

Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2010 November Paper 1 Variant 3: Calculator
2 hours

The total number of marks available is 80.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You must write down all the stages in your working.

1. Show that

$$\sec x - \cos x \equiv \sin x \tan x.$$

(3)

Solution

Well,

$$\begin{aligned}\sec x - \cos x &\equiv \frac{1}{\cos x} - \cos x \\ &\equiv \frac{1}{\cos x} - \frac{1}{\cos^2 x} \cos x \\ &\equiv \frac{1 - \cos^2 x}{\cos x} \\ &\equiv \frac{\sin^2 x}{\cos x} \\ &\equiv \sin x \cdot \frac{\sin x}{\cos x} \\ &\equiv \underline{\underline{\sin x \tan x}},\end{aligned}$$

as required.

2. A 4-digit number is formed by using four of the seven digits 2, 3, 4, 5, 6, 7, and 8. No digit can be used more than once in any one number.

Find how many different 4-digit numbers can be formed if

- (a) there are no restrictions,

(2)

Solution

$$7 \times 6 \times 5 \times 4 = \underline{\underline{840}}.$$

(b) the number is even.

(2)

Solution

There are 4 choices: 2, 4, 6, or 8. So

$$4 \times 6 \times 5 \times 4 = \underline{480}.$$

3. The line

(4)

$$y = mx + 2$$

is a tangent to the curve

$$y = x^2 + 12x + 18.$$

Find the possible values of m .

Solution

Well,

$$x^2 + 12x + 18 = mx + 2 \Rightarrow x^2 + (12 - m)x + 16 = 0.$$

Now, $a = 1$, $b = 12 - m$, and $c = 16$:

$$\begin{aligned} b^2 - 4ac &= 0 \Rightarrow (12 - m)^2 - 4(1)(16) = 0 \\ &\Rightarrow (12 - m)^2 = 64 \\ &\Rightarrow 12 - m = -8 \text{ or } 12 - m = 8 \\ &\Rightarrow \underline{m = 20} \text{ or } \underline{m = 4}. \end{aligned}$$

4. The remainder when the expression

(4)

$$x^3 + kx^2 - 5x - 3$$

is divided by $(x - 2)$ is 5 times the remainder when the expression is divided by $(x + 1)$.

Find the value of k .

Solution

We use synthetic division twice:

$$\begin{array}{r|rrrr} 2 & 1 & k & -5 & -3 \\ & \downarrow & & & \\ & 1 & k+2 & 2k-1 & 4k-5 \end{array}$$

and

$$\begin{array}{r|rrrr} -1 & 1 & k & -5 & -3 \\ & \downarrow & & & \\ & 1 & k-1 & -k-4 & k+1 \end{array}$$

So,

$$\begin{aligned} 4k - 5 &= 5(k + 1) \Rightarrow 4k - 5 = 5k + 5 \\ &\Rightarrow \underline{\underline{k = -10}}. \end{aligned}$$

5. Solve the simultaneous equations

(5)

$$\log_3 a = 2 \log_3 b$$

$$\log_3(2a - b) = 1.$$

Solution

Well,

$$\begin{aligned} \log_3 a = 2 \log_3 b &\Rightarrow \log_3 a = \log_3 b^2 \\ &\Rightarrow a = b^2 \quad (1) \end{aligned}$$

and

$$\log_3(2a - b) = 1 \Rightarrow 2a - b = 3 \quad (2).$$

Insert (1) into (2):

$$2(b^2) - b = 3 \Rightarrow 2b^2 - b - 3 = 0$$

$$\left. \begin{array}{l} \text{add to:} \\ \text{multiply to: } (+2) \times (-3) = -6 \end{array} \right\} -3, +2$$

e.g.,

$$\begin{aligned} \Rightarrow 2b^2 - 3b + 2b - 3 &= 0 \\ \Rightarrow b(2b - 3) + 1(2b - 3) &= 0 \\ \Rightarrow (b + 1)(2b - 3) &= 0 \\ \Rightarrow b = -1 \text{ (cannot have — why?) or } b &= \frac{3}{2} \\ \Rightarrow a = \frac{9}{4}; \end{aligned}$$

hence,

$$\underline{\underline{a = \frac{9}{4} \text{ and } b = \frac{3}{2}.}}$$

6. Solve the equation

$$3x^3 + 7x^2 - 22x - 8 = 0.$$

(6)

Solution

Let

$$f(x) = 3x^3 + 7x^2 - 22x - 8.$$

Then,

$$\begin{aligned} f(1) &= 3 + 7 - 22 - 8 = -20, \\ f(-1) &= -3 + 7 + 22 - 8 = 18, \\ f(2) &= 24 + 28 - 44 - 8 = 0, \end{aligned}$$

so $(x - 2)$ is a factor.

