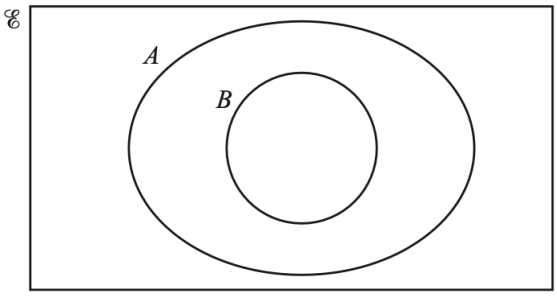
Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2013 June Paper 1 Variant 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. The Venn diagram shows the universal set $\mathscr{E}$, the set A, and the set B.



Given that

- $\mathrm{n}(B) = 5$,
- $\mathrm{n}(A') = 10$, and
- $\mathrm{n}(\mathscr{E}) = 26$,

find

(a) $\mathrm{n}(A \cap B)$, (1)

Solution

$\underline{\underline{5}}$.

(b) $\mathrm{n}(A)$, (1)

Solution

$$26 - 10 = \underline{\underline{16}}.$$

(c) $\mathrm{n}(B' \cap A)$. (1)

Solution

$$16 - 5 = \underline{\underline{11}}.$$

2. A 4-digit number is to be formed from the digits 1, 2, 5, 7, 8, and 9.

Each digit may only be used once.

Find the number of different 4-digit numbers that can be formed if

(a) there are no restrictions, (1)

Solution

$$6 \times 5 \times 4 \times 3 = \underline{\underline{360}}.$$

(b) the 4-digit numbers are divisible by 5, (2)

Solution

$$5 \times 4 \times 3 \times 1 = \underline{\underline{60}}.$$

(c) the 4-digit number are divisible by 5 and are greater than 7 000. (2)

Solution

$$3 \times 4 \times 3 \times 1 = \underline{\underline{36}}.$$

3. Show that (3)

$$(1 - \cos\theta - \sin\theta)^2 - 2(1 - \sin\theta)(1 - \cos\theta) \equiv 0.$$

Solution

Now,

$\times$	1	$-\cos\theta$	$-\sin\theta$
1	1	$-\cos\theta$	$-\sin\theta$
$-\cos\theta$	$-\cos\theta$	$+\cos^2\theta$	$+\sin\theta\cos\theta$
$-\sin\theta$	$-\sin\theta$	$+\sin\theta\cos\theta$	$+\sin^2\theta$

and

$\times$	1	$-\sin\theta$
1	1	$-\sin\theta$
$-\cos\theta$	$-\cos\theta$	$+\sin\theta\cos\theta$

so

$$\begin{aligned} & (1-\cos\theta-\sin\theta)^2-2(1-\sin\theta)(1-\cos\theta) \\ \equiv\ & (1-2\cos\theta-2\sin\theta+2\sin\theta\cos\theta+\cos^2\theta+\sin^2\theta) \\ & \quad -2(1-\cos\theta-\sin\theta+\sin\theta\cos\theta) \\ \equiv\ & 2-2\cos\theta-2\sin\theta+2\sin\theta\cos\theta \\ & \quad -2+2\cos\theta+2\sin\theta-2\sin\theta\cos\theta \\ \equiv\ & \underline{\underline{0}}, \end{aligned}$$

as required.

4. Find the set of values of k for which the curve (4)

$$y = 2x^2 + kx + 2k - 6$$

lies above the x-axis for all values of x.

Solution

Quadratic formula: $a = 2$, $b = k$, and $c = 2k - 6$:

$$\begin{aligned} b^2 - 4ac < 0 &\Rightarrow k^2 - 4(2)(2k-6) < 0 \\ &\Rightarrow k^2 - 16k - 48 < 0 \end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -16\\ \text{multiply to:} & +48\end{array}\right\} -12, -4$$

$$\Rightarrow (k-12)(k-4) < 0.$$

We need a 'table of signs':

	$k<4$	$k=4$	$4<k<12$	$k=12$	$k>12$
$(k-4)$	$-$	0	$+$	$+$	$+$
$(k-12)$	$-$	$-$	$-$	0	$+$
$(k-4)(k-12)$	$+$	0	$-$	0	$+$

Hence, the set of values of k is

$$\underline{\underline{4 < k < 12}}.$$

5. The line (6)

$$3x + 4y = 15$$

cuts the curve

$$2xy = 9$$

at the points A and B.

