Dr Oliver Mathematics
Further Mathematics
First Order Differential Equations
Past Examination Questions

This booklet consists of 34 questions across a variety of examination topics.
The total number of marks available is 312.

1. (a) Find the general solution of the differential equation (8)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \cos^3 x,$$

expressing y in terms of x.

Solution

$$\text{IF} = \mathrm{e}^{\int \tan x \,\mathrm{d}x} = \mathrm{e}^{\ln \sec x} = \sec x.$$

Hence

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \cos^3 x &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}\sec x + y\sec x\tan x = \cos^2 x \\
&\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\sec x) = \cos^2 x \\
&\Rightarrow y\sec x = \int \cos^2 x \mathrm{d}x \\
&\Rightarrow y\sec x = \int \left(\tfrac{1}{2}\cos 2x + \tfrac{1}{2}\right) \mathrm{d}x \\
&\Rightarrow y\sec x = \tfrac{1}{4}\sin 2x + \tfrac{1}{2}x + c \\
&\Rightarrow \underline{\underline{y = \cos x\left(\tfrac{1}{4}\sin 2x + \tfrac{1}{2}x + c\right)}},
\end{aligned}$$

or equivalent.

(b) Find the particular solution for which $y = 2$ when $x = \pi$. (2)

Solution

$$x = \pi, y = 2 \Rightarrow 2 = -1\left(\tfrac{1}{2}\pi + c\right) \Rightarrow c = -2 - \tfrac{1}{2}\pi.$$

Hence

$$\underline{\underline{y = \cos x\left(\tfrac{1}{4}\sin 2x + \tfrac{1}{2}x - 2 - \tfrac{1}{2}\pi\right)}}.$$

2. (a) Use the substitution $z = x + y$ to show that the differential equation (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y+3}{x+y-1} \qquad (\dagger)$$

may be written in the form

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2(z+1)}{z-1}.$$

Solution

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}$$

and so

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y+3}{x+y-1} &\Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x} - 1 = \frac{z+3}{z-1} \\
&\Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{z+3}{z-1} + 1 \\
&\Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{z+3}{z-1} + \frac{z-1}{z-1} \\
&\Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2z+2}{z-1} \\
&\Rightarrow \underline{\underline{\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2(z+1)}{z-1}}},
\end{aligned}$$

as required.

(b) Hence find the general solution of the differential equation (†). (4)

Solution

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2(z+1)}{z-1} &\Rightarrow \frac{z-1}{z+1}\,\mathrm{d}z = 2\,\mathrm{d}x \\
&\Rightarrow \int \frac{z-1}{z+1}\,\mathrm{d}z = \int 2\,\mathrm{d}x \\
&\Rightarrow \int \frac{(z+1)-2}{z+1}\,\mathrm{d}z = \int 2\,\mathrm{d}x \\
&\Rightarrow \int \left(1 - \frac{2}{z+1}\right)\,\mathrm{d}z = \int 2\,\mathrm{d}x \\
&\Rightarrow z - 2\ln|z+1| = 2x + c \\
&\Rightarrow x + y - 2\ln|x+y+1| = 2x + c \\
&\Rightarrow \underline{\underline{y - 2\ln|x+y+1| = x + c}},
\end{aligned}$$

or equivalent.

3. (a) Show that (2)

$$\sqrt{\frac{1+x}{1-x}}$$

is an integrating factor for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{1-x^2}=\sqrt{1-x},\ |x|<1.$$

Solution

$$\begin{aligned}\int\frac{1}{1-x^2}\,\mathrm{d}x&=\frac{1}{2}\int\left(\frac{1}{1+x}-\frac{1}{1-x}\right)\,\mathrm{d}x\\&=\frac{1}{2}\left(\ln(1+x)-\ln(1-x)\right)+c\text{ (since }|x|<1)\\&=\ln\sqrt{\frac{1+x}{1-x}}+c\end{aligned}$$

and hence

$$\text{IF}=\mathrm{e}^{\int\frac{1}{1-x^2}\,\mathrm{d}x}=\underline{\underline{\sqrt{\frac{1+x}{1-x}}}},$$

as required.

(b) Hence find the solution of the differential equation for which $y=2$ when $x=0$. Give your answer in the form $y=\mathrm{f}(x)$. (6)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{1-x^2}=\sqrt{1-x}&\Rightarrow\sqrt{\frac{1+x}{1-x}}\frac{\mathrm{d}y}{\mathrm{d}x}+\sqrt{\frac{1+x}{1-x}}\frac{y}{1-x^2}=\sqrt{\frac{1+x}{1-x}}\sqrt{1-x}\\&\Rightarrow\sqrt{\frac{1+x}{1-x}}\frac{\mathrm{d}y}{\mathrm{d}x}+\sqrt{\frac{1+x}{1-x}}\frac{y}{1-x^2}=\sqrt{1+x}\\&\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}\left(y\sqrt{\frac{1+x}{1-x}}\right)=\sqrt{1+x}\\&\Rightarrow y\sqrt{\frac{1+x}{1-x}}=\tfrac{2}{3}(1+x)^{\frac{3}{2}}+c\\&\Rightarrow y=\sqrt{\frac{1-x}{1+x}}\left(\tfrac{2}{3}(1+x)^{\frac{3}{2}}+c\right).\end{aligned}$$

Now

$$x = 0, y = 2 \Rightarrow 2 = \tfrac{2}{3} + c \Rightarrow c = \tfrac{4}{3}$$

and hence

$$\underline{\underline{y = \sqrt{\frac{1-x}{1+x}}\left(\tfrac{2}{3}(1+x)^{\frac{3}{2}} + \tfrac{4}{3}\right)}},$$

or equivalent.

4. (a) Use the substitution $y = xz$ to find the general solution of the differential equation (6)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x\cos\left(\frac{y}{x}\right).$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = z + x\frac{\mathrm{d}z}{\mathrm{d}x}$$

and so

$$\begin{aligned}
x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x\cos\left(\frac{y}{x}\right) &\Rightarrow x\left(z + x\frac{\mathrm{d}z}{\mathrm{d}x}\right) - xz = x\cos z \\
&\Rightarrow xz + x^2\frac{\mathrm{d}z}{\mathrm{d}x} - xz = x\cos z \\
&\Rightarrow x^2\frac{\mathrm{d}z}{\mathrm{d}x} = x\cos z \\
&\Rightarrow x\frac{\mathrm{d}z}{\mathrm{d}x} = \cos z \\
&\Rightarrow \sec z\,\mathrm{d}z = \frac{1}{x}\,\mathrm{d}x \\
&\Rightarrow \int \sec z\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x \\
&\Rightarrow \ln|\sec z + \tan z| = \ln|x| + c \\
&\Rightarrow \ln|\sec z + \tan z| = \ln|x| + \ln A \\
&\Rightarrow |\sec z + \tan z| = A|x| \\
&\Rightarrow \underline{\underline{\left|\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right)\right| = A|x|}},
\end{aligned}$$

or equivalent.

(b) Find the solution of the differential equation for which $y = \pi$ when $x = 4$. (2)

Solution

Using the values for x and y,

$$\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4} = 4A \Rightarrow A = \frac{1+\sqrt{2}}{4}$$

and so

$$\underline{\underline{\left|\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right)\right| = \frac{1+\sqrt{2}}{4}|x|}}.$$

5. The substitution $y = u^k$, where k is an integer, is used to solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = x^2y^2 \qquad (\dagger)$$

by changing it into an equation (‡) in the variables u and x.

