - 3 \times 25)$$
$$= 7 \times 25 - 2 \times 87$$
$$= 7(634 - 7 \times 87) - 2 \times 87$$
$$= \underline{\underline{7 \times 634 - 51 \times 87}};$$

hence, $\underline{\underline{a = 7}}$ and $\underline{\underline{b = -51}}$.

4. Use integration by parts to find (3)

$$\int (x+2)(2x+7)^{\frac{1}{2}}\,\mathrm{d}x.$$

Solution

$$u = x + 2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = (2x+7)^{\frac{1}{2}} \Rightarrow v = \tfrac{1}{3}(2x+7)^{\frac{3}{2}}$$

Hence,

$$\begin{aligned}\int (x+2)(2x+7)^{\frac{1}{2}}\,\mathrm{d}x &= \tfrac{1}{3}(x+2)(2x+7)^{\frac{3}{2}} - \int \tfrac{1}{3}(2x+7)^{\frac{3}{2}}\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{3}(x+2)(2x+7)^{\frac{3}{2}} - \tfrac{1}{15}(2x+7)^{\frac{5}{2}}\,\mathrm{d}x + c}}.\end{aligned}$$

5. Matrix $\mathbf{A}$ is given by (3)

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & k & 3 \\ k & 18 & -7 \end{pmatrix}.$$

Find the values of k so that the matrix $\mathbf{A}$ is singular.

Solution

$$\begin{aligned}\det \mathbf{A} = 0 &\Rightarrow 1(-7k-54) - 3(-14-3k) + 1(36-k^2) = 0 \\ &\Rightarrow -7k - 54 + 42 + 9k + 36 - k^2 = 0 \\ &\Rightarrow k^2 - 2k - 24 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -2 \\ \text{multiply to:} & -24\end{array}\right\} -6,\ +4$$

$$\begin{aligned}&\Rightarrow (k-6)(k+4) = 0 \\ &\Rightarrow \underline{\underline{k = 6 \text{ or } k = -4}}.\end{aligned}$$

6. The first three terms of a sequence are defined algebraically by

$$x+5,\ 3x+2,\ 5x-1,$$

where $x \in \mathbb{N}$.

(a) Show that these three terms form the start of an arithmetic sequence. (2)

Solution

$$(3x+2)-(x+5)=2x-3$$

and

$$(5x-1)-(3x+2)=2x-3.$$

So, these three terms form the start of an arithmetic sequence.

(b) Find a simplified expression for the 15th term of this sequence. (2)

Solution

$$\begin{aligned}\text{15th term} &= a+14d\\ &= (x+5)+14(2x-3)\\ &= \underline{\underline{29x-37}}.\end{aligned}$$

(c) Given that the sum of the first 20 terms of this sequence is 1 130, find the value of x. (2)

Solution

$$\begin{aligned}\tfrac{1}{2}(20)[2(x+5)+19(2x-3)] = 1\,130 &\Rightarrow 40x-47=113\\ &\Rightarrow 40x=160\\ &\Rightarrow \underline{\underline{x=4}}.\end{aligned}$$

7. The complex number

$$z=3+\mathrm{i}$$

is a root of

$$z^2-6z+a=0,$$

where a is a real number.

(a) State the second root of (1)

$$z^2-6z+a=0.$$

Solution

$\underline{\underline{z = 3 - \mathrm{i}}}$.

(b) Hence, or otherwise, find the value of a. (2)

Solution

Difference of two squares:

$$\begin{aligned}(3+\mathrm{i})(3-\mathrm{i}) &= 3^2 - (\mathrm{i})^2 \\ &= 9 - (-1) \\ &= \underline{\underline{10}}.\end{aligned}$$

The expression

$$z^2 - 6z + a$$

is a factor of

$$z^3 - z^2 - 20z + b,$$

where b is a real number.

(c) Find the value of b. (1)

Solution

Let us call the third root c. Then

$$z^3 - z^2 - 20z + b = (z - c)(z^2 - 6z + 10).$$

The z-term:

$$\begin{aligned}-20 = 6c + 10 &\Rightarrow -30 = 6c \\ &\Rightarrow c = -5\end{aligned}$$

and

$$b = 10(-5) = \underline{\underline{50}}.$$

8. (a) Differentiate (2)

$$x \ln x - x$$

with respect to x.

