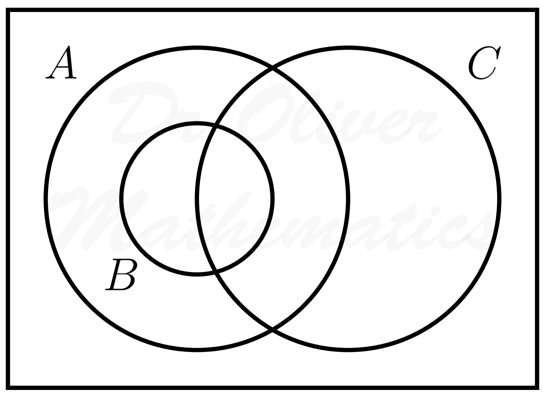
Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2007 June Paper 1: Calculator
2 hours

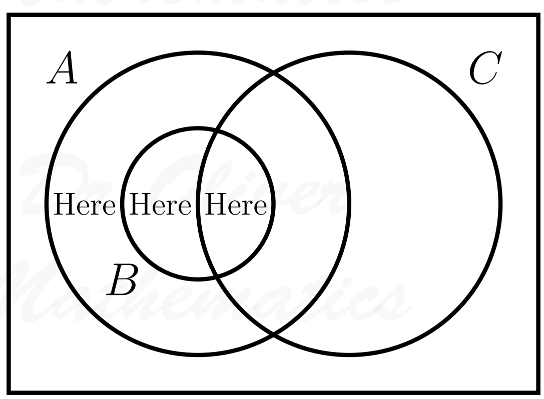
The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. (a) The diagram above shows a universal set $\mathscr{E}$ and the three sets A, B, and C.

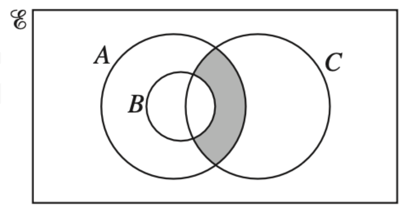


(i) Copy the above diagram and shade the region representing $(A \cup C') \cup B$. (1)

Solution

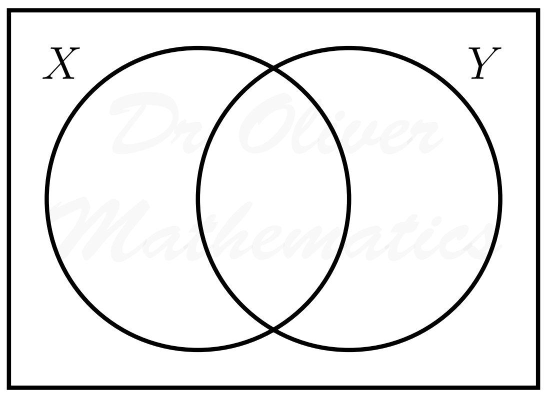


(ii) Express, in set notation, the set represented by the shaded region in the diagram below. (1)



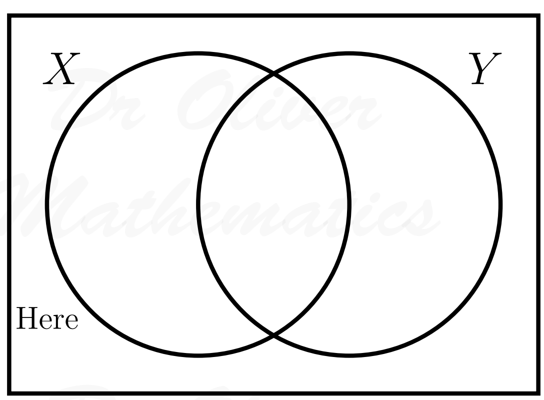
Solution
$\underline{\underline{A \cap B' \cap C}}$.

(b) The diagram shows a universal set $\mathscr{E}$ and the sets X and Y. (2)

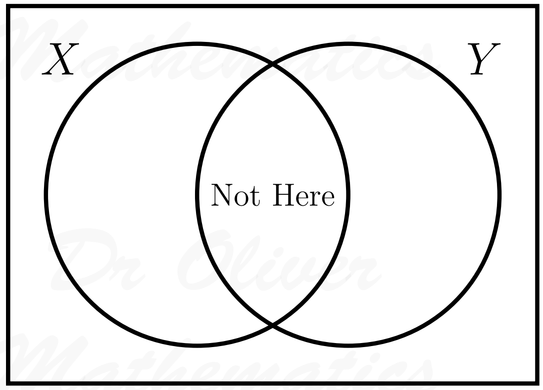


Show, by means of two diagrams, that the set $(X \cup Y)'$ is not the same as the set $X' \cup Y'$.