(a) Show that equation (‡) may be written in the form (4)

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{3u}{kx} = \frac{1}{k}xu^{k+1}.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(u^k) = \frac{\mathrm{d}}{\mathrm{d}u}(u^k) \times \frac{\mathrm{d}u}{\mathrm{d}x} = ku^{k-1}\frac{\mathrm{d}u}{\mathrm{d}x}$$

and hence

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = x^2y^2 &\Rightarrow x\left(ku^{k-1}\frac{\mathrm{d}u}{\mathrm{d}x}\right) + 3u^k = x^2(u^k)^2 \\ &\Rightarrow ku^{k-1}x\frac{\mathrm{d}u}{\mathrm{d}x} + 3u^k = x^2u^{2k} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{3u}{kx} = \frac{1}{k}xu^{k+1}}},\end{aligned}$$

as required.

(b) Write down the value of k for which the integrating factor method may be used to solve equation (‡). (1)

Solution

The RHS needs to be just a function of x and so $\underline{\underline{k = -1}}$ is the value that is required.

(c) Using this value of k, solve equation (‡) and hence find the general solution of equation (†), giving your answer in the form $y = \mathrm{f}(x)$. (4)

Solution

If $k = -1$ then the differential equation is

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{3u}{x} = -x.$$

The integrating factor is

$$\mathrm{e}^{\int -\frac{3}{x}\,\mathrm{d}x} = \mathrm{e}^{-3\ln x} = x^{-3}$$

and so

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{3u}{x} = -x &\Rightarrow x^{-3}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{3u}{x^4} = -x^{-2}\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-3}u\right) = -x^{-2}\\ &\Rightarrow x^{-3}u = -x^{-1} + c\\ &\Rightarrow u = x^2 + cx^3\\ &\Rightarrow \frac{1}{y} = x^2 + cx^3\\ &\Rightarrow \underline{\underline{y = \frac{1}{x^2 + cx^3}}}.\end{aligned}$$

6. The variables x and y are related by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^2 + y^2}{xy}. \qquad (\dagger)$$

(a) Use the substitution $y = ux$, where u is a function of x to transform the differential equation (†) into (3)

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{u}.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(ux) = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^2 + y^2}{xy} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{y} + \frac{y}{x}$$
$$\Rightarrow u + x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{u} + u$$
$$\Rightarrow \underline{\underline{x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{u}}},$$

as required.

(b) Hence find the general solution of differential equation (†), giving your answer in the form $y^2 = \mathrm{f}(x)$. (4)

Solution

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{u} \Rightarrow u\,\mathrm{d}u = \frac{2}{x}\,\mathrm{d}x$$
$$\Rightarrow \int u\,\mathrm{d}u = \int \frac{2}{x}\,\mathrm{d}x$$
$$\Rightarrow \tfrac{1}{2}u^2 = 2\ln|x| + c$$
$$\Rightarrow u^2 = 4\ln|x| + 2c$$
$$\Rightarrow \underline{\underline{y^2 = x^2(4\ln|x| + 2c)}},$$

or any equivalent expression that begins '$y^2 =$'. (Note that there is nothing specified in the question about the possible values of x and so we must have the modulus signs in our answer.)

7. Find the solution of the differential equation (9)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = 2x$$

for which $y = 2$ when $x = \frac{\pi}{6}$. Give your answer in the form $y = \mathrm{f}(x)$.

Solution

$$\mathrm{IF} = \mathrm{e}^{\int \cot x\,\mathrm{d}x}\mathrm{e}^{\ln\sin x} = \sin x$$

and hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}\sin x + y\cos x = 2x\sin x$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\sin x) = 2x\sin x$$
$$\Rightarrow y\sin x = \int 2x\sin x\,\mathrm{d}x.$$

This requires integration by parts:

$$u = 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x.$$

Hence

$$y\sin x = \int 2x\sin x\,\mathrm{d}x \Rightarrow y\sin x = -2x\cos x + \int 2\cos x\,\mathrm{d}x$$
$$\Rightarrow y\sin x = -2x\cos x + 2\sin x + c$$
$$\Rightarrow y = -2x\cot x + 2 + c\,\mathrm{cosec}\,x.$$

Finally,

$$x = \tfrac{\pi}{6}, y = 2 \Rightarrow 2 = -2 \times \tfrac{\pi}{6} \times \sqrt{3} + 2 + 2c \Rightarrow c = \tfrac{\pi\sqrt{3}}{6}$$

and so

$$\underline{\underline{y = -2x\cot x + 2 + \tfrac{\pi\sqrt{3}}{6}\,\mathrm{cosec}\,x}},$$

or equivalent.

8. Solve the differential equation (8)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = x^4\mathrm{e}^{2x}$$

for y in terms of x, given that $y = 0$ when $x = 1$.

Solution

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = x^4\mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{3}{x}y = x^3\mathrm{e}^{2x}$$

and so

$$\mathrm{IF} = \mathrm{e}^{\int -\frac{3}{x}\,\mathrm{d}x} = \mathrm{e}^{-3\ln x} = \mathrm{e}^{\ln x^{-3}} = x^{-3}.$$

So

$$\frac{dy}{dx} - \frac{3}{x}y = x^3e^{2x} \Rightarrow x^{-3}\frac{dy}{dx} - \frac{3}{x^4}y = e^{2x}$$
$$\Rightarrow \frac{d}{dx}\left(x^{-3}y\right) = e^{2x}$$
$$\Rightarrow x^{-3}y = \int e^{2x}\,dx$$
$$\Rightarrow x^{-3}y = \tfrac{1}{2}e^{2x} + c$$
$$\Rightarrow y = x^3(\tfrac{1}{2}e^{2x} + c).$$

Now

$$x = 0, y = 1 \Rightarrow 1 = \tfrac{1}{2} + c \Rightarrow c = \tfrac{1}{2}$$

and so

$$\underline{\underline{y = \tfrac{1}{2}x^3(e^{2x} + 1).}}$$

9. The differential equation (8)

$$3xy^2\frac{dy}{dx} + 2y^3 = \frac{\cos x}{x}$$

is to be solved for $x > 0$. Use the substitution $u = y^3$ to find the general solution for y in terms of x.

Solution

$$\frac{du}{dx} = \frac{d}{dx}(y^3) = \frac{d}{dy}(y^3) \times \frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

and hence

$$3xy^2\frac{dy}{dx} + 2y^3 = \frac{\cos x}{x} \Rightarrow x\frac{du}{dx} + 2u = \frac{\cos x}{x}$$
$$\Rightarrow \frac{du}{dx} + \frac{2u}{x} = \frac{\cos x}{x^2}.$$

Now

$$\text{IF} = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$$

and so

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{2u}{x} = \frac{\cos x}{x^2} &\Rightarrow x^2\frac{\mathrm{d}u}{\mathrm{d}x} + 2xu = \cos x\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(x^2u) = \cos x\\ &\Rightarrow x^2u = \sin x + c\\ &\Rightarrow u = \frac{\sin x + c}{x^2}\\ &\Rightarrow y^3 = \frac{\sin x + c}{x^2}\\ &\Rightarrow \underline{\underline{y = \sqrt[3]{\frac{\sin x + c}{x^2}}}}.\end{aligned}$$

10. (a) By using an integrating factor, find the general solution of the differential equation (7)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4y}{2x+1} = 4(2x+1)^5.$$

Solution

$$\text{IF} = \mathrm{e}^{\int \frac{4}{2x+1}\,\mathrm{d}x} = \mathrm{e}^{2\ln(2x+1)} = (2x+1)^2$$

and so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4y}{2x+1} = 4(2x+1)^5 &\Rightarrow (2x+1)^2\frac{\mathrm{d}y}{\mathrm{d}x} + 4(2x+1)y = 4(2x+1)^7\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(y(2x+1)^2\right) = 4(2x+1)^7\\ &\Rightarrow y(2x+1)^2 = \int 4(2x+1)^7\,\mathrm{d}x\\ &\Rightarrow y(2x+1)^2 = \tfrac{1}{4}(2x+1)^8 + c\\ &\Rightarrow \underline{\underline{y = \tfrac{1}{4}(2x+1)^6 + c(2x+1)^{-2}}},\end{aligned}$$

or equivalent.