We use synthetic division:

$$\begin{array}{r|rrrr} 2 & 3 & 7 & -22 & -8 \\ & \downarrow & 6 & 26 & 8 \\ \hline & 3 & 13 & 4 & 0 \end{array}$$

and

$$f(x) = (x - 2)(3x^2 + 13x + 4).$$

Next,

$$\left. \begin{array}{l} \text{add to:} \quad +13 \\ \text{multiply to: } (+3) \times (+4) = +12 \end{array} \right\} +1, +12$$

e.g.,

$$\begin{aligned} 3x^2 + 13x + 4 = 0 &\Rightarrow 3x^2 + x + 12x + 4 = 0 \\ &\Rightarrow x(3x + 1) + 4(3x + 1) = 0 \\ &\Rightarrow (x + 4)(3x + 1) = 0 \\ &\Rightarrow x = -4 \text{ or } x = -\frac{1}{3}. \end{aligned}$$

Hence, the solutions are

$$\underline{\underline{x = -4, x = -\frac{1}{3}, \text{ or } x = 2.}}$$

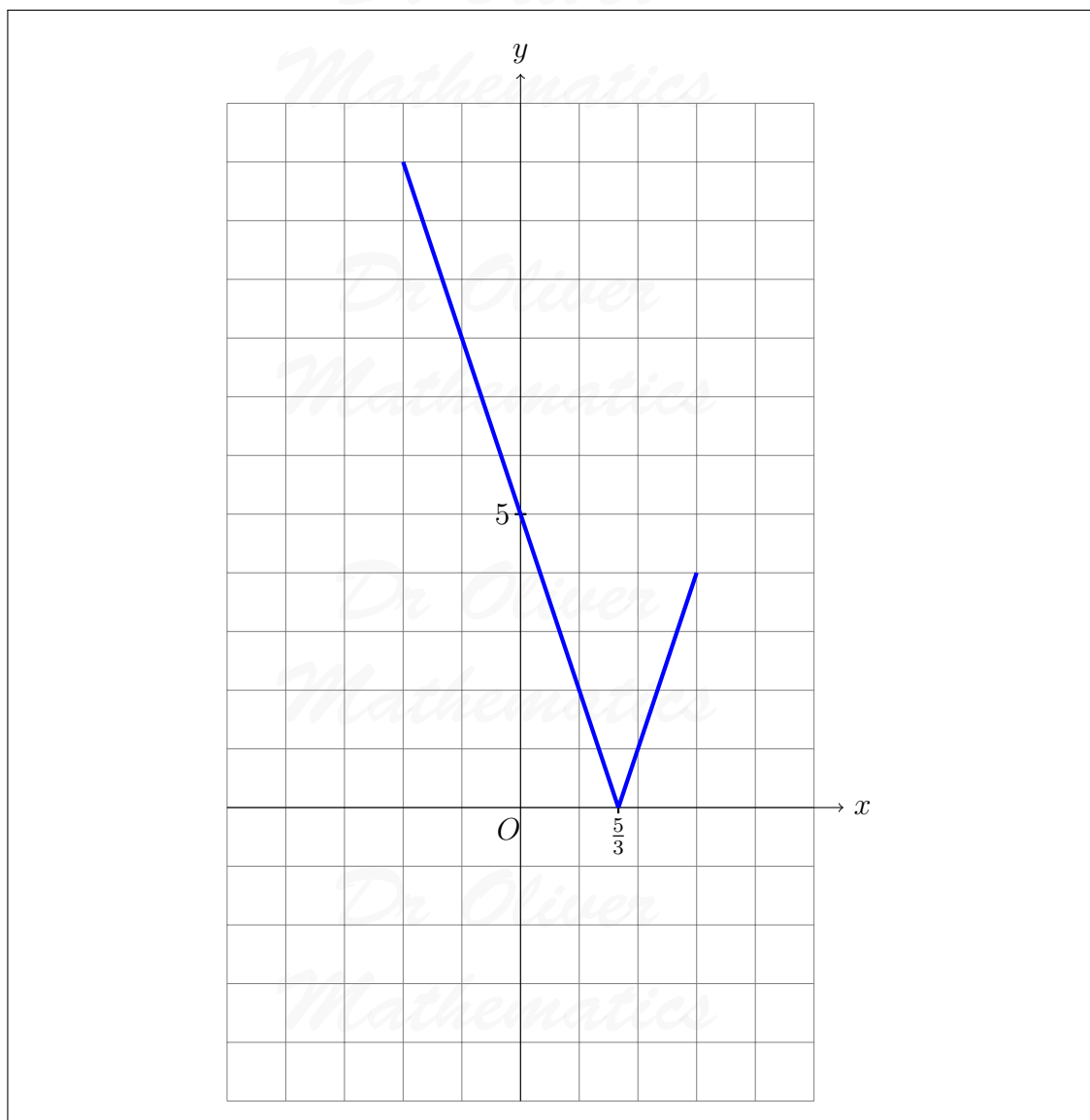
7. (a) Sketch the graph of

$$y = |3x - 5| \text{ for } -2 \leq x \leq 3,$$

(3)

showing the coordinates of the points where the graph meets the axes.