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x\ln x - x) &= x\cdot\frac{1}{x} + 1\cdot\ln x - 1\\ &= 1+\ln x - 1\\ &= \underline{\underline{\ln x}}.\end{aligned}$$

(b) Hence find the general solution of the differential equation (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\ln x = x^{-x}.$$

Solution

$$\mathrm{IF} = \mathrm{e}^{\int \ln x} = \mathrm{e}^{x\ln x - x}$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + y\ln x = x^{-x} &\Rightarrow \mathrm{e}^{x\ln x-x}\frac{\mathrm{d}y}{\mathrm{d}x} + y\mathrm{e}^{x\ln x-x}\ln x = \mathrm{e}^{x\ln x-x}x^{-x}\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\mathrm{e}^{x\ln x-x}) = \mathrm{e}^{x\ln x-x}x^{-x}\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\mathrm{e}^{x\ln x-x}) = \mathrm{e}^{x\ln x}\mathrm{e}^{-x}x^{-x}\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\mathrm{e}^{x\ln x-x}) = \mathrm{e}^{\ln x^{x}}\mathrm{e}^{-x}x^{-x}\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(yx^{x}\mathrm{e}^{-x}) = x^{x}\mathrm{e}^{-x}x^{-x}\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(yx^{x}\mathrm{e}^{-x}) = \mathrm{e}^{-x}\\ &\Rightarrow yx^{x}\mathrm{e}^{-x} = -\mathrm{e}^{-x} + c\\ &\Rightarrow \underline{\underline{y = \frac{-\mathrm{e}^{-x}+c}{x^{x}\mathrm{e}^{-x}}}}.\end{aligned}$$

9. The matrix $\mathbf{A}$ is given by (5)

$$\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}.$$

Prove by induction that

$$\mathbf{A}^n = \begin{pmatrix} 3^n & 1-3^n \\ 0 & 1 \end{pmatrix}.$$

Solution

$\underline{n = 1}$:

$$\begin{aligned} \begin{pmatrix} 3^1 & 1-3^1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 3 & 1-3 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} \\ &= \mathbf{A}, \end{aligned}$$

and so the result is true for $n = 1$.

Now, suppose it is true for $n = k$, i.e.,

$$\mathbf{A}^k = \begin{pmatrix} 3^k & 1-3^k \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{A}^{k+1} &= \mathbf{A}^k \mathbf{A} \\ &= \begin{pmatrix} 3^k & 1-3^k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{k+1} & -2(3^k) + (1-3^k) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{k+1} & 1 - 2 \cdot 3^k - 3^k \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{k+1} & 1 - 3^k(2+1) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{k+1} & 1 - 3^k \cdot 3 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{k+1} & 1 - 3^{k+1} \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the result is true for all positive integers n.

10. Solve the differential equation (9)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 9\sin x + 13\cos x,$$

given that $y = 5$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

Solution

Complementary function:

$$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2 \text{ (repeated)}$$

and hence the complementary function is

$$y = (Ax + B)\mathrm{e}^{2x}.$$

Particular integral: try

$$\begin{aligned} y = C\sin x + D\cos x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= C\cos x - D\sin x \\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -C\sin x - D\cos x. \end{aligned}$$

Now,

$$\begin{aligned} &\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 9\sin x + 13\cos x \\ \Rightarrow\ &(-C\sin x - D\cos x) - 4(C\cos x - D\sin x) + 4(C\sin x + D\cos x) \\ &= 9\sin x + 13\cos x \end{aligned}$$

and

$$\begin{aligned} -C + 4D + 4C = 9 &\Rightarrow 3C + 4D = 9 \quad (1) \\ -D - 4C + 4D = 13 &\Rightarrow -4C + 3D = 13 \quad (2). \end{aligned}$$

Next,

$$\begin{aligned} 4 \times (1):&\quad 12C + 16D = 36 \quad (3) \\ 3 \times (2):&\quad -12C + 9D = 39 \quad (4) \end{aligned}$$

and (3) + (4):

$$\begin{aligned} 25D = 75 &\Rightarrow D = 3 \\ &\Rightarrow 3C + 4(3) = 9 \\ &\Rightarrow 3C = -3 \\ &\Rightarrow C = -1. \end{aligned}$$

The particular integral is $y = -\sin x + 3\cos x$.