(b) The gradient of a curve at any point (x, y) on the curve is given by the differential equation (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4y}{2x+1} = 4(2x+1)^5.$$

The point whose x-coordinate is zero is a stationary point of the curve. Using your answer to part (a), find the equation of the curve.

Solution
Using the differential equation,

$$x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow y = 1.$$

Now, using part (a),

$$x = 0, y = 1 \Rightarrow 1 = \tfrac{1}{4} + c \Rightarrow c = \tfrac{3}{4}$$

and hence

$$\underline{\underline{y = \tfrac{1}{4}(2x+1)^6 + \tfrac{3}{4}(2x+1)^{-2}}}.$$

11. (a) Differentiate $\ln(\ln x)$ with respect to x. (1)

Solution
You may find it easiest to see the answer by writing out a full chain rule statement. Let $y = \ln u$ where $u = \ln x$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(\ln x)) = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{u} \times \frac{1}{x} = \underline{\underline{\frac{1}{x \ln x}}}.$$

(b) (i) Show that $\ln x$ is an integrating factor for the first-order differential equation (2)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x \ln x} y = 9x^2,\ x > 1.$$

Solution
By part (a),

$$\text{IF} = \mathrm{e}^{\int \frac{1}{x \ln x}\,\mathrm{d}x} = \mathrm{e}^{\ln(\ln x)} = \underline{\underline{\ln x}},$$

as required.

(ii) Hence find the solution of this differential equation, given that $y = 4\mathrm{e}^3$ when $x = \mathrm{e}$. (6)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x \ln x} y = 9x^2 \Rightarrow \ln x \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x} y = 9x^2 \ln x$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y \ln x) = 9x^2 \ln x$$
$$\Rightarrow y \ln x = \int 9x^2 \ln x \, \mathrm{d}x.$$

This requires integration by parts:

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 9x^2 \Rightarrow v = 3x^3.$$

Hence

$$y \ln x = \int 9x^2 \ln x \, \mathrm{d}x \Rightarrow y \ln x = 3x^3 \ln x - \int 3x^2 \, \mathrm{d}x$$
$$\Rightarrow y \ln x = 3x^3 \ln x - x^3 + c.$$

Now,

$$x = \mathrm{e}, y = 4\mathrm{e}^3 \Rightarrow 4\mathrm{e}^3 = 3\mathrm{e}^3 - \mathrm{e}^3 + c \Rightarrow c = 2\mathrm{e}^3$$

and so

$$y \ln x = 3x^3 \ln x - x^3 + 4\mathrm{e}^3 \Rightarrow \underline{\underline{y = \frac{3x^3 \ln x - x^3 + 2\mathrm{e}^3}{\ln x}}},$$

or equivalent.

12. By using an integrating factor, find the general solution of the differential equation (6)

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2x}{x^2 + 4} u = 3(x^2 + 4),$$

giving your answer in the form $u = \mathrm{f}(x)$.

Solution

$$\mathrm{IF} = \mathrm{e}^{\int -\frac{2x}{x^2+4} \, \mathrm{d}x} = \mathrm{e}^{\ln -(x^2+4)} = (x^2 + 4)^{-1}.$$

Hence

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2x}{x^2+4}u = 3(x^2+4) \Rightarrow \frac{1}{x^2+4}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2xu}{(x^2+4)^2} = 3$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{x^2+4}\right) = 3$$
$$\Rightarrow \frac{u}{x^2+4} = \int 3\,\mathrm{d}x$$
$$\Rightarrow \frac{u}{x^2+4} = 3x + c$$
$$\Rightarrow \underline{\underline{u = (3x+c)(x^2+4)}},$$

or equivalent.

13. (a) Find the particular values of the constants a, b, and c for which $a + b\sin 2x + c\cos 2x$ satisfies the differential equation (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 20 - 20\cos 2x.$$

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(a + b\sin 2x + c\cos 2x) = 2b\cos 2x - 2c\sin 2x$$

and so, substituting into the differential equation gives

$$(2b+4c)\cos 2x + (4b-2c)\sin 2x + 4a \equiv 20 - 20\cos 2x.$$

Hence $\underline{\underline{a = 5,\ b = -2,\ \text{and } c = -4}}$.

(b) Hence find the solution of this differential equation, given that $y = 4$ when $x = 0$. (4)

Solution

The 'complementary function' is $y = A\mathrm{e}^{-4x}$ and so the general solution is

$$y = A\mathrm{e}^{-4x} + 5 - 2\sin 2x - 4\cos 2x.$$

Since $y = 4$ when $x = 0$ we have $A = 3$ and hence

$$\underline{\underline{y = 3\mathrm{e}^{-4x} + 5 - 2\sin 2x - 4\cos 2x}}.$$

Note that you can solve this problem using the integrating factor approach: the IF is clearly e^{4x} and so

$$\begin{aligned}\frac{dy}{dx} + 4y = 20 - 20\cos 2x \Rightarrow e^{4x}\frac{dy}{dx} + 4e^{4x}y &= 20e^{4x} - 20e^{4x}\cos 2x\\ \Rightarrow \frac{d}{dx}\left(ye^{4x}\right) &= 20e^{4x} - 20e^{4x}\cos 2x\\ \Rightarrow ye^{4x} &= \int \left(20e^{4x} - 20e^{4x}\cos 2x\right)\, dx\\ \Rightarrow ye^{4x} &= 5e^{4x} + c - 20\int e^{4x}\cos 2x\, dx.\end{aligned}$$

You should be able to recognise that this integral is a classic case of having to do integration by parts twice. Let $I = \int e^{4x}\cos 2x\, dx$. Then

$$u = e^{4x} \Rightarrow \frac{du}{dx} = 4e^{4x} \text{ and } \frac{dv}{dx} = \cos 2x \Rightarrow v = \tfrac{1}{2}\sin 2x.$$

So

$$I = \tfrac{1}{2}e^{4x}\sin 2x - 2\int e^{4x}\sin 2x\, dx.$$

Now for the second round of integration by parts:

$$u = e^{4x} \Rightarrow \frac{du}{dx} = 4e^{4x} \text{ and } \frac{dv}{dx} = \sin 2x \Rightarrow v = -\tfrac{1}{2}\cos 2x$$

and so

$$\begin{aligned}I &= \tfrac{1}{2}e^{4x}\sin 2x - 2\left\{-\tfrac{1}{2}e^{4x}\cos 2x + 2\int e^{4x}\cos 2x\, dx\right\}\\ &= \tfrac{1}{2}e^{4x}\sin 2x + e^{4x}\cos 2x - 4I,\end{aligned}$$

and hence

$$I = \tfrac{1}{10}e^{4x}\sin 2x + \tfrac{1}{5}e^{4x}\cos 2x.$$

So

$$\begin{aligned} & ye^{4x} = 5e^{4x} + c - 20\left(\tfrac{1}{10}e^{4x}\sin 2x + \tfrac{1}{5}e^{4x}\cos 2x\right)\\ \Rightarrow\quad & ye^{4x} = 5e^{4x} + c - 2e^{4x}\sin 2x - 4e^{4x}\cos 2x\\ \Rightarrow\quad & y = 5 + ce^{-4x} - 2\sin 2x - 4\cos 2x,\end{aligned}$$

and we can now find the value of c as before.

14. Find the general solution of the differential equation (7)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\cot 2x = \sin x,\ 0 < x < \tfrac{\pi}{2},$$

giving your answer in the form $y = \mathrm{f}(x)$.