Solution
Well, $(X \cup Y)'$ is



whereas $X' \cup Y'$ is



2. Find the equation of the normal to the curve (5)

$$y = \frac{2x+4}{x-2}$$

at the point where $x = 4$.

Solution

Quotient rule:

$$u = 2x + 4 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
$$v = x - 2 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-2)(2) - (2x+4)(1)}{(x-2)^2} = \frac{2x - 4 - 2x - 4}{(x-2)^2} = \frac{-8}{(x-2)^2}.$$

Now,

$$x = 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2$$

which means

$$m_{\text{normal}} = \tfrac{1}{2}.$$

Next,

$$x = 4 \Rightarrow y = 6$$

and the equation of the normal is

$$y - 6 = \tfrac{1}{2}(x - 4) \Rightarrow y - 6 = \tfrac{1}{2}x - 2$$
$$\Rightarrow \underline{\underline{y = \tfrac{1}{2}x + 4}}.$$

3. The straight line (6)

$$3x = 2y + 18$$

intersects the curve

$$2x^2 - 23x + 2y + 50 = 0$$

at the points A and B.

Given that A lies below the x-axis and that the point P lies on AB such that

$$AP : PB = 1 : 2,$$

find the coordinates of P.

Solution

Well,

$$3x = 2y + 18 \Rightarrow 2y = 3x - 18$$

and insert the line into the curve:

$$\begin{aligned} 2x^2 - 23x + 2y + 50 = 0 &\Rightarrow 2x^2 - 23x + (3x - 18) + 50 = 0 \\ &\Rightarrow 2x^2 - 20x + 32 = 0 \\ &\Rightarrow 2(x^2 - 10x + 16) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -10 \\ \text{multiply to:} & +16 \end{array}\right\} -8, -2$$

$$\begin{aligned} &\Rightarrow 2(x - 8)(x - 2) = 0 \\ &\Rightarrow x - 8 = 0 \text{ or } x - 2 = 0 \\ &\Rightarrow x = 8 \text{ or } x = 2 \\ &\Rightarrow y = 3 \text{ or } y = -6; \end{aligned}$$

so, $A(2, -6)$ and $B(8, 3)$. Now,

$$8 - 2 = 6$$

and

$$AP : PB = 1 : 2 = 2 : 4$$

which leads to the x-coordinate of P is

$$2 + 2 = 4.$$

Next,

$$x = 4 \Rightarrow y = -3$$

and $\underline{\underline{P(4, -3)}}$.

4. (a) Find the first three terms, in ascending powers of u, in the expansion of (2)

$$(2 + u)^5.$$

Solution

$$\begin{aligned}(2+u)^5 &= (2)^5 + \binom{5}{1}(2)^4(u)^1 + \binom{5}{2}(2)^3(u)^2 + \dots \\ &= \underline{\underline{32 + 80u + 80u^2 + \dots}}.\end{aligned}$$

(b) By replacing u with (4)

$$2x - 5x^2,$$

find the coefficient of x^2 in the expansion of

$$(2 + 2x - 5x^2)^5.$$

Solution

Replacing u:

$$(2 + u)^5 = 32 + 80(2x - 5x^2) + 80(2x - 5x^2)^2 + \dots$$

$\times$	$2x$	$-5x^2$
$2x$	$4x^2$	$\ldots$
$-5x^2$	$\ldots$	$\ldots$

$$\begin{aligned} &= 32 + 160x - 400x^2 + 80(4x^2 + \ldots) + \ldots \\ &= 32 + 160x - 80x^2 + \ldots; \end{aligned}$$

hence, the coefficient of x^2 is $\underline{\underline{-80}}$.