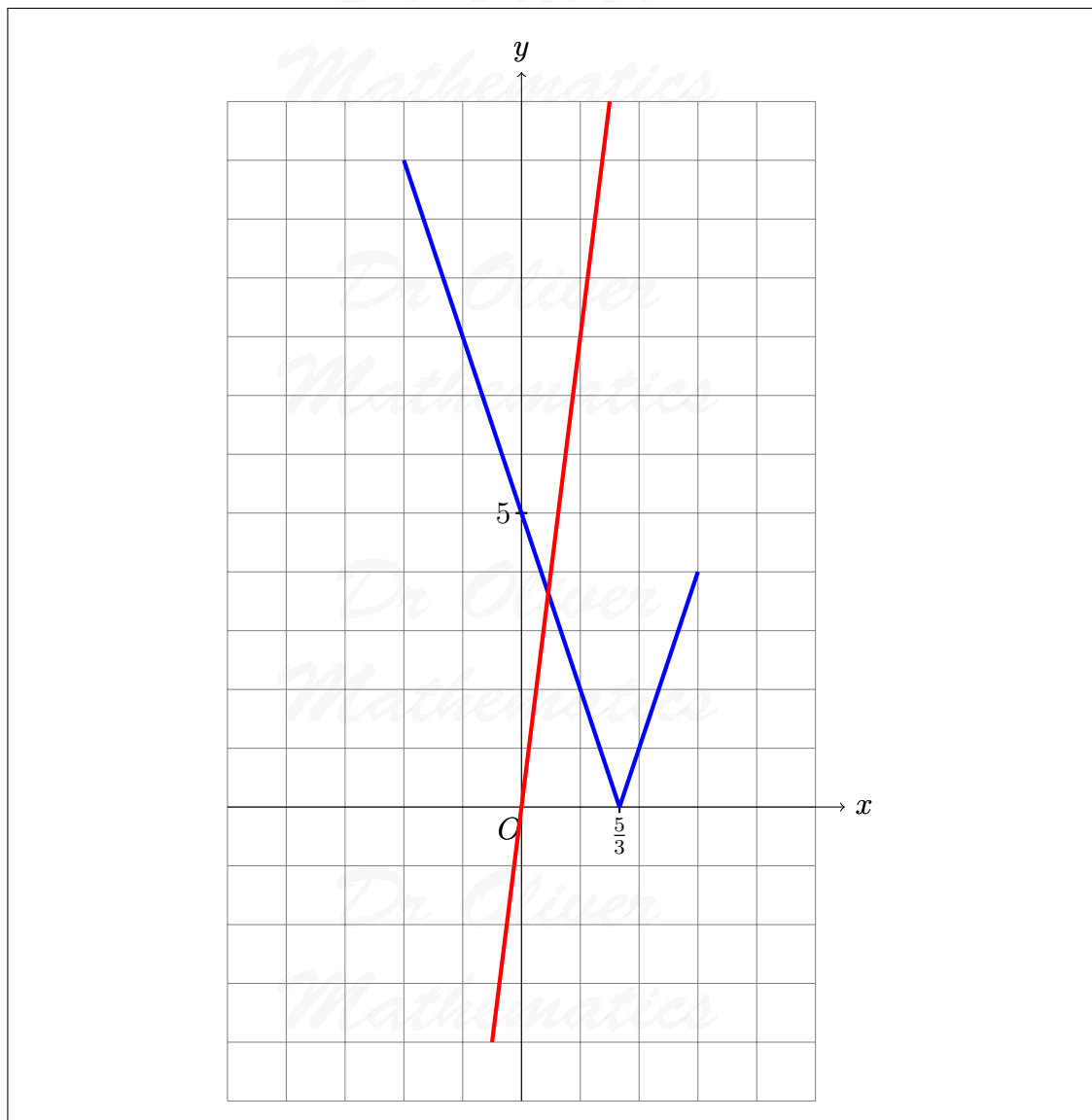
Solution



(b) On the same diagram, sketch the graph of (1)

$$y = 8x.$$

Solution



(c) Solve the equation

$$8x = |3x - 5|.$$

(3)

Solution

$8x = 3x - 5$:

$$8x = 3x - 5 \Rightarrow 5x = -5$$

$$\Rightarrow x = -1 \text{ (can't happen — why?).}$$

$$\underline{8x = -(3x - 5)} :$$

$$8x = -(3x - 5) \Rightarrow 8x = -3x + 5$$

$$\Rightarrow 11x = 5$$

$$\Rightarrow \underline{\underline{x = \frac{5}{11}}}.$$

8. A function f is defined, for $x \in \mathbb{R}$, by

$$f(x) = x^2 + 4x - 6.$$

- (a) (i) Find the least value of $f(x)$ and the value of x for which it occurs. (2)

Solution

Well,

$$\begin{aligned} f(x) &= x^2 + 4x - 6 \\ &= [(x^2 + 4x + 4) - 4] - 6 \\ &= (x + 2)^2 - 10. \end{aligned}$$

Hence, the least value of $f(x)$ is -10 and the value of x for which it occurs is $x = -2$.

- (ii) Hence write down a suitable domain for $f(x)$ in order that $f^{-1}(x)$ exists. (1)

Solution

E.g., $x > -2$.

(b) Functions g and h are defined, for $x \in \mathbb{R}$, by

$$g(x) = \frac{1}{2}x - 1,$$

$$h(x) = x^2 - x.$$

- (i) Find $g^{-1}(x)$. (2)

Solution

Well,

$$\begin{aligned} y &= \frac{1}{2}x - 1 \Rightarrow y + 1 = \frac{1}{2}x \\ &\Rightarrow 2(y + 1) = x \end{aligned}$$

and