Solution

$$\mathrm{IF} = \mathrm{e}^{\int 2\cot 2x\,\mathrm{d}x} = \mathrm{e}^{\ln\sin 2x} = \sin 2x.$$

Hence

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\cot 2x = \sin x &\Rightarrow \sin 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\cos 2x = \sin 2x\sin x\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\sin 2x) = 2\sin^2 x\cos x\\ &\Rightarrow y\sin 2x = \tfrac{2}{3}\sin^3 x + c\\ &\Rightarrow \underline{\underline{y = \frac{2\sin^3 x + 3c}{3\sin 2x}}}.\end{aligned}$$

15. Find the general solution of the differential equation (7)

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \frac{1}{x},$$

giving your answer in the form $y = \mathrm{f}(x)$.

Solution

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \frac{1}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x+1}y = \frac{1}{x(x+1)}.$$

Now,

$$\mathrm{IF} = \mathrm{e}^{\int \frac{2}{x+1}\,\mathrm{d}x} = \mathrm{e}^{2\ln(x+1)} = \mathrm{e}^{\ln(x+1)^2} = (x+1)^2$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x+1}y = \frac{1}{x(x+1)} \Rightarrow (x+1)^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2(x+1)y = \frac{x+1}{x}$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[y(x+1)^2\right] = \frac{x+1}{x}$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[y(x+1)^2\right] = 1 + \frac{1}{x}$$
$$\Rightarrow y(x+1)^2 = x + \ln|x| + c$$
$$\Rightarrow \underline{\underline{y = \frac{x + \ln|x| + c}{(x+1)^2}}}.$$

16. During an industrial process, the mass of salt, S kg, dissolved in a liquid t minutes after the process begins is modelled by the differential equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{2S}{120-t} = \tfrac{1}{4},\ 0 \leqslant t < 120.$$

Given that $S = 6$ when $t = 0$,

(a) find S in terms of t, (8)

Solution

$$\text{IF} = \mathrm{e}^{\int \frac{2}{120-t}\,\mathrm{d}x} = \mathrm{e}^{-2\ln(120-t)} = \mathrm{e}^{\ln(120-t)^{-2}} = (120-t)^{-2}$$

and

$$\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{2S}{120-t} = \tfrac{1}{4} \Rightarrow \frac{1}{(120-t)^2}\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{2S}{(120-t)^3} = \frac{1}{4(120-t)^2}$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{S}{(120-t)^2}\right] = \frac{1}{4(120-t)^2}$$
$$\Rightarrow \frac{S}{(120-t)^2} = \frac{1}{4(120-t)} + c$$
$$\Rightarrow S = \tfrac{1}{4}(120-t) + c(120-t)^2.$$

Now,

$$S = 6, t = 0 \Rightarrow 6 = 30 + 14400c \Rightarrow c = -\tfrac{1}{600}$$

and

$$\underline{\underline{S = \tfrac{1}{4}(120-t) - \tfrac{1}{600}(120-t)^2}}.$$

(b) calculate the maximum mass of salt that the model predicts will be dissolved in the liquid at any one time during the process. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} = 0 &\Rightarrow -\tfrac{1}{4} + \tfrac{1}{300}(120 - t) = 0 \\ &\Rightarrow \tfrac{1}{300}(120 - t) = \tfrac{1}{4} \\ &\Rightarrow 120 - t = 75 \\ &\Rightarrow t = 45 \\ &\Rightarrow \underline{\underline{S = \tfrac{75}{8}}}.\end{aligned}$$

17. Obtain the general solution of the differential equation (8)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cos x,\ x > 0,$$

giving your answer in the form $y = \mathrm{f}(x)$.

Solution

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cos x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = \frac{\cos x}{x}$$

and

$$\mathrm{IF} = \mathrm{e}^{\int \frac{2}{x}\,\mathrm{d}x} = \mathrm{e}^{2\ln x} = x^2.$$

Hence

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = \frac{\cos x}{x} &\Rightarrow x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x\cos x \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(x^2 y) = x\cos x \\ &\Rightarrow x^2 y = \int x\cos x\,\mathrm{d}x.\end{aligned}$$

Now,

$$u = x,\ \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1,\ v = \sin x$$

and

$$x^2y = \int x\cos x\,\mathrm{d}x \Rightarrow x^2y = x\sin x - \int \sin x\,\mathrm{d}x$$
$$\Rightarrow x^2y = x\sin x + \cos x + c$$
$$\Rightarrow y = \frac{x\sin x + \cos x + c}{x^2}.$$

18. (7)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = 2\sec^3 x.$$

Giving that $y = 3$ and $x = 0$, find y in terms of x.

Solution

$$\mathrm{IF} = \mathrm{e}^{-\int \tan x\,\mathrm{d}x} = \mathrm{e}^{\ln\cos x} = \cos x.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = 2\sec^3 x \Rightarrow \cos x\frac{\mathrm{d}y}{\mathrm{d}x} - y\sin x = 2\sec^2 x$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\cos x) = 2\sec^2 x$$
$$\Rightarrow y\cos x = 2\tan x + c.$$

Now,

$$x = 0, y = 3 \Rightarrow 3 = 0 + c \Rightarrow c = 3$$

and

$$y\cos x = 2\tan x + 3 \Rightarrow y = \frac{2\tan x + 3}{\cos x}.$$

19. Solve the differential equation (7)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = x,$$

to obtain y as a function of x.

Solution

$$\text{IF} = e^{-\int 3\,dx} = e^{-3x}$$

and

$$\begin{aligned}\frac{dy}{dx} - 3y = x &\Rightarrow e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = xe^{-3x}\\ &\Rightarrow \frac{d}{dx}(e^{-3x}y) = xe^{-3x}\\ &\Rightarrow e^{-3x}y = \int xe^{-3x}\,dx.\end{aligned}$$

Now,

$$u = x,\ \frac{dv}{dx} = e^{-3x} \Rightarrow \frac{du}{dx} = 1,\ v = -\tfrac{1}{3}e^{-3x}$$

and

$$\begin{aligned}e^{-3x}y = \int xe^{-3x}\,dx &\Rightarrow e^{-3x}y = -\tfrac{1}{3}xe^{-3x} + \tfrac{1}{3}\int e^{-3x}\,dx\\ &\Rightarrow e^{-3x}y = -\tfrac{1}{3}xe^{-3x} - \tfrac{1}{9}e^{-3x} + c\\ &\Rightarrow \underline{\underline{y = -\tfrac{1}{3}x - \tfrac{1}{9} + ce^{3x}}}.\end{aligned}$$

20. (a) Show that the substitution $y = vx$ transforms the differential equation (3)

$$\frac{dy}{dx} = \frac{x}{y} + \frac{3y}{x},\ x > 0,\ y > 0 \quad (\dagger)$$

into the differential equation

$$x\frac{dv}{dx} = 2v + \frac{1}{v} \quad (\ddagger).$$

Solution

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

and

$$\frac{dy}{dx} = \frac{x}{y} + \frac{3y}{x} \Rightarrow v + x\frac{dv}{dx} = \frac{x}{xv} + \frac{3vx}{x}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{1}{v} + 3v$$
$$\Rightarrow x\frac{dv}{dx} = 2v + \frac{1}{v}.$$

(b) By solving differential equation (‡), find a general solution of the differential equation (†) in the form $y = f(x)$. (7)

Solution

$$\text{IF} = e^{-2\int \frac{1}{x}\,dx} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

and

$$x\frac{dv}{dx} = 2v + \frac{1}{v} \Rightarrow x\frac{dv}{dx} = \frac{2v^2+1}{v}$$
$$\Rightarrow \frac{v}{2v^2+1}\,dv = \frac{1}{x}\,dx$$
$$\Rightarrow \int \frac{v}{2v^2+1}\,dv = \int \frac{1}{x}\,dx$$
$$\Rightarrow \tfrac{1}{4}\ln(2v^2+1) = \ln x + c$$
$$\Rightarrow \ln(2v^2+1)^{\frac{1}{4}} - \ln x = c$$
$$\Rightarrow \ln\left[\frac{(2v^2+1)^{\frac{1}{4}}}{x}\right] = c$$
$$\Rightarrow \frac{(2v^2+1)^{\frac{1}{4}}}{x} = e^c$$
$$\Rightarrow (2v^2+1)^{\frac{1}{4}} = xe^c$$
$$\Rightarrow 2v^2+1 = Ax^4$$

where $A = \mathrm{e}^{4c}$

$$\begin{aligned}
&\Rightarrow 2v^2 = Ax^4 - 1 \\
&\Rightarrow v^2 = \frac{Ax^4 - 1}{2} \\
&\Rightarrow v = \sqrt{\frac{Ax^4 - 1}{2}} \\
&\Rightarrow \frac{y}{x} = \sqrt{\frac{Ax^4 - 1}{2}} \\
&\Rightarrow \underline{\underline{y = x\sqrt{\frac{Ax^4 - 1}{2}}}}.
\end{aligned}$$