The general solution is

$$y = (Ax + B)\mathrm{e}^{2x} - \sin x + 3\cos x.$$

Now,

$$\begin{aligned} x = 0, y = 5 &\Rightarrow 5 = B + 3 \\ &\Rightarrow B = 2. \end{aligned}$$

Next,

$$\begin{aligned} & y = (Ax + 2)\mathrm{e}^{2x} - \sin x + 3\cos x \\ \Rightarrow \quad & \frac{\mathrm{d}y}{\mathrm{d}x} = A\mathrm{e}^{2x} + 2(Ax + 2)\mathrm{e}^{2x} - \cos x - 3\sin x \end{aligned}$$

and

$$\begin{aligned} x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 0 = A + 4 - 1 \\ &\Rightarrow A = -3. \end{aligned}$$

Hence,

$$\underline{\underline{y = (-3x + 2)\mathrm{e}^{2x} - \sin x + 3\cos x}}.$$

11. A curve defined parametrically has the following properties: (4)

- $x = \tan^{-1} 2t$,
- $\frac{\mathrm{d}y}{\mathrm{d}x} = 6t(1 + 4t^2)$, and
- $y = 5$ when $t = 1$.

Find y in terms of t.

Solution

Now,

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1 + (2t)^2} &\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1 + 4t^2} \\ &\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1 + 4t^2}{2} \end{aligned}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} \Rightarrow 6t(1+4t^2) = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{1+4t^2}{2}$$
$$\Rightarrow 6t = \tfrac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}t}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 12t$$
$$\Rightarrow y = 6t^2 + c$$

for some constant c. Now,

$$t = 1, y = 5 \Rightarrow 5 = 6(1^2) + c$$
$$\Rightarrow c = -1;$$

hence,

$$\underline{\underline{y = 6t^2 - 1}}.$$

12. Let

$$z = \cos\theta + \mathrm{i}\sin\theta.$$

(a) Use de Moivre's theorem to state an expression for z^4. (1)

Solution

$$z^4 = (\cos\theta + \mathrm{i}\sin\theta)^4$$
$$= \underline{\underline{\cos 4\theta + \mathrm{i}\sin 4\theta}}.$$

(b) State and simplify the binomial expansion of (3)

$$(\cos\theta + \mathrm{i}\sin\theta)^4.$$

Solution

$$
\begin{aligned}
& (\cos\theta + \mathrm{i}\sin\theta)^4 \\
= \;& \cos^4\theta + 4(\cos\theta)^3(\mathrm{i}\sin\theta) + 6(\cos\theta)^2(\mathrm{i}\sin\theta)^2 \\
& \qquad + 4(\cos\theta)(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4 \\
= \;& \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta \\
= \;& \underline{\underline{(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + 4\mathrm{i}(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)}}.
\end{aligned}
$$

(c) Hence show that:

(i) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$ (2)

Solution

Equating the real parts:

$$
\begin{aligned}
\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\
&= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\
&= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + (1 - 2\cos^2\theta + \cos^4\theta) \\
&= \underline{\underline{8\cos^4\theta - 8\cos^2\theta + 1}},
\end{aligned}
$$

as required.

(ii) $\sin\theta\cot 4\theta$ can be written in terms of $\cos\theta$ only. (2)