5. A curve has the equation

$$y = \sqrt{x} + \frac{9}{\sqrt{x}}.$$

(a) Find expressions for (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

Solution

$$\begin{aligned} y = \sqrt{x} + \frac{9}{\sqrt{x}} &\Rightarrow y = x^{\frac{1}{2}} + 9x^{-\frac{1}{2}} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2}x^{-\frac{1}{2}} - \tfrac{9}{2}x^{-\frac{3}{2}}}} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\tfrac{1}{4}x^{-\frac{3}{2}} + \tfrac{27}{4}x^{-\frac{5}{2}}}}. \end{aligned}$$

(b) Show that the curve has a stationary value when $x = 9$. (1)

Solution

Well,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \tfrac{1}{2}x^{-\frac{1}{2}} - \tfrac{9}{2}x^{-\frac{3}{2}} = 0$$
$$\Rightarrow x^{-\frac{1}{2}} - 9x^{-\frac{3}{2}} = 0$$
$$\Rightarrow x^{-\frac{3}{2}}(x-9) = 0$$
$$\Rightarrow \underline{\underline{x = 9}},$$

as required.

(c) Find the nature of this stationary value. (2)

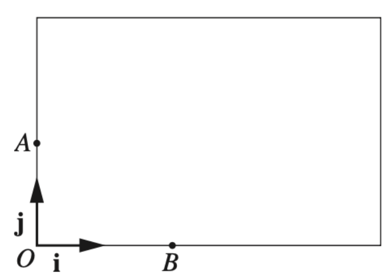
Solution

Now,

$$x = 9 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \tfrac{1}{54} > 0$$

and so this stationary value is a minimum.

6. The diagram shows a large rectangular television screen in which one corner is taken as the origin O and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along two of the edges.



In a game, an alien spacecraft appears at the point A with position vector $12\mathbf{j}$ cm and moves across the screen with velocity $(40\mathbf{i} + 15\mathbf{j})$ cm per second.

A player fires a missile from a point B; the missile is fired 0.5 seconds after the spacecraft appears on the screen.

The point B has position vector $46\mathbf{i}$ cm and the velocity of the missile is $(k\mathbf{i} + 30\mathbf{j})$ cm per second, where k is a constant.

Given that the missile hits the spacecraft,

(a) show that the spacecraft moved across the screen for 1.8 seconds before impact, (4)

Solution
Vertically,
Alien:

$$12 + (1.8 \times 15) = 12 + 27 = 39.$$

Missile:

$$(1.8 - 0.5) \times 30 = 1.3 \times 30 = 39.$$

Hence, the spacecraft moved across the screen for 1.8 seconds before impact

(b) find the value of k. (3)

Solution
Horizontally,
Alien:

$$1.8 \times 40 = 72$$

Missile:

$$46 + (1.8 - 0.5) \times k = 46 + 1.3k.$$

So,

$$46 + 1.3k = 72 \Rightarrow 1.3k = 26$$
$$\Rightarrow k = 20.$$

7. (a) Use the substitution $u = 5^x$ to solve the equation (5)

$$5^{x+1} = 8 + 4(5^{-x}).$$

Solution

$$5^{x+1} = 8 + 4(5^{-x}) \Rightarrow 5^1 \cdot 5^x = 8 + 4(5^x)^{-1}$$
$$\Rightarrow 5u = 8 + \frac{4}{u}$$

multiply by u:

$$\Rightarrow 5u^2 = 8u + 4$$
$$\Rightarrow 5u^2 - 8u - 4 = 0$$

$$\begin{array}{ll} \text{add to:} & -8 \\ \text{multiply to:} & (+5)\times(-4) = -20 \end{array} \Bigg\} -10,\ +2$$

e.g.,

$$\begin{aligned}
&\Rightarrow 5u^2 - 10u + 2u - 4 = 0 \\
&\Rightarrow 5u(u-2) + 2(u-2) = 0 \\
&\Rightarrow (5u+2)(u-2) = 0 \\
&\Rightarrow 5u + 2 = 0 \text{ or } u - 2 = 0 \\
&\Rightarrow u = -\tfrac{2}{5} \text{ or } u = 2
\end{aligned}$$

but $5^x \neq -\frac{2}{5}$!

$$\begin{aligned}
&\Rightarrow 5^x = 2 \\
&\Rightarrow \underline{\underline{x = \log_5 2 \text{ or } 0.431 \text{ (3 sf)}}}.
\end{aligned}$$

(b) Given that (3)

$$\log(p-q) = \log p - \log q,$$

express p in terms of q.