Given that $y = 3$ and $x = 1$,

(c) find the particular solution of differential equation (†). (2)

Solution

Now,

$$x = 1, y = 3 \Rightarrow 3 = \sqrt{\frac{A - 1}{2}} \Rightarrow A = 19$$

and

$$\underline{\underline{y = x\sqrt{\frac{19x^4 - 1}{2}}}}.$$

21. (a) Show that the substitution $y = \frac{1}{t}$ transforms the differential equation (4)

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} + y \cos x = y^2, \; 0 < x < \pi \quad (\dagger)$$

into the differential equation

$$\frac{\mathrm{d}t}{\mathrm{d}x} - t \cot x = -\operatorname{cosec} x, \; 0 < x < \pi \quad (\ddagger).$$

Solution

$$y = \frac{1}{t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{t^2}\,\frac{\mathrm{d}t}{\mathrm{d}x}$$

and

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = y^2 \Rightarrow -\frac{1}{t^2}\sin x\frac{\mathrm{d}t}{\mathrm{d}x} + \frac{1}{t}\cos x = \frac{1}{t^2}$$
$$\Rightarrow \sin x\frac{\mathrm{d}t}{\mathrm{d}x} - t\cos x = -1$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}t}{\mathrm{d}x} - t\cot x = -\operatorname{cosec} x}}.$$

(b) Solve the differential equation (‡). (5)

Solution

$$\text{IF} = \mathrm{e}^{-\int \cot x\,\mathrm{d}x} = \mathrm{e}^{-\ln\sin x} = \mathrm{e}^{\ln(\sin x)^{-1}} = \operatorname{cosec} x$$

and so

$$\frac{\mathrm{d}t}{\mathrm{d}x} - t\cot x = -\operatorname{cosec} x \Rightarrow \operatorname{cosec} x\frac{\mathrm{d}t}{\mathrm{d}x} - t\operatorname{cosec} x\cot x = -\operatorname{cosec}^2 x$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(t\operatorname{cosec} x) = -\operatorname{cosec}^2 x$$
$$\Rightarrow t\operatorname{cosec} x = \cot x + c$$
$$\Rightarrow \underline{\underline{t = \sin x(\cot x + c)}}.$$

(c) Hence show that (2)

$$y = \frac{1}{\cos x + c\sin x},$$

where c is an arbitrary constant, is a general solution of the differential equation (†).

Solution

$$t = \sin x(\cot x + c) \Rightarrow \frac{1}{y} = \sin x(\cot x + c)$$
$$\Rightarrow y = \frac{1}{\sin x(\cot x + c)}$$
$$\Rightarrow \underline{\underline{y = \frac{1}{\cos x + c\sin x}}}.$$

Given that $y = \frac{\sqrt{2}}{3}$ at $x = \frac{\pi}{4}$,

(d) find the value of y at $x = \frac{\pi}{2}$. (3)

Solution

$$\begin{aligned} x = \tfrac{\pi}{4}, y = \tfrac{\sqrt{2}}{3} \Rightarrow \tfrac{\sqrt{2}}{3} &= \frac{1}{\frac{1}{\sqrt{2}} + \frac{c}{\sqrt{2}}} \\ \Rightarrow \frac{\sqrt{2}}{3} &= \frac{\sqrt{2}}{1+c} \\ \Rightarrow c &= 2, \end{aligned}$$

and we have

$$y = \frac{1}{\cos\frac{\pi}{2} + 2\sin\frac{\pi}{2}} = \underline{\underline{\tfrac{1}{2}}}.$$

22. (a) Find, in the form $y = \mathrm{f}(x)$, the general solution of the equation (6)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = \sin x,\ 0 < x < \pi.$$

Solution

$$\text{IF} = \mathrm{e}^{\int \cot x\,\mathrm{d}x} = \mathrm{e}^{\ln\sin x} = \sin x$$

and so

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = \sin x &\Rightarrow \sin x\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = \sin^2 x \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\sin x) = \sin^2 x \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\sin x) = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x \\ &\Rightarrow y\sin x = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c \\ &\Rightarrow \underline{\underline{y = \frac{\frac{1}{2}x - \frac{1}{4}\sin 2x + c}{\sin x}}}. \end{aligned}$$

Given that $y = 1$ at $x = \frac{\pi}{2}$,

(b) show that, at $x = \frac{\pi}{4}$, (4)

$$y = \frac{(6-\pi)\sqrt{2}}{8}.$$

Solution

$$x = \tfrac{\pi}{2}, y = 1 \Rightarrow 1 = \frac{\frac{\pi}{4} - \frac{1}{4}\sin\pi + c}{\sin\frac{\pi}{2}} \Rightarrow c = 1 - \tfrac{\pi}{4}$$

and

$$\begin{aligned} y &= \frac{\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} + 1 - \frac{\pi}{4}}{\sin\frac{\pi}{4}} \\ &= \frac{\frac{3}{4} - \frac{\pi}{8}}{\frac{1}{\sqrt{2}}} \\ &= \frac{\frac{1}{8}(6-\pi)}{\frac{1}{\sqrt{2}}} \\ &= \underline{\underline{\frac{(6-\pi)\sqrt{2}}{8}}}. \end{aligned}$$

23. (a) Show that the substitution $z = \dfrac{1}{y^2}$ transforms the differential equation (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = 4xy^3,\ y > 0 \quad (\dagger)$$

into the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} - 2z = -8x \quad (\ddagger).$$

Solution

$$z = \frac{1}{y^2} \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{2}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x}$$

and we have

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + y = 4xy^3 &\Rightarrow -\frac{y^3}{2}\frac{\mathrm{d}z}{\mathrm{d}x} + y = 4xy^3 \\ &\Rightarrow y^2\frac{\mathrm{d}z}{\mathrm{d}x} - 2 = -8xy^2 \\ &\Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x} - \frac{2}{y^2} = -8x \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}z}{\mathrm{d}x} - 2z = -8x}}. \end{aligned}$$

(b) Hence find the solution of the differential equation (†) in the form $y = \mathrm{f}(x)$. (7)

Solution

$$\text{IF} = \mathrm{e}^{-\int 2\,\mathrm{d}x} = \mathrm{e}^{-2x}$$

and so

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} - 2z = -8x &\Rightarrow \mathrm{e}^{-2x}\frac{\mathrm{d}z}{\mathrm{d}x} - 2\mathrm{e}^{-2x}z = -8x\mathrm{e}^{-2x} \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(z\mathrm{e}^{-2x}) = -8x\mathrm{e}^{-2x} \\ &\Rightarrow z\mathrm{e}^{-2x} = -\int 8x\mathrm{e}^{-2x}\,\mathrm{d}x.\end{aligned}$$

Now,

$$u = -8x, \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-2x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -8, v = -\tfrac{1}{2}\mathrm{e}^{-2x}$$

and

$$\begin{aligned}z\mathrm{e}^{-2x} = -\int 8x\mathrm{e}^{-2x}\,\mathrm{d}x &\Rightarrow z\mathrm{e}^{-2x} = 4x\mathrm{e}^{-2x} - \int 4\mathrm{e}^{-2x}\,\mathrm{d}x \\ &\Rightarrow z\mathrm{e}^{-2x} = 4x\mathrm{e}^{-2x} + 2\mathrm{e}^{-2x} + c \\ &\Rightarrow z = 4x + 2 + c\mathrm{e}^{2x} \\ &\Rightarrow \frac{1}{y^2} = 4x + 2 + c\mathrm{e}^{2x} \\ &\Rightarrow y^2 = \frac{1}{4x + 2\mathrm{e}^{-2x} + c\mathrm{e}^{2x}} \\ &\Rightarrow \underline{\underline{y = \sqrt{\frac{1}{-4x - 2 + c\mathrm{e}^{2x}}}}}.\end{aligned}$$

The stationary point of the graph of a particular solution of the differential equation (†) is (x_1, y_1), $x_1 > 0$.

