Dr Oliver Mathematics
Determinant, Eigenvalues, and Trace

In this note, we will illustrate some facts concerning the determinant, eigenvalues, and trace of a square matrix and this will give us a way of checking whether our determinant and eigenvalues are correct.

Suppose $\mathbf{A} = (a_{ij})$ is an $n \times n$ matrix.

Definition 1
An *eigenvalue* of $\mathbf{A}$ is a number λ such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

for some nonzero vector $\mathbf{v}$.

Definition 2
An *eigenvector* of $\mathbf{A}$ is a nonzero vector $\mathbf{v}$ such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

for some number λ.

Definition 3
The *trace*, $\operatorname{trace}(\mathbf{A})$, is given by

$$\operatorname{trace}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}.$$

Definition 4
The *determinant*, $\det \mathbf{A}$, is given by

(a) if

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$\det \mathbf{A} = ad - bc,$$

(b) if

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

then

$$\det \mathbf{A} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

Definition 5
The determinant

$$\mathbf{A} - \lambda\mathbf{I}$$

is called the *characteristic polynomial* of $\mathbf{A}$.

Definition 6
The equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

is called the *characteristic equation* of $\mathbf{A}$.

Then we need three theorems, each of which will we will state without proof.

Theorem 1
The matrix $\mathbf{A}$ has n eigenvalues (including each according to its multiplicity).

Theorem 2
The sum of the n eigenvalues of $\mathbf{A}$ is the same as the trace of $\mathbf{A}$, i.e.,

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}.$$

Theorem 3
The product of the n eigenvalues of $\mathbf{A}$ is the same as the determinant of $\mathbf{A}$, i.e.,

$$\prod_{i=1}^{n} \lambda_i = \det \mathbf{A}.$$

We will now do five examples.

1.
$$\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 5 & 1 \end{pmatrix}.$$

(a) Find $\det \mathbf{A}$.

Solution

$$\begin{aligned} \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} &= (2 \times 1) - (4 \times 5) \\ &= \underline{\underline{-18}}. \end{aligned}$$

(b) Solve the characteristic polynomial of $\mathbf{A}$.

Solution

$$\begin{aligned} \begin{vmatrix} 2-\lambda & 4 \\ 5 & 1-\lambda \end{vmatrix} &= 0 \\ \Rightarrow \quad (2-\lambda)(1-\lambda) - 20 &= 0 \\ \Rightarrow \quad (\lambda^2 - 3\lambda + 2) - 20 &= 0 \\ \Rightarrow \quad \lambda^2 - 3\lambda - 18 &= 0 \\ \Rightarrow \quad (\lambda - 6)(\lambda + 3) &= 0 \\ \Rightarrow \quad \underline{\underline{\lambda = -3 \text{ or } 6}}. \end{aligned}$$

(c) Find trace($\mathbf{A}$).

Solution

$$\text{trace}(\mathbf{A}) = 2 + 1 = \underline{\underline{3}}.$$

(d) Verify that the sum of the two eigenvalues of $\mathbf{A}$ is the same as the trace of $\mathbf{A}$.

Solution

$$\text{Sum of the eigenvalues} = (-3) + 6 = 3$$

which is the same as the trace. ✓

(e) Verify that the product of the two eigenvalues of $\mathbf{A}$ is the same as the determinant of $\mathbf{A}$.

Solution

$$\text{Product of the eigenvalues} = (-3) \times 6 = -18$$

which is the same as the determinant. ✓

2.

$$\mathbf{B} = \begin{pmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

(a) Find $\det \mathbf{B}$.

Solution

$$\begin{vmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 3(0-2) - 2(-1-1) + 2(-2-0)$$
$$= \underline{\underline{-6}}.$$

(b) Solve the characteristic polynomial of $\mathbf{B}$.