$$\underline{g^{-1}(x) = 2(x + 1)}.$$

(ii) Solve

$$g h(x) = g^{-1}(x).$$

(3)

Solution

Now,

$$\begin{aligned} g h(x) &= g(h(x)) \\ &= g(x^2 - x) \\ &= \frac{1}{2}(x^2 - x) - 1 \end{aligned}$$

and

$$\begin{aligned} g h(x) = g^{-1}(x) &\Rightarrow \frac{1}{2}(x^2 - x) - 1 = 2(x + 1) \\ &\Rightarrow x^2 - x - 2 = 4(x + 1) \\ &\Rightarrow x^2 - x - 2 = 4x + 4 \\ &\Rightarrow x^2 - 5x - 6 = 0 \end{aligned}$$

$$\begin{array}{l} \text{add to:} \\ \text{multiply to:} \end{array} \left. \begin{array}{l} -5 \\ -6 \end{array} \right\} -6, +1$$

$$\begin{aligned} &\Rightarrow (x - 6)(x + 1) = 0 \\ &\Rightarrow \underline{x = 6 \text{ or } x = -1}. \end{aligned}$$

9. (a) Find

$$\int \left(x^{\frac{1}{3}} - 3 \right)^2 dx.$$

(3)

Solution

Well,

$$\begin{array}{r|rr}
 \times & x^{\frac{1}{3}} & -3 \\
 \hline
 x^{\frac{1}{3}} & x^{\frac{2}{3}} & -3x^{\frac{1}{3}} \\
 -3 & -3x^{\frac{1}{3}} & +9 \\
 \hline
 \end{array}$$

and so

$$\begin{aligned}
 \int \left(x^{\frac{1}{3}} - 3\right)^2 dx &= \int \left(x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 9\right) dx \\
 &= \underline{\underline{\frac{3}{5}x^{\frac{5}{3}} - \frac{9}{2}x^{\frac{4}{3}} + 9x + c.}}
 \end{aligned}$$

(b) (i) Given that

$$y = x\sqrt{x^2 + 6},$$

(3)

find $\frac{dy}{dx}$.