(c) Show that $y_1 = \dfrac{1}{2\sqrt{x_1}}$. (2)

Solution

So, we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow y_1 = 4x_1y_1^3 \\ &\Rightarrow 1 = 4x_1y_1^2 \\ &\Rightarrow y_1^2 = \frac{1}{4x_1} \\ &\Rightarrow \underline{\underline{y_1 = \frac{1}{2\sqrt{x_1}}}}.\end{aligned}$$

24. Find the general solution of the differential equation (7)

$$\sin\frac{\mathrm{d}y}{\mathrm{d}x} - y\cos x = \sin 2x \sin x,$$

giving your answer in the form $y = \mathrm{f}(x)$.

Solution

$$\sin\frac{\mathrm{d}y}{\mathrm{d}x} - y\cos x = \sin 2x\sin x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} - y\cot x = \sin^2$$

and

$$\mathrm{IF} = \mathrm{e}^{-\int \cot x\,\mathrm{d}x} = \mathrm{e}^{-\ln\sin x} = \mathrm{e}^{\ln(\sin x)^{-1}} = \frac{1}{\sin x}.$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} - y\cot x = \sin^2 x &\Rightarrow \frac{1}{\sin x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\cos x}{\sin^2 x}y = \frac{\sin^2 x}{\sin x} \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y}{\sin x}\right] = \frac{2\sin x\cos x}{\sin x} \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y}{\sin x}\right] = 2\cos x \\ &\Rightarrow \frac{y}{\sin x} = 2\sin x + c \\ &\Rightarrow \underline{\underline{y = 2\sin^2 x + c\sin x}}.\end{aligned}$$

25. (a) Show that the substitution $z = y^{\frac{1}{2}}$ transforms the differential equation (5)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 4y\tan x = 2y^{\frac{1}{2}} \quad (\dagger)$$

into the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} - 2z\tan x = 1 \quad (\ddagger).$$

Solution

$$z = y^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x} = \tfrac{1}{2}y^{-\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}$$

and so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} - 4y\tan x = 2y^{\frac{1}{2}} &\Rightarrow \tfrac{1}{2}y^{-\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}x} - 2y^{\frac{1}{2}}\tan x = 1\\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}z}{\mathrm{d}x} - 2z\tan x = 1.}}\end{aligned}$$

(b) Solve the differential equation (‡) to find z as a function of x. (6)

Solution

$$\mathrm{IF} = \mathrm{e}^{-2\int \tan x\,\mathrm{d}x} = \mathrm{e}^{-2\ln\sec x} = \mathrm{e}^{\ln(\sec x)^{-2}} = \cos^2 x$$

and

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} - 2z\tan x = 1 &\Rightarrow \cos^2 x\frac{\mathrm{d}z}{\mathrm{d}x} - 2z\sin x\cos x = \cos^2 x\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(z\cos^2 x) = \cos^2 x\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(z\cos^2 x) = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x\\ &\Rightarrow z\cos^2 x = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c\\ &\Rightarrow \underline{\underline{z = \frac{\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c}{\cos^2 x}}}.\end{aligned}$$

(c) Hence obtain the general solution of the differential equation (†). (1)

Solution

$$\begin{aligned}z = y^{\frac{1}{2}} &\Rightarrow y^{\frac{1}{2}} = \frac{\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c}{\cos^2 x}\\ &\Rightarrow \underline{\underline{y = \left(\frac{\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c}{\cos^2 x}\right)^2}}.\end{aligned}$$

26. Find the general solution of the differential equation (8)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = \frac{\ln x}{x},\ x > 0,$$

giving your answer in the form $y = \mathrm{f}(x)$.

Solution

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = \frac{\ln x}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{5}{x}y = \frac{\ln x}{x^2}.$$

Now,

$$\mathrm{IF} = \mathrm{e}^{\int \frac{5}{x}\,\mathrm{d}x} = \mathrm{e}^{5\ln x} = \mathrm{e}^{\ln x^5} = x^5$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{5}{x}y = \frac{\ln x}{x} &\Rightarrow x^5\frac{\mathrm{d}y}{\mathrm{d}x} + 5x^4y = x^3\ln x \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(x^5y) = x^3\ln x \\ &\Rightarrow x^5y = \int x^3\ln x\,\mathrm{d}x.\end{aligned}$$

Now,

$$u = \ln x, \frac{\mathrm{d}v}{\mathrm{d}x} = x^3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}, v = \tfrac{1}{4}x^4$$

and we have

$$\begin{aligned}x^5y = \int x^3\ln x\,\mathrm{d}x &\Rightarrow x^5y = \tfrac{1}{4}x^4\ln x - \int \tfrac{1}{4}x^3\,\mathrm{d}x \\ &\Rightarrow x^5y = \tfrac{1}{4}x^4\ln x - \tfrac{1}{16}x^4 + c \\ &\Rightarrow y = \frac{\tfrac{1}{4}x^4\ln x - \tfrac{1}{16}x^4 + c}{x^5} \\ &\Rightarrow \underline{\underline{y = \frac{\ln x}{4x} - \frac{1}{16x} + \frac{c}{x^5}}}.\end{aligned}$$

27. (a) Show that the substitution $y = vx$ transforms the differential equation (3)

$$3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + y^3 \quad (\dagger)$$

into the differential equation

$$3v^2x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 - 2v^3 \quad (\ddagger).$$

Solution

$$y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

and we have

$$\begin{aligned} 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + y^3 &\Rightarrow 3x(vx)^2\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = x^3 + (vx)^3 \\ &\Rightarrow 3v^3x^3 + 3v^2x^4\frac{\mathrm{d}v}{\mathrm{d}x} = x^3 + v^3x^3 \\ &\Rightarrow 3v^3 + 3v^2x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v^3 \\ &\Rightarrow \underline{\underline{3v^2x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 - 2v^3}}. \end{aligned}$$

(b) By solving differential equation (‡), find a general solution of the differential equation (†) in the form $y = \mathrm{f}(x)$. (6)

Solution

For this, we need $x > 0$ and $y > 0$:

$$\begin{aligned}
3v^2 x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 - 2v^3 &\Rightarrow \frac{3v^2}{1-2v^3}\,\mathrm{d}v = \frac{1}{x}\,\mathrm{d}x \\
&\Rightarrow \int \frac{3v^2}{1-2v^3}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x \\
&\Rightarrow -\tfrac{1}{2}\ln(1-2v^3) = \ln x + c \\
&\Rightarrow -\ln(1-2v^3) = 2\ln x + 2c \\
&\Rightarrow -\ln(1-2v^3) = \ln x^2 + \ln a \text{ (where } \ln a = 2c) \\
&\Rightarrow -\ln(1-2v^3) = \ln(ax^2) \\
&\Rightarrow \frac{1}{1-2v^3} = ax^2 \\
&\Rightarrow 1 - 2v^3 = \frac{1}{ax^2} \\
&\Rightarrow 2v^3 = 1 - \frac{1}{ax^2} \\
&\Rightarrow v^3 = \frac{ax^2-1}{2ax^2} \\
&\Rightarrow \frac{y}{x} = \sqrt[3]{\frac{ax^2-1}{2ax^2}} \\
&\Rightarrow \underline{\underline{y = x\sqrt[3]{\frac{ax^2-1}{2ax^2}}}}.
\end{aligned}$$