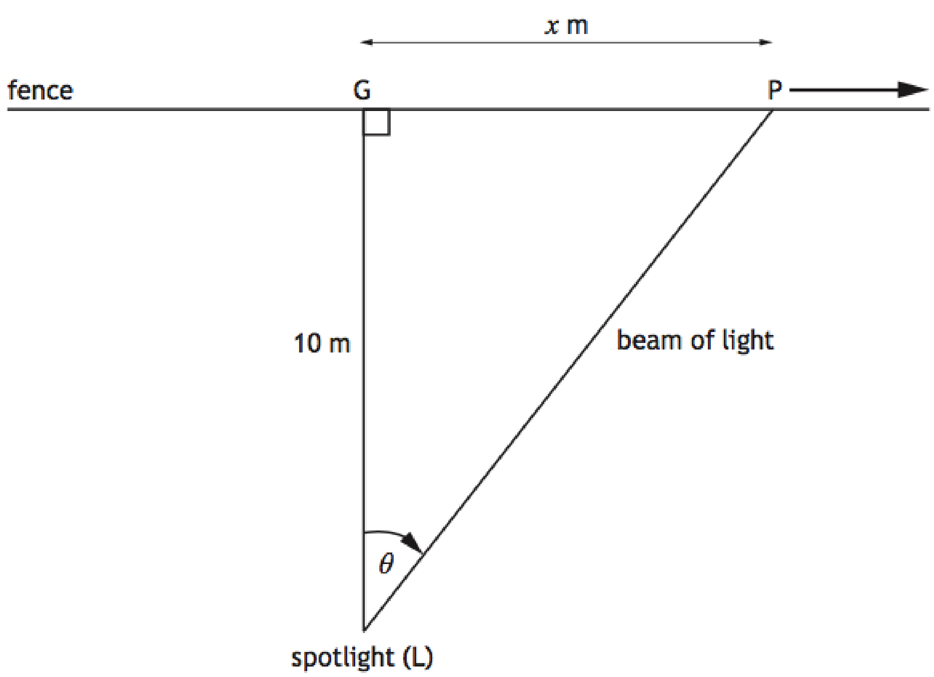
Solution

$$
\begin{aligned}
\sin\theta\cot 4\theta &= \frac{\sin\theta\cos 4\theta}{\sin 4\theta} \\
&= \frac{\sin\theta\cos 4\theta}{4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)} \\
&= \frac{\sin\theta\cos 4\theta}{4\sin\theta(\cos^3\theta - \cos\theta\sin^2\theta)} \\
&= \frac{\cos 4\theta}{4(\cos^3\theta - \cos\theta\sin^2\theta)} \\
&= \frac{8\cos^4\theta - 8\cos^2\theta + 1}{4\cos^3\theta - 4\cos\theta(1-\cos^2\theta)} \\
&= \frac{8\cos^4\theta - 8\cos^2\theta + 1}{4\cos^3\theta - 4\cos\theta + 4\cos^3\theta} \\
&= \underline{\underline{\frac{8\cos^4\theta - 8\cos^2\theta + 1}{8\cos^3\theta - 4\cos\theta}}}.
\end{aligned}
$$

13. A security spotlight is situated 10 metres from a straight fence. The spotlight rotates at a constant speed and makes one full revolution every 12 seconds.

The situation at time t seconds is modelled in the diagram below, where:

- L is the position of the spotlight,
- G is the point on the fence nearest to the spotlight,
- P is the position where the light hits the fence,
- θ is the angle between LG and LP, and
- x is the distance in metres from G to P.



(a) Show that:

(i) $\dfrac{d\theta}{dt} = \frac{1}{6}\pi$ radians per second, (1)

Solution

$$\frac{d\theta}{dt} = \frac{2\pi}{12} = \underline{\underline{\tfrac{1}{6}\pi \text{ radians per second.}}}$$

(ii) $\dfrac{dx}{dt} = \frac{5}{3}\pi \sec^2\theta$ metres per second. (4)

Solution

$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan\theta = \frac{x}{10}$$
$$\Rightarrow x = 10\tan\theta$$
$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = 10\sec^2\theta$$

and

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t} \\ &= 10\sec^2\theta \times \tfrac{1}{6}\pi \\ &= \underline{\underline{\tfrac{5}{3}\pi\sec^2\theta \text{ metres per second}}},\end{aligned}$$

as required.

(b) Prove that (1)

$$1 + \tan^2\theta = \sec^2\theta.$$

Solution

$$\begin{aligned}1 + \tan^2\theta &\equiv \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} \\ &\equiv \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} \\ &\equiv \frac{1}{\cos^2\theta} \\ &\equiv \underline{\underline{\sec^2\theta}},\end{aligned}$$

as required.

(c) Hence, or otherwise, find the exact value of $\frac{\mathrm{d}x}{\mathrm{d}t}$ when P is 5 metres from G. (3)

Solution

$$x = 5 \Rightarrow \tan\theta = \tfrac{1}{2}$$

and

$$
\begin{aligned}
1 + (\tfrac{1}{2})^2 = \sec^2\theta &\Rightarrow \sec^2\theta = \tfrac{5}{4} \\
&\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \tfrac{5}{3}\pi \times \tfrac{5}{4} \\
&\Rightarrow \underline{\underline{\frac{\mathrm{d}x}{\mathrm{d}t} = \tfrac{25}{12}\pi \text{ m/s}}}.
\end{aligned}
$$