Solution

Product rule:

$$\begin{aligned}
 u = x &\Rightarrow \frac{du}{dx} = 1 \\
 v = (x^2 + 6)^{\frac{1}{2}} &\Rightarrow \frac{dv}{dx} = x(x^2 + 6)^{-\frac{1}{2}}
 \end{aligned}$$

and so

$$\begin{aligned}
 \frac{dy}{dx} &= (x)[x(x^2 + 6)^{-\frac{1}{2}}] + (1)[(x^2 + 6)^{\frac{1}{2}}] \\
 &= x^2(x^2 + 6)^{-\frac{1}{2}} + (x^2 + 6)^{\frac{1}{2}} \\
 &= (x^2 + 6)^{-\frac{1}{2}}[x^2 + (x^2 + 6)] \\
 &= \underline{\underline{\frac{2x + 6}{(x^2 + 6)^{\frac{1}{2}}}}}
 \end{aligned}$$

(ii) Hence find

$$\int \frac{x^2 + 3}{\sqrt{x^2 + 6}} dx.$$

(2)

Solution

Now,

$$\begin{aligned}\int \frac{x^2 + 3}{\sqrt{x^2 + 6}} dx &= \frac{1}{2} \int \frac{2x^2 + 6}{\sqrt{x^2 + 6}} dx \\ &= \underline{\underline{\frac{1}{2}x\sqrt{x^2 + 6} + c.}}\end{aligned}$$

10. A particle travels in a straight line so that, t s after passing through a fixed point O , its displacement s m from O is given by

$$s = \ln(t^2 + 1).$$

- (a) Find the value of t when $s = 5$.

(2)

Solution

Well,

$$\begin{aligned}s = 5 &\Rightarrow \ln(t^2 + 1) = 5 \\ &\Rightarrow t^2 + 1 = e^5 \\ &\Rightarrow t^2 = e^5 - 1 \\ &\Rightarrow \underline{\underline{t = \sqrt{e^5 - 1}}}.\end{aligned}$$

- (b) Find the distance travelled by the particle during the third second.

(2)

Solution

Now,

$$\begin{aligned}\text{distance travelled} &= s(3) - s(2) \\ &= \ln(3^2 + 1) - \ln(2^2 + 1) \\ &= \ln 10 - \ln 5 \\ &= \ln \left(\frac{10}{5} \right) \\ &= \underline{\underline{\ln 2}}.\end{aligned}$$

- (c) Show that, when $t = 2$, the velocity of the particle is 0.8 ms^{-1} .

(2)

Solution

Now,

$$s = \ln(t^2 + 1) \Rightarrow v = \frac{2t}{t^2 + 1}$$

and

$$t = 2 \Rightarrow \underline{v = 0.8 \text{ ms}^{-1}}.$$

- (d) Find the acceleration of the particle when
- $t = 2$
- .

(3)

Solution

Quotient rule:

$$u = 2t \Rightarrow \frac{du}{dt} = 2$$

$$v = t^2 + 1 \Rightarrow \frac{dv}{dt} = 2t$$

and so

$$a = \frac{(t^2 + 1)(2) - (2t)(2t)}{(t^2 + 1)^2}.$$

Finally,

$$t = 2 \Rightarrow a = \frac{(2^2 + 1)(2) - (4)(4)}{(2^2 + 1)^2}$$

$$\Rightarrow \underline{a = -\frac{6}{25} \text{ ms}^{-2}}.$$

11. Solve the equation

(a) $3 \sin x - 4 \cos x = 0$, for $0^\circ \leq x \leq 360^\circ$,

(3)