Find the length of the line AB.

Solution

Well,

$$3x + 4y = 15 \Rightarrow 4y = 15 - 3x \quad (1)$$

and

$$\begin{aligned} 2xy = 9 &\Rightarrow 4xy = 18\\ &\Rightarrow x(4y) = 18 \end{aligned}$$

insert equation (1):

$$\begin{aligned}
&\Rightarrow x(15-3x) = 18\\
&\Rightarrow 15x - 3x^2 = 18\\
&\Rightarrow 3x^2 - 15x = -18\\
&\Rightarrow 3x^2 - 15x + 18 = 0\\
&\Rightarrow 3(x^2 - 5x + 6) = 0
\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -5\\ \text{multiply to:} & +6\end{array}\right\} -3,\ -2$$

$$\begin{aligned}
&\Rightarrow 3(x-3)(x-2) = 0\\
&\Rightarrow x - 3 = 0 \text{ or } x - 2 = 0\\
&\Rightarrow x = 3 \text{ or } x = 2\\
&\Rightarrow y = 1\tfrac{1}{2} \text{ or } y = 2\tfrac{1}{4};
\end{aligned}$$

so, the points A and B are $(3, 1\frac{1}{2})$ and $(2, 2\frac{1}{4})$.

Finally,

$$\begin{aligned}
AB &= \sqrt{(2-3)^2 + (1\tfrac{1}{2} - 2\tfrac{1}{4})^2}\\
&= \sqrt{1 + \tfrac{9}{16}}\\
&= \sqrt{\tfrac{25}{16}}\\
&= \underline{\underline{1\tfrac{1}{4}}}.
\end{aligned}$$

6. The normal to the curve (6)

$$y + 2 = 3\tan x,$$

at the point on the curve where $x = \frac{3}{4}\pi$, cuts the y-axis at the point P.

Find the coordinates of P.

Solution

Well,

$$y + 2 = 3\tan x \Rightarrow y = 3\tan x - 2$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3\sec^2 x.$$

Now,

$$x = \tfrac{3}{4}\pi \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6$$
$$\Rightarrow m_{\text{normal}} = -\tfrac{1}{6}.$$

Next,

$$x = \tfrac{3}{4}\pi \Rightarrow y = -5$$

and the equation is

$$y + 5 = -\tfrac{1}{6}(x - \tfrac{3}{4}\pi).$$

Finally,

$$x = 0 \Rightarrow y + 5 = -\tfrac{1}{6}(0 - \tfrac{3}{4}\pi)$$
$$\Rightarrow y + 5 = \tfrac{1}{8}\pi$$
$$\Rightarrow y = \tfrac{1}{8}\pi - 5;$$

hence, $\underline{\underline{P(0, \tfrac{1}{8}\pi - 5)}}$.

7. It is given that

$$\mathrm{f}(x) = 6x^3 - 5x^2 + ax + b$$

has a factor of $(x + 2)$ and leaves a remainder of 27 when divided by $(x - 1)$.

(a) Show that $b = 40$ and find the value of a. (4)

Solution

We use synthetic division twice:

-2	6	-5	a	b
	$\downarrow$	-12	34	$-2a - 68$
	6	-17	$a + 34$	$-2a + b - 68$

and so

$$-2a + b - 68 \Rightarrow -2a + b = 68 \quad (1).$$

Next,

1	6	-5	a	b
	$\downarrow$	6	1	$a+1$
	6	1	$a+1$	$a+b+1$

and so

$$a+b+1=27 \Rightarrow a+b=26 \quad (2).$$

Do (2) − (1):

$$3a=-42 \Rightarrow \underline{\underline{a=-14}}$$

insert a into (2):

$$\Rightarrow -14+b=26$$
$$\Rightarrow \underline{\underline{b=40}},$$

as required.