Solution

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ -1 & -\lambda & 1 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$
$$\Rightarrow \quad (3-\lambda)[-\lambda(1-\lambda)-2] - 2[-(1-\lambda)-1] + 2[-2+\lambda] = 0$$
$$\Rightarrow \quad (3-\lambda)(\lambda^2-\lambda-2) - 2[\lambda-2] + 2[-2+\lambda] = 0$$
$$\Rightarrow \quad (3-\lambda)(\lambda^2-\lambda-2) = 0$$
$$\Rightarrow \quad (3-\lambda)(\lambda-2)(\lambda+1) = 0$$
$$\Rightarrow \quad \underline{\underline{\lambda = -1, 2, \text{ or } 3}}.$$

(c) Find trace($\mathbf{B}$).

Solution

$$\text{trace}(\mathbf{B}) = 3 + 0 + 1 = \underline{\underline{4}}.$$

(d) Verify that the sum of the three eigenvalues of $\mathbf{B}$ is the same as the trace of $\mathbf{B}$.

Solution

$$\text{Sum of the eigenvalues} = (-1) + 2 + 3 = 4$$

which is the same as the trace. ✓

(e) Verify that the product of the three eigenvalues of $\mathbf{B}$ is the same as the determinant of $\mathbf{B}$.

Solution

$$\text{Product of the eigenvalues} = (-1) \times 2 \times 3 = -6$$

which is the same as the determinant. ✓

We can even do $\mathbb{R}$ ...

3.
$$\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

(a) Find $\det \mathbf{C}$.

Solution

$$\begin{aligned} \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} &= (1 \times 4) - (2 \times 3) \\ &= \underline{\underline{-2}}. \end{aligned}$$

(b) Solve the characteristic polynomial of $\mathbf{C}$.

Solution

$$\begin{aligned} & \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0 \\ \Rightarrow\ & (1-\lambda)(4-\lambda) - 6 = 0 \\ \Rightarrow\ & (\lambda^2 - 5\lambda + 4) - 6 = 0 \\ \Rightarrow\ & \lambda^2 - 5\lambda = 2 \\ \Rightarrow\ & \lambda^2 - 5\lambda + 6\tfrac{1}{4} = 2 + 6\tfrac{1}{4} \\ \Rightarrow\ & (\lambda - \tfrac{5}{2})^2 = \tfrac{33}{4} \\ \Rightarrow\ & \lambda - \tfrac{5}{2} = \pm\tfrac{\sqrt{33}}{2} \\ \Rightarrow\ & \underline{\underline{\lambda = \tfrac{5\pm\sqrt{33}}{2}}}. \end{aligned}$$

(c) Find trace($\mathbf{C}$).

Solution

$$\text{trace}(\mathbf{C}) = 1 + 4 = \underline{\underline{5}}.$$

(d) Verify that the sum of the two eigenvalues of $\mathbf{C}$ is the same as the trace of $\mathbf{C}$.

Solution

$$\text{Sum of the eigenvalues} = (\tfrac{5-\sqrt{89}}{2}) + (\tfrac{5+\sqrt{89}}{2}) = 5$$

which is the same as the trace. ✓

(e) Verify that the product of the two eigenvalues of $\mathbf{C}$ is the same as the determinant of $\mathbf{C}$.

Solution

$\times$	$\frac{5}{2}$	$-\frac{\sqrt{33}}{2}$
$\frac{5}{2}$	$\frac{25}{4}$	$-\frac{5\sqrt{33}}{2}$
$+\frac{\sqrt{33}}{2}$	$+\frac{5\sqrt{33}}{2}$	$-\frac{33}{4}$

Finally,

$$\text{product of the eigenvalues} = \tfrac{25}{4} - \tfrac{33}{4} = -2$$

which is the same as the determinant. ✓

. . . or $\mathbb{C}$ . . .

4.

$$\mathbf{D} = \begin{pmatrix} 2 & -6 \\ 3 & -4 \end{pmatrix}.$$

(a) Find $\det \mathbf{D}$.

Solution

$$\begin{aligned} \begin{vmatrix} 2 & -6 \\ 3 & -4 \end{vmatrix} &= [2 \times (-4)] - [3 \times (-6)] \\ &= \underline{\underline{10}}. \end{aligned}$$

(b) Solve the characteristic polynomial of $\mathbf{D}$.