Solution

$$3 \sin x - 4 \cos x = 0 \Rightarrow 3 \sin x = 4 \cos x$$

$$\Rightarrow \tan x = \frac{4}{3}$$

$$\Rightarrow x = 53.130\ 102\ 35, 233.130\ 102\ 4 \text{ (FCD)}$$

$$\Rightarrow \underline{x = 53.1, 233 \text{ (3 sf)}}.$$

(b) $11 \sin y + 1 = 4 \cos^2 y$, for $0^\circ \leq y \leq 360^\circ$,

(4)

Solution

$$\begin{aligned} 11 \sin y + 1 = 4 \cos^2 y &\Rightarrow 11 \sin y + 1 = 4(1 - \sin^2 y) \\ &\Rightarrow 11 \sin y + 1 = 4 - 4 \sin^2 y \\ &\Rightarrow 4 \sin^2 y + 11 \sin y - 3 = 0 \end{aligned}$$

$$\left. \begin{array}{l} \text{add to:} \\ \text{multiply to: } (+4) \times (-3) = -12 \end{array} \right\} +11, -1$$

e.g.,

$$\begin{aligned} &\Rightarrow 4 \sin^2 y + 12 \sin y - \sin y - 3 = 0 \\ &\Rightarrow 4 \sin y(\sin y + 3) - 1(\sin y + 3) = 0 \\ &\Rightarrow (4 \sin y - 1)(\sin y + 3) = 0 \\ &\Rightarrow \sin y = \frac{1}{4} \text{ or } \sin y = -3 \text{ (cannot happen)} \\ &\Rightarrow y = 14.47751219, 165.5224878 \text{ (FCD)} \\ &\Rightarrow \underline{y = 14.5, 166 \text{ (3 sf)}}. \end{aligned}$$

(c) $\sec(2z + \frac{1}{3}\pi) = -2$, for $0 \leq z \leq \pi$ radians.

(4)

Solution

Well,

$$\begin{aligned} 0 \leq z \leq \pi &\Rightarrow 0 \leq 2z \leq 2\pi \\ &\Rightarrow \frac{1}{3}\pi \leq 2z + \frac{1}{3}\pi \leq \frac{13}{6}\pi \end{aligned}$$

and

$$\begin{aligned} \sec(2z + \frac{1}{3}\pi) = -2 &\Rightarrow \cos(2z + \frac{1}{3}\pi) = -\frac{1}{2} \\ &\Rightarrow 2z + \frac{1}{3}\pi = \frac{2}{3}\pi, \frac{4}{3}\pi \\ &\Rightarrow 2z = \frac{1}{3}\pi, \pi \\ &\Rightarrow \underline{z = \frac{1}{6}\pi, \frac{1}{2}\pi}. \end{aligned}$$

EITHER

12. A curve has the equation

$$y = A \sin 2x + B \cos 3x.$$

The curve passes through the point with coordinates $(\frac{1}{12}\pi, 3)$ and has a gradient of -4 when $x = \frac{1}{3}\pi$.

- (a) Show that $A = 4$ and find the value of B . (6)

Solution

Well,

$$\begin{aligned} x = \frac{1}{12}\pi, y = 3 &\Rightarrow A \sin \frac{1}{6}\pi + B \cos \frac{1}{4}\pi = 3 \\ &\Rightarrow \frac{1}{2}A + \frac{\sqrt{2}}{2}B = 3 \\ &\Rightarrow A + \sqrt{2}B = 6. \end{aligned}$$

Now,

$$y = A \sin 2x + B \cos 3x \Rightarrow \frac{dy}{dx} = 2A \cos 2x - 3B \sin 3x$$

and

$$\begin{aligned} x = \frac{1}{3}\pi, \frac{dy}{dx} = -4 &\Rightarrow 2A \cos \frac{2}{3}\pi - 3B \sin \pi = -4 \\ &\Rightarrow 2A(-\frac{1}{2}) = -4 \\ &\Rightarrow -A = -4 \\ &\Rightarrow \underline{\underline{A = 4}}. \end{aligned}$$

Finally,

$$\begin{aligned} 4 + \sqrt{2}B = 6 &\Rightarrow \sqrt{2}B = 2 \\ &\Rightarrow \underline{\underline{B = \sqrt{2}}}. \end{aligned}$$

- (b) Given that, for $0 \leq x \leq \frac{1}{3}\pi$, the curve lies above the x -axis, find the area of the region enclosed by the curve, the y -axis, and the line $x = \frac{1}{3}\pi$. (5)