(b) Show that (2)

$$\mathrm{f}(x)=(x+2)(px^2+qx+r),$$

where p, q, and r are integers to be found.

Solution

So,

-2	6	-5	-14	40
	$\downarrow$	-12	34	-40
	6	-17	20	0

and so

$$\underline{\underline{\mathrm{f}(x)=(x+2)(6x^2-17x+20)}};$$

so, $\underline{\underline{p=6}}$, $\underline{\underline{q=-17}}$, and $\underline{\underline{r=20}}$.

(c) Hence solve $\mathrm{f}(x)=0$. (2)

Solution
Now,

$$\begin{aligned} b^2 - 4ac &= 17^2 - 4(6)(20) \\ &= -191 \\ &< 0 \end{aligned}$$

so $6x^2 - 17x + 20 = 0$ has no real roots. Hence,

$$\mathrm{f}(x) = 0 \Rightarrow \underline{\underline{x = -2}}.$$

8. (a) Given that the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 3 & -5 \end{pmatrix},$$

find

(i) $\mathbf{A}^2$, (2)

Solution
Well,

$$\begin{aligned} \mathbf{A}^2 &= \begin{pmatrix} 4 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 3 & -5 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 22 & -2 \\ -3 & 31 \end{pmatrix}}}. \end{aligned}$$

(ii) $3\mathbf{A} + 4\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. (2)

Solution

$$\begin{aligned} 3\mathbf{A} + 4\mathbf{I} &= 3 \begin{pmatrix} 4 & 2 \\ 3 & -5 \end{pmatrix} + 4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 6 \\ 9 & -15 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 16 & 6 \\ 9 & -11 \end{pmatrix}}}. \end{aligned}$$

(b) (i) Find the inverse matrix of (2)

$$\begin{pmatrix} 6 & 1 \\ -9 & 3 \end{pmatrix}.$$

Solution
Well,

$$\det = 18 - (-9) = 27$$

and so the inverse is

$$\underline{\underline{\tfrac{1}{27}\begin{pmatrix} 3 & -1 \\ 9 & 6 \end{pmatrix}.}}$$

(ii) Hence solve the equations (3)

$$6x + y = 5,$$
$$-9x + 3y = \tfrac{3}{2}.$$

Solution
Convert it into matrix form:

$$\begin{pmatrix} 6 & 1 \\ -9 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ \frac{3}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ -9 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ \frac{3}{2} \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{27}\begin{pmatrix} 3 & -1 \\ 9 & 6 \end{pmatrix}\begin{pmatrix} 5 \\ \frac{3}{2} \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{27}\begin{pmatrix} \frac{27}{2} \\ 54 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 2 \end{pmatrix};$$

hence,

$$\underline{\underline{x = \tfrac{1}{2},\ y = 2.}}$$

9. (a) Given that n is a positive integer, find the first 3 terms in the expansion of (2)

$$\left(1 + \tfrac{1}{2}x\right)^n$$

in ascending powers of x.

Solution

Well,

$$\left(1+\tfrac{1}{2}x\right)^n = (1)^n + \binom{n}{1}(1)^{n-1}(\tfrac{1}{2}x) + \binom{n}{2}(1)^{n-2}(\tfrac{1}{2}x)^2 + \dots$$
$$= \underline{\underline{1 + \tfrac{1}{2}nx + \tfrac{1}{8}n(n-1)x^2 + \dots}}$$

(b) Given that the coefficient of x^2 in the expansion of (5)

$$(1-x)\left(1+\tfrac{1}{2}x\right)^n$$

is $\frac{25}{4}$, find the value of n.