Solution

$$\begin{aligned} \begin{vmatrix} 2-\lambda & -6 \\ 3 & -4-\lambda \end{vmatrix} &= 0 \\ \Rightarrow \quad (2-\lambda)(-4-\lambda) + 18 &= 0 \\ \Rightarrow \quad (\lambda^2 + 2\lambda - 8) + 18 &= 0 \\ \Rightarrow \quad \lambda^2 + 2\lambda &= -10 \\ \Rightarrow \quad \lambda^2 + 2\lambda + 1 &= -10 + 1 \\ \Rightarrow \quad (\lambda+1)^2 &= -9 \\ \Rightarrow \quad \lambda + 1 &= \pm 3\mathrm{i} \\ \Rightarrow \quad \underline{\underline{\lambda = -1 \pm 3\mathrm{i}}}. \end{aligned}$$

(c) Find trace($\mathbf{D}$).

Solution

$$\operatorname{trace}(\mathbf{D}) = 2 + (-4) = \underline{\underline{-2}}.$$

(d) Verify that the sum of the two eigenvalues of $\mathbf{D}$ is the same as the trace of $\mathbf{D}$.

Solution

$$\text{Sum of the eigenvalues} = (-1 - 3\mathrm{i}) + (-1 + 3\mathrm{i}) = -2$$

which is the same as the trace. ✓

(e) Verify that the product of the two eigenvalues of $\mathbf{D}$ is the same as the determinant of $\mathbf{D}$.

Solution

$\times$	-1	$+3\mathrm{i}$
-1	1	$+3\mathrm{i}$
$-3\mathrm{i}$	$-3\mathrm{i}$	$+9$

Finally,

$$\text{product of the eigenvalues} = 1 + 9 = 10$$

which is the same as the determinant. ✓

...or even repeated roots!

5.
$$\mathbf{E} = \begin{pmatrix} 5+a & a \\ -a & 5-a \end{pmatrix}.$$

(a) Find $\det \mathbf{E}$.

Solution

$$\begin{aligned} \begin{vmatrix} 5+a & a \\ -a & 5-a \end{vmatrix} &= (5+a)(5-a) - [(-a) \times a)] \\ &= (25 - a^2) + a^2 \\ &= \underline{\underline{25}}. \end{aligned}$$

(b) Solve the characteristic polynomial of $\mathbf{E}$.

Solution

$$\begin{aligned} & \begin{vmatrix} 5+a-\lambda & a \\ -a & 5-a-\lambda \end{vmatrix} = 0 \\ \Rightarrow \quad & (5+a-\lambda)(5-a-\lambda) + a^2 = 0 \end{aligned}$$

$\times$	5	$+a$	$-\lambda$
5	25	$+5a$	-5λ
$-a$	$-5a$	$-a^2$	$+a\lambda$
$-\lambda$	-5λ	$-a\lambda$	$+\lambda^2$

$$\begin{aligned}
&\Rightarrow \quad (\lambda^2 - 10\lambda - a^2 + 25) + a^2 = 0 \\
&\Rightarrow \quad \lambda^2 - 10\lambda + 25 = 0 \\
&\Rightarrow \quad (\lambda - 5)^2 = 0 \\
&\Rightarrow \quad \underline{\underline{\lambda = 5 \text{ (multiplicity of 2)}}}.
\end{aligned}$$

(c) Find trace($\mathbf{E}$).

Solution

$$\text{trace}(\mathbf{E}) = (5 + a) + (5 - a) = \underline{\underline{10}}.$$

(d) Verify that the sum of the two eigenvalues of $\mathbf{E}$ is the same as the trace of $\mathbf{E}$.

Solution

$$\text{Sum of the eigenvalues} = 5 + 5 = 10$$

which is the same as the trace. ✓

(e) Verify that the product of the two eigenvalues of $\mathbf{E}$ is the same as the determinant of $\mathbf{E}$.

Solution

$$\text{Product of the eigenvalues} = 5 \times 5 = 25$$

which is the same as the determinant. ✓