Given that $y = 2$ at $x = 1$,

(c) find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1$. (2)

Solution

$$x = 1, y = 2 \Rightarrow 3 \times 1 \times 2^2 \times \frac{\mathrm{d}y}{\mathrm{d}x} = 1^3 + 2^3 \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{3}{4}}}.$$

28. (a) Find, in the form $y = \mathrm{f}(x)$, the general solution of the equation (6)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\tan x = \sin 2x, 0 < x < \tfrac{\pi}{2}.$$

Solution

$$\text{IF} = \mathrm{e}^{\int 2\tan x \, \mathrm{d}x} = \mathrm{e}^{2\ln\sec x} = \mathrm{e}^{\ln\sec^2 x} = \sec^2 x$$

and so

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + 2y\tan x = \sin 2x &\Rightarrow \sec^2 x \frac{\mathrm{d}y}{\mathrm{d}x} + 2y\sec^2 x\tan x = \sin 2x \sec^2 x \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(y\sec^2 x\right) = (2\sin x\cos x)\sec^2 x \\ &\Rightarrow y\sec^2 x = \int 2\tan x \, \mathrm{d}x \\ &\Rightarrow y\sec^2 x = 2\ln\sec x + c \\ &\Rightarrow \underline{\underline{y = \cos^2 x(2\ln\sec x + c)}}, \end{aligned}$$

or equivalent.

Given that $y = 2$ at $x = \frac{\pi}{3}$,

(b) find the value of y at $x = \frac{\pi}{6}$, giving your answer in the form $a + k\ln b$ where a and b are integers and k is rational. (4)

Solution

$$x = \tfrac{\pi}{3}, y = 2 \Rightarrow 2 = \tfrac{1}{4}(2\ln 2 + c) \Rightarrow c = 8 - \ln 4.$$

Hence

$$x = \tfrac{\pi}{6} \Rightarrow y = \tfrac{3}{4}\left(2\ln\tfrac{2}{\sqrt{3}} + 8 - \ln 4\right) = \underline{\underline{6 - \tfrac{3}{4}\ln 3}}.$$

29. (a) Find the general solution of the differential equation (5)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x^2.$$

Solution

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = 4x$$

and hence

$$\text{IF} = \mathrm{e}^{\int \frac{2}{x}\,\mathrm{d}x} = \mathrm{e}^{2\ln x} = \mathrm{e}^{\ln x^2} = x^2.$$

So

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = 4x &\Rightarrow x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 4x^3 \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(yx^2) = 4x^3 \\ &\Rightarrow yx^2 = \int 4x^3\,\mathrm{d}x \\ &\Rightarrow yx^2 = x^4 + c \\ &\Rightarrow \underline{\underline{y = \frac{x^4 + c}{x^2}}},\end{aligned}$$

or equivalent.

(b) Find the particular solution for which $y = 5$ at $x = 1$, giving your answer in the form $y = \mathrm{f}(x)$. (2)

Solution

$$x = 1, y = 5 \Rightarrow 5 = \frac{1 + c}{1} \Rightarrow c = 4$$

and hence

$$\underline{\underline{y = \frac{x^4 + 4}{x^2}}}.$$

(c) (i) Find the exact values of the coordinates of the turning points of the curve with equation $y = \mathrm{f}(x)$, making your method clear. (2)

Solution

$$y = x^2 + 4x^{-2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 8x^{-3}.$$

So

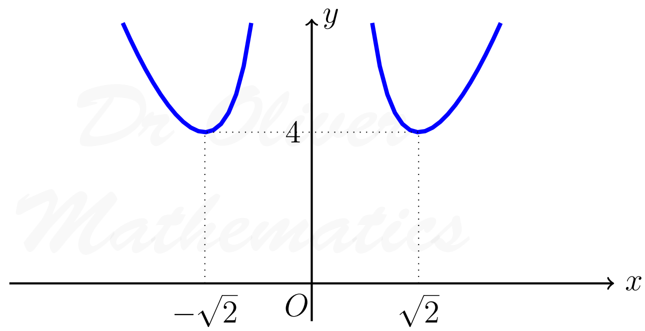
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 2x - 8x^{-3} = 0 \\ &\Rightarrow x^4 = 4 \\ &\Rightarrow x = \pm\sqrt{2}.\end{aligned}$$

So the coordinates of the stationary points are

$$\underline{\underline{(\sqrt{2}, 4)}} \text{ and } \underline{\underline{(-\sqrt{2}, 4)}}.$$

(ii) Sketch the curve with equation $y = \mathrm{f}(x)$, showing the coordinates of the turning points. (3)

Solution



30. (a) Show that the substitution $v = y^{-3}$ transforms the differential equation (5)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2x^4y^4 \qquad (\dagger)$$

into the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{3v}{x} = -6x^3. \qquad (\ddagger)$$

Solution

$$v = y^{-3} \Rightarrow y = v^{-\frac{1}{3}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{1}{3}v^{-\frac{4}{3}}\frac{\mathrm{d}v}{\mathrm{d}x}.$$

So,

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2x^4y^4 &\Rightarrow x\left(-\tfrac{1}{3}v^{-\frac{4}{3}}\frac{\mathrm{d}v}{\mathrm{d}x}\right) + v^{-\frac{1}{3}} = 2x^4v^{-\frac{4}{3}} \\ &\Rightarrow xv^{-\frac{4}{3}}\frac{\mathrm{d}v}{\mathrm{d}x} - 3v^{-\frac{1}{3}} = -6x^4v^{-\frac{4}{3}} \\ &\Rightarrow v^{-\frac{4}{3}}\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{3}{x}v^{-\frac{1}{3}} = -6x^3v^{-\frac{4}{3}} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{3v}{x} = -6x^3}}, \end{aligned}$$

as required.

(b) By solving the differential equation ($\ddagger$), find a general solution of differential equation ($\dagger$) in the form $y^3 = \mathrm{f}(x)$. (6)

Solution

$$\text{IF} = e^{\int -\frac{3}{x}\,dx} = e^{-3\ln x} = x^{-3}$$

and so

$$\begin{aligned}\frac{dv}{dx} - \frac{3v}{x} = -6x^3 &\Rightarrow x^{-3}\frac{dv}{dx} - \frac{3v}{x^4} = -6\\ &\Rightarrow \frac{d}{dx}(x^{-3}v) = -6\\ &\Rightarrow x^{-3}v = c - 6x\\ &\Rightarrow v = cx^3 - 6x^4\\ &\Rightarrow y^{-3} = cx^3 - 6x^4\\ &\Rightarrow \underline{\underline{y^3 = \frac{1}{cx^3 - 6x^4}}}.\end{aligned}$$

31. (a) Find the general solution of the differential equation (6)

$$\frac{dy}{dx} + 2y\tan x = e^{4x}\cos^2 x, -\tfrac{\pi}{2} < x < \tfrac{\pi}{2},$$

giving your answer in the form $y = f(x)$.