Solution

Now,

$\times$	$+\frac{1}{2}nx$	$+\frac{1}{8}n(n-1)x^2$
1	$\dots$	$+\frac{1}{8}n(n-1)x^2$
$-x$	$-\frac{1}{2}nx^2$	$\dots$

and so

$$\tfrac{1}{8}n(n-1) - \tfrac{1}{2}n = \tfrac{25}{4} \Rightarrow n(n-1) - 4n = 50$$
$$\Rightarrow (n^2 - n) - 4n = 50$$
$$\Rightarrow n^2 - 5n - 50 = 0$$

add to: -5
multiply to: -50 $\Big\}$ $-10, +5$

$$\Rightarrow (n-10)(n+5) = 0$$
$$\Rightarrow n - 10 = 0 \text{ or } n + 5 = 0$$
$$\Rightarrow n = 10 \text{ or } n = -5.$$

But $n \neq -5$ (why?) so $\underline{\underline{n = 10}}$.

10. (a) (i) Find (2)

$$\int \sqrt{2x-5}\,\mathrm{d}x.$$

Solution
Well,

$$\begin{aligned}\int \sqrt{2x-5}\,\mathrm{d}x &= \int (2x-5)^{\frac{1}{2}}\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{3}(2x-5)^{\frac{3}{2}} + c}}.\end{aligned}$$

(ii) Hence evaluate (2)

$$\int_3^{15} \sqrt{2x-5}\,\mathrm{d}x.$$

Solution
Now,

$$\begin{aligned}\int_3^{15} \sqrt{2x-5}\,\mathrm{d}x &= \left[\tfrac{1}{3}(2x-5)^{\frac{3}{2}}\right]_{x=3}^{15} \\ &= \tfrac{125}{3} - \tfrac{1}{3} \\ &= \underline{\underline{41\tfrac{1}{3}}}.\end{aligned}$$

(b) (i) Find (2)

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3 \ln x).$$

Solution
Product rule:

$$\begin{aligned}u = x^3 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2 \\ v = \ln x &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}\end{aligned}$$

and so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x^3 \ln x) &= (x^3)\left(\frac{1}{x}\right) + (3x^2)(\ln x) \\ &= \underline{\underline{x^2 + 3x^2 \ln x}}.\end{aligned}$$

(ii) Hence find (3)

$$\int x^2 \ln x \, \mathrm{d}x.$$

Solution
Now,

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3 \ln x) = x^2 + 3x^2 \ln x \Rightarrow 3x^2 \ln x = \frac{\mathrm{d}}{\mathrm{d}x}(x^3 \ln x) - x^2$$

$$\Rightarrow 3\int x^2 \ln x \, \mathrm{d}x = x^3 \ln x - \int x^2 \, \mathrm{d}x + c$$

$$\Rightarrow 3\int x^2 \ln x \, \mathrm{d}x = x^3 \ln x - \tfrac{1}{3}x^3 + c$$

$$\Rightarrow \underline{\underline{\int x^2 \ln x \, \mathrm{d}x = \tfrac{1}{3}x^3 \ln x - \tfrac{1}{9}x^3 + c.}}$$

11. (a) Solve (5)

$$\cos 2x + 2\sec 2x + 3 = 0 \text{ for } 0^\circ \leqslant x \leqslant 360^\circ.$$

Solution
Well,

$$0^\circ \leqslant x \leqslant 360^\circ \Rightarrow 0^\circ \leqslant 2x \leqslant 720^\circ$$

and

$$\cos 2x + 2\sec 2x + 3 = 0 \Rightarrow \cos 2x + \frac{2}{\cos 2x} + 3 = 0$$

multiply by $\cos 2x$:

$$\Rightarrow \cos^2 2x + 3\cos 2x + 2 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & +3 \\ \text{multiply to:} & +2 \end{array}\right\} +1, +2$$