Solution

Well,

$$\begin{aligned}
\log(p-q) = \log p - \log q &\Rightarrow \log(p-q) = \log\left(\frac{p}{q}\right) \\
&\Rightarrow p - q = \frac{p}{q} \\
&\Rightarrow p - \frac{p}{q} = q \\
&\Rightarrow p\left(1 - \frac{1}{q}\right) = q \\
&\Rightarrow p\left(\frac{q-1}{q}\right) = q \\
&\Rightarrow \underline{\underline{p = \frac{q^2}{q-1}}}.
\end{aligned}$$

8. (a) Solve, for $0 \leqslant x \leqslant 2$, the equation (3)

$$1 + 5\cos 3x = 0,$$

giving your answer in radians correct to 2 decimal places.

Solution

Well,

$$0 \leqslant x \leqslant 2 \Rightarrow 0 \leqslant 3x \leqslant 6$$

and

$$\begin{aligned} 1 + 5\cos 3x = 0 &\Rightarrow 5\cos 3x = -1 \\ &\Rightarrow \cos 3x = -\tfrac{1}{5} \\ &\Rightarrow 3x = 1.772\,154\,48,\ 4.511\,031\,06 \text{ (FCD)} \\ &\Rightarrow x = 0.590\,718\,082\,5,\ 1.503\,677\,02 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 0.59,\ 1.50 \text{ (2 dp)}}}. \end{aligned}$$

(b) Find all the angles between 0° and 360° such that (5)

$$\sec y + 5\tan y = 3\cos y.$$

Solution

$$\begin{aligned} \sec y + 5\tan y = 3\cos y &\Rightarrow \frac{1}{\cos y} + \frac{5\sin y}{\cos y} = 3\cos y \\ &\Rightarrow \frac{1 + 5\sin y}{\cos y} = 3\cos y \\ &\Rightarrow 1 + 5\sin y = 3\cos^2 y \\ &\Rightarrow 1 + 5\sin y = 3(1 - \sin^2 y) \\ &\Rightarrow 1 + 5\sin y = 3 - 3\sin^2 y) \\ &\Rightarrow 3\sin^2 y + 5\sin y - 2 = 0 \end{aligned}$$

e.g.,

$$\left.\begin{array}{ll}\text{add to:} & +5 \\ \text{multiply to:} & (+3)\times(-2) = -6\end{array}\right\} +5,\ -1$$

$$\begin{aligned}
&\Rightarrow 3\sin^2 y + 6\sin y - \sin y - 2 = 0 \\
&\Rightarrow 3\sin y(\sin y + 2) - 1(\sin y + 2) = 0 \\
&\Rightarrow (3\sin y - 1)(\sin y + 2) = 0 \\
&\Rightarrow 3\sin y - 1 = 0 \text{ or } \sin y + 2 = 0 \\
&\Rightarrow \sin y = \tfrac{1}{3} \text{ or } \sin y = -2 \text{ (can't happen!)} \\
&\Rightarrow y = 19.471\,220\,63,\ 160.528\,779\,4 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{y = 19.5,\ 161 \text{ (3 sf)}}}.
\end{aligned}$$

9. The table below shows experimental values of the variables x and y.

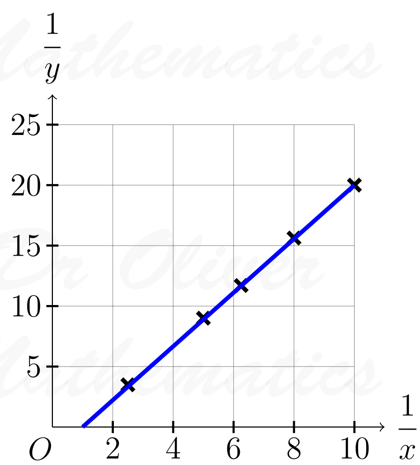
x	0.100	0.125	0.160	0.200	0.400
y	0.050	0.064	0.085	0.111	0.286

(a) On graph paper draw the graph of $\dfrac{1}{y}$ against $\dfrac{1}{x}$. (3)

Solution

$\frac{1}{x}$	10	8	6.25	5	2.5
$\frac{1}{y}$	20	15.625	11.7...	9.00...	3.49...

so we have



Hence,

(b) express y in terms of x, (4)

Solution

Well, we have the line of best fit goes through $(0, 1)$ and $(10, 20)$:

$$\begin{aligned}\text{gradient} &= \frac{20-1}{10-0} \\ &= 1.9\end{aligned}$$

and the equation is

$$\begin{aligned}\frac{1}{y} - 0 = 1.9\left(\frac{1}{x} - 1\right) &\Rightarrow \frac{1}{y} = 1.9\left(\frac{1-x}{x}\right) \\ &\Rightarrow y = \frac{x}{1.9(1-x)} \\ &\Rightarrow \underline{\underline{y = \frac{10x}{19(1-x)}}}.\end{aligned}$$