Solution

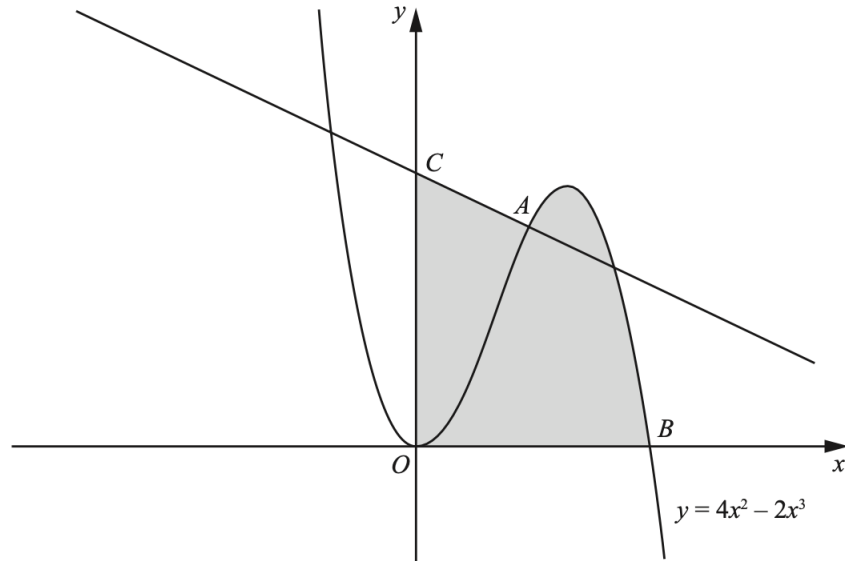
Now,

$$\begin{aligned} \int_0^{\frac{1}{3}\pi} (4 \sin 2x - 3 \cos 3x) dx &= [-2 \cos 2x - \sin 3x]_{x=0}^{\frac{1}{3}\pi} \\ &= (1 - 0) - (-2 - 0) \\ &= \underline{\underline{3}}. \end{aligned}$$

OR

13. The diagram shows the curve

$$y = 4x^2 - 2x^3.$$



- The point A lies on the curve and the x -coordinate of A is 1.
 - The curve crosses the x -axis at the point B .
 - The normal to the curve at the point A crosses the y -axis at the point C .
- (a) Show that the coordinates of C are $(0, 2.5)$. (5)

Solution

Now,

$$y = 4x^2 - 2x^3 \Rightarrow \frac{dy}{dx} = 8x - 6x^2$$

and

$$\begin{aligned} x = 1 &\Rightarrow \frac{dy}{dx} = 2 \\ &\Rightarrow m_{\text{normal}} = -\frac{1}{2}. \end{aligned}$$

Next,

$$x = 1 \Rightarrow y = 2$$

and the equation of the line normal to A is

$$y - 2 = -\frac{1}{2}(x - 1).$$

Finally,

$$\begin{aligned}x = 0 &\Rightarrow y - 2 = -\frac{1}{2}(0 - 1) \\ &\Rightarrow y - 2 = 0.5 \\ &\Rightarrow y = 2.5;\end{aligned}$$

hence, the coordinates of C are $(0, 2.5)$.

(b) Find the area of the shaded region.

(6)

Solution

Well,

$$y = 4x^2 - 2x^3 = 2x^2(2 - x)$$

which means $B(2, 0)$. Next,

$$\begin{aligned}y - 2 &= -\frac{1}{2}(x - 1) \Rightarrow y - 2 = -\frac{1}{2}x + \frac{1}{2} \\ &\Rightarrow y = -\frac{1}{2}x + \frac{5}{2}.\end{aligned}$$

Let $D(1, 0)$. Now,

shaded area = area of $OCAD$ + integral

$$\begin{aligned}&= \frac{1}{2}(2.5 + 2)(1) + \int_1^2 (4x^2 - 2x^3) dx \\ &= 2\frac{1}{4} + \left[\frac{4}{3}x^3 - \frac{1}{2}x^4 \right]_{x=1}^2 \\ &= 2\frac{1}{4} + \left(\frac{4}{3} - \frac{1}{2} \right) \\ &= 2\frac{1}{4} + 1\frac{5}{6} \\ &= \underline{\underline{4\frac{1}{12}}}.\end{aligned}$$