$$\Rightarrow (\cos 2x + 1)(\cos 2x + 2) = 0$$

$$\Rightarrow \cos 2x + 1 = 0 \text{ or } \cos 2x + 2 = 0$$

$$\Rightarrow \cos 2x = -1 \text{ or } \cos 2x = -2 \text{ (cannot happen)}$$

$$\Rightarrow 2x = 180, 540$$

$$\Rightarrow \underline{\underline{x = 90, 270.}}$$

(b) Solve (4)

$$2\sin^2(y - \tfrac{1}{6}\pi) = 1 \text{ for } 0 \leqslant y \leqslant \pi.$$

Solution

Now,

$$0 \leqslant y\pi \leqslant \pi \Rightarrow -\tfrac{1}{6}\pi \leqslant y - \tfrac{1}{6}\pi \leqslant \tfrac{5}{6}\pi$$

and

$$\begin{aligned} 2\sin^2(y - \tfrac{1}{6}\pi) = 1 &\Rightarrow \sin^2(y - \tfrac{1}{6}\pi) = \tfrac{1}{2} \\ &\Rightarrow \sin(y - \tfrac{1}{6}\pi) = \pm\tfrac{1}{\sqrt{2}}. \end{aligned}$$

$\underline{\sin(y - \frac{1}{6}\pi) = \frac{1}{\sqrt{2}}}$:

$$\begin{aligned} \sin(y - \tfrac{1}{6}\pi) = \tfrac{1}{\sqrt{2}} &\Rightarrow y - \tfrac{1}{6}\pi = \tfrac{1}{4}\pi,\ \tfrac{3}{4}\pi \\ &\Rightarrow \underline{\underline{y = \tfrac{5}{12}\pi,\ \tfrac{11}{12}\pi}}. \end{aligned}$$

$\underline{\sin(y - \frac{1}{6}\pi) = -\frac{1}{\sqrt{2}}}$:

$$\sin(y - \tfrac{1}{6}\pi) = -\tfrac{1}{\sqrt{2}} \Rightarrow y - \tfrac{1}{6}\pi = \text{does not have any solutions.}$$

12. A particle P moves in a straight line such that, t s after leaving a point O, its velocity v ms^{-1} is given by

$$v = 36t - 3t^2, \text{ for } t \geqslant 0.$$

(a) Find the value of t when the velocity of P stops increasing. (2)

Solution

Well,

$$v = 36t - 3t^2 \Rightarrow a = 36 - 6t$$

and

$$\begin{aligned} a = 0 &\Rightarrow 36 - 6t = 0 \\ &\Rightarrow 6t = 36 \\ &\Rightarrow \underline{\underline{t = 6}}. \end{aligned}$$

(b) Find the value of t when P comes instantaneous rest. (2)

Solution
Now,

$$\begin{aligned} v = 0 &\Rightarrow 36t - 3t^2 = 0 \\ &\Rightarrow 3t(12 - t) = 0 \\ &\Rightarrow t = 0 \text{ or } 12 - t = 0 \\ &\Rightarrow t = 0 \text{ or } t = 12; \end{aligned}$$

hence, $\underline{\underline{t = 12}}$.

(c) Find the distance of P from O when P is at instantaneous rest. (3)

Solution
Now,

$$\begin{aligned} \int_0^{12} (36t - 3t^2)\,\mathrm{d}t &= \left[18t^2 - t^3\right]_{t=0}^{12} \\ &= (2\,592 - 1\,728) - (0 - 0) \\ &= \underline{\underline{864 \text{ m}}}. \end{aligned}$$

(d) Find the speed of P when P is again at O. (4)

Solution
Well,

$$\begin{aligned} 18t^2 - t^3 = 0 &\Rightarrow t^2(18 - t) = 0 \\ &\Rightarrow t = 0 \text{ or } 18 - t = 0 \\ &\Rightarrow t = 0 \text{ or } t = 18. \end{aligned}$$

Now,

$$t = 18 \Rightarrow v = -324$$

so

$$\text{speed} = \underline{\underline{324 \text{ ms}^{-1}}}.$$