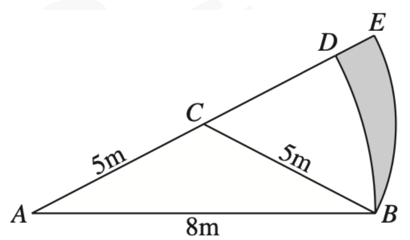
(c) find the value of x for which $y = 0.15$. (2)

Solution

Now,

$$
\begin{aligned}
y = 0.15 \Rightarrow 0.15 &= \frac{10x}{19(1-x)} \\
\Rightarrow 2.85(1-x) &= 10x \\
\Rightarrow 2.85 - 2.85x &= 10x \\
\Rightarrow 2.85 &= 7.15x \\
\Rightarrow x &= 0.\dot{3}98\,60\dot{1} \\
\Rightarrow \underline{\underline{x = 0.399 \text{ (3 sf)}}}&.
\end{aligned}
$$

10. The diagram shows an isosceles triangle ABC in which $AB = 8$ m and $BC = CA = 5$ m.



- $ABDA$ is a sector of the circle, centre A and radius 8 m.
- $CBEC$ is a sector of the circle, centre C and radius 5 m.

(a) Show that angle BCE is 1.287 radians, correct to 3 decimal places. (2)

Solution

$$
\begin{aligned}
& AB^2 = AC^2 + BC^2 - 2 \times AC \times BC \times \cos BCA \\
\Rightarrow\quad & 8^2 = 5^2 + 5^2 - 2 \times 5 \times 5 \times \cos BCA \\
\Rightarrow\quad & 64 = 50 - 50\cos BCA \\
\Rightarrow\quad & 50\cos BCA = -14 \\
\Rightarrow\quad & \cos BCA = -\tfrac{7}{25} \\
\Rightarrow\quad & \cos BCA = 1.854\,590\,436 \text{ (FCD)} \\
\Rightarrow\quad & \cos BCE = \pi - 1.854\ldots \\
\Rightarrow\quad & \angle BCE = 1.287\,002\,218 \text{ (FCD)} \\
\Rightarrow\quad & \underline{\underline{\angle BCA = 1.287 \text{ (3 dp)}}},
\end{aligned}
$$

as required.

(b) Find the perimeter of the shaded region. (4)

Solution

Arc BD:

$$\begin{aligned}\angle CAB &= \tfrac{1}{2}(\pi - 1.854\ldots)\\ &= 0.645\,501\,108\,8 \text{ (FCD)}\end{aligned}$$

and

$$\begin{aligned}\text{arc } BD &= 8 \times 0.645\ldots\\ &= 5.148\,008\,87 \text{ (FCD)}.\end{aligned}$$

Arc BE:

$$\begin{aligned}\text{arc } BE &= 5 \times 1.287\ldots\\ &= 6.435\,011\,09 \text{ (FCD)}.\end{aligned}$$

DE:

$$DE = 10 - 8 = 2.$$

Add them all up:

$$\begin{aligned}\text{perimeter} &= 5.148\ldots + 6.435\ldots + 2\\ &= 13.583\,019\,96 \text{ (FCD)}\\ &= \underline{\underline{13.6\text{cm (3 sf)}}}.\end{aligned}$$

(c) Find the area of the shaded region. (4)

Solution

$$\begin{aligned}\text{Area of } ABC &= \tfrac{1}{2} \times 5 \times 8 \times \sin 0.645\ldots\\ &= 12.\end{aligned}$$

$$\begin{aligned}\text{Sector } ABD &= \tfrac{1}{2} \times 5 \times 5 \times 1.287\ldots\\ &= 16.085\,527\,73 \text{ (FCD)}.\end{aligned}$$

$$\text{Sector } CBE = \tfrac{1}{2} \times 8 \times 8 \times 0.645\ldots$$
$$= 20.656\,035\,48 \text{ (FCD)}.$$

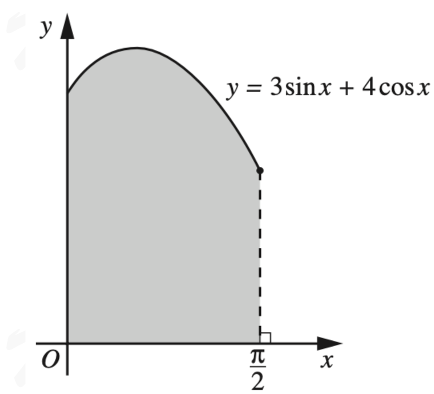
Finally,

$$\begin{aligned} \text{shaded area} &= 12 + 16.085\ldots - 20.656\ldots \\ &= 7.431\,492\,243 \text{ (FCD)} \\ &= \underline{\underline{7.43 \text{ m}^2 \text{ (3 sf)}}}. \end{aligned}$$

EITHER

11. The graph shows part of the curve

$$y = 3\sin x + 4\cos x \text{ for } 0 \leqslant x \leqslant \tfrac{1}{2}\pi \text{ radians.}$$



(a) Find the coordinates of the maximum point of the curve. (5)

Solution

Now,

$$y = 3\sin x + 4\cos x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos x - 4\sin x$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3\cos x - 4\sin x = 0 \\ &\Rightarrow 4\sin x = 3\cos x \\ &\Rightarrow \tan x = \tfrac{3}{4} \\ &\Rightarrow x = 0.643\,501\,108\,8 \text{ (FCD)} \end{aligned}$$

and

$$x = 0.643\ldots \Rightarrow y = 5;$$

hence, the coordinates of the maximum point of the curve are $\underline{\underline{(0.644, 5)\ (3\text{ sf})}}$.

(b) Find the area of the shaded region. (5)

Solution

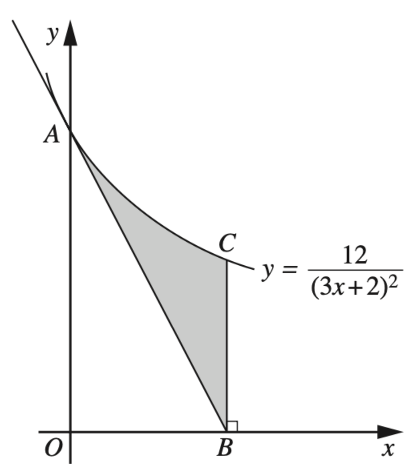
$$\begin{aligned}\text{Area} &= \int_0^{\frac{1}{2}\pi} (3\sin x + 4\cos x)\,\mathrm{d}x \\ &= [-3\cos x + 4\sin x]_{x=0}^{\frac{1}{2}\pi} \\ &= (0+4) - (-3-0) \\ &= 4+3 \\ &= \underline{\underline{7}}.\end{aligned}$$

OR

12. The diagram, which is not drawn to scale, shows part of the curve

$$y = \frac{12}{(3x+2)^2},$$

intersecting the y-axis at A.



The tangent to the curve at A meets the x-axis at B.
The point C lies on the curve and BC is parallel to the y-axis.

(a) Find the x-coordinate of B. (4)

Solution

$$y = \frac{12}{(3x+2)^2} \Rightarrow y = 12(3x+2)^{-2}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 12(3x+2)^{-3} \times (-2) \times 3$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -72(3x+2)^{-3}.$$

Now,

$$x = 0 \Rightarrow y = 3,\ \frac{\mathrm{d}y}{\mathrm{d}x} = -9$$

so $A(0,3)$. Next,

$$y - 3 = -9(x-0) \Rightarrow y = -9x + 3$$

and, finally, the x-coordinate of B is

$$y = 0 \Rightarrow -9x + 3 = 0$$
$$\Rightarrow -9x = -3$$
$$\Rightarrow \underline{\underline{x = \tfrac{1}{3}}}.$$

(b) Find the area of the shaded region. (6)

Solution

$$\text{Shaded region} = \text{integral} - \text{triangle } OAB$$
$$= \int_0^{\frac{1}{3}} 12(3x+2)^{-2}\,\mathrm{d}x - \left(\tfrac{1}{2} \times 3 \times \tfrac{1}{3}\right)$$
$$= \left[-4(3x+2)^{-1}\right]_{x=0}^{\frac{1}{3}} - \tfrac{1}{2}$$
$$= \left[-\tfrac{4}{3} - (-2)\right] - \tfrac{1}{2}$$
$$= \underline{\underline{\tfrac{1}{6}}}.$$