Solution

$$\text{IF} = e^{\int 2\tan x\,dx} = e^{2\ln(\sec x)} = e^{\ln(\sec^2 x)} = \sec^2 x.$$

Hence

$$\begin{aligned}\frac{dy}{dx} + 2y\tan x = e^{4x}\cos^2 x &\Rightarrow \frac{dy}{dx}\sec^2 x + 2y\tan x\sec^2 x = e^{4x}\\ &\Rightarrow \frac{d}{dx}(y\sec^2 x) = e^{4x}\\ &\Rightarrow y\sec^2 x = \tfrac{1}{4}e^{4x} + c\\ &\Rightarrow \underline{\underline{y = \cos^2 x\left(\tfrac{1}{4}e^{4x} + c\right)}}.\end{aligned}$$

(b) Find the particular solution for which $y = 1$ at $x = 0$. (2)

Solution

$$x = 0, y = 1 \Rightarrow 1 = \tfrac{1}{4} + c \Rightarrow c = \tfrac{3}{4}$$

and so

$$\underline{\underline{y = \tfrac{1}{4}\cos^2 x\left(e^{4x} + 3\right)}}.$$

32. Find, in the form $y = \mathrm{f}(x)$, he general solution of the differential equation (6)

$$\tan x \frac{\mathrm{d}y}{\mathrm{d}x} + y = 3\cos 2x \tan x,\ 0 < x < \tfrac{\pi}{2}.$$

Solution

$$\tan x \frac{\mathrm{d}y}{\mathrm{d}x} + y = 3\cos 2x \tan x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = 3\cos 2x$$

and we have

$$\text{IF} = \mathrm{e}^{\int \cot x\,\mathrm{d}x} = \mathrm{e}^{\ln \sin x} = \sin x.$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = 3\cos 2x &\Rightarrow \sin x\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = 3\cos 2x \sin x\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\sin x) = 3\cos 2x\sin x\\ &\Rightarrow y\sin x = \int 3\cos 2x \sin x\,\mathrm{d}x\\ &\Rightarrow y\sin x = \int 3(2\cos^2 - 1)\sin x\,\mathrm{d}x\\ &\Rightarrow y\sin x = 3(-\tfrac{2}{3}\cos^3 x + \cos x) + c\\ &\Rightarrow y\sin x = -2\cos^3 x + 3\cos x + c\\ &\Rightarrow y = \frac{-2\cos^3 x + 3\cos x + c}{\sin x}.\end{aligned}$$

33.

$$p\frac{\mathrm{d}x}{\mathrm{d}t} + qx = r, \text{ where } p, q, \text{ and } r \text{ are constants.}$$

(a) Given that $x = 0$ when $t = 0$,

(i) find x in terms of t, (4)

Solution

$$p\frac{\mathrm{d}x}{\mathrm{d}t} + qx = r \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{q}{p}x = \frac{r}{p}$$

and

$$\text{IF} = \mathrm{e}^{\int \frac{q}{p}\,\mathrm{d}x} = \mathrm{e}^{\frac{q}{p}t}.$$

Now,

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{q}{p}x = \frac{r}{p} \Rightarrow \mathrm{e}^{\frac{q}{p}t}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{q}{p}\mathrm{e}^{\frac{q}{p}t}x = \frac{r}{p}\mathrm{e}^{\frac{q}{p}t}$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(x\mathrm{e}^{\frac{q}{p}t}) = \tfrac{r}{p}\mathrm{e}^{\frac{q}{p}t}$$
$$\Rightarrow x\mathrm{e}^{\frac{q}{p}t} = \int \tfrac{r}{p}\mathrm{e}^{\frac{q}{p}t}\,\mathrm{d}t$$
$$\Rightarrow x\mathrm{e}^{\frac{q}{p}t} = \tfrac{r}{q}\mathrm{e}^{\frac{q}{p}t} + c$$
$$\Rightarrow x = \tfrac{r}{q} + c\mathrm{e}^{-\frac{q}{p}t}.$$

Also,

$$x = 0, t = 0 \Rightarrow 0 = \tfrac{r}{q} + c \Rightarrow c = -\tfrac{r}{q}$$

and we have

$$\underline{\underline{x = \tfrac{r}{q} - \tfrac{r}{q}\mathrm{e}^{-\frac{q}{p}t}}}.$$

(ii) find the limiting value of x as $t \to \infty$. (1)

Solution

$$t \to \infty, \underline{\underline{x \to \tfrac{r}{q}}}.$$

(b) (7)

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} + 2y = \sin\theta.$$

Given that $y = 0$ when $\theta = 0$, find y in terms of θ.

Solution

$$\mathrm{IF} = \mathrm{e}^{\int 2\,\mathrm{d}\theta} = \mathrm{e}^{2\theta}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} + 2y = \sin\theta \Rightarrow \mathrm{e}^{2\theta}\frac{\mathrm{d}y}{\mathrm{d}\theta} + 2\mathrm{e}^{2\theta}y = \mathrm{e}^{2\theta}\sin\theta$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\theta}(y\mathrm{e}^{2\theta}) = \mathrm{e}^{2\theta}\sin\theta$$
$$\Rightarrow y\mathrm{e}^{2\theta} = \int \mathrm{e}^{2\theta}\sin\theta\,\mathrm{d}\theta.$$

Now, we are going to have to apply integration by parts twice:

$$u = \mathrm{e}^{2\theta}, \frac{\mathrm{d}v}{\mathrm{d}\theta} = \sin\theta \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}\theta} = 2\mathrm{e}^{2\theta}, v = -\cos\theta,$$

$$u = 2e^{2\theta}, \frac{dv}{d\theta} = \cos\theta \Rightarrow \frac{du}{d\theta} = 4e^{2\theta}, v = \sin\theta,$$

and

$$\begin{aligned} ye^{2\theta} = \int e^{2\theta}\sin\theta\,d\theta \Rightarrow ye^{2\theta} &= -e^{2\theta}\cos\theta + \int 2e^{2\theta}\cos\theta\,d\theta \\ \Rightarrow ye^{2\theta} &= -e^{2\theta}\cos\theta + \left[2e^{2\theta}\sin\theta - \int 4e^{2\theta}\sin\theta\,d\theta\right] \\ \Rightarrow ye^{2\theta} &= -e^{2\theta}\cos\theta + 2e^{2\theta}\sin\theta - 4\int e^{2\theta}\sin\theta\,d\theta \\ \Rightarrow ye^{2\theta} &= -e^{2\theta}\cos\theta + 2e^{2\theta}\sin\theta - 4ye^{2\theta} + c \\ \Rightarrow 5ye^{2\theta} &= -e^{2\theta}\cos\theta + 2e^{2\theta}\sin\theta + c. \end{aligned}$$

Now,

$$y = 0, \theta = 0 \Rightarrow 0 = -1 + c \Rightarrow c = 1$$

and

$$\begin{aligned} & 5ye^{2\theta} = -e^{2\theta}\cos\theta + 2e^{2\theta}\sin\theta + 1 \\ \Rightarrow\quad & ye^{2\theta} = \tfrac{1}{5}e^{2\theta}(2\sin\theta - \cos\theta) + \tfrac{1}{5} \\ \Rightarrow\quad & \underline{\underline{y = \tfrac{1}{5}(2\sin\theta - \cos\theta) + \tfrac{1}{5}e^{-2\theta}}}. \end{aligned}$$

34. (a) Find, in the form $y = f(x)$, the general solution of the equation (8)

$$\cos x\frac{dy}{dx} + 2\sin x = 2\cos^3 x\sin x + 1, 0 < x < \tfrac{\pi}{2}.$$

Solution

$$\cos x\frac{dy}{dx} + 2\sin x = 2\cos^3 x\sin x + 1 \Rightarrow \frac{dy}{dx} + 2\tan x = 2\cos^2 x\sin x + \sec x$$

and we have

$$\text{IF} = e^{\int \tan x\,dx} = e^{\ln\sec x} = \sec x.$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2\tan x = 2\cos^2 x \sin x + 1 \Rightarrow \sec x \frac{\mathrm{d}y}{\mathrm{d}x} + 2\sec x \tan x = 2\cos x \sin x + \sec^2 x$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y \sec x) = 2\cos x \sin x + \sec^2 x$$
$$\Rightarrow y \sec x = \int (\sin 2x + \sec^2 x)\,\mathrm{d}x$$
$$\Rightarrow y \sec x = -\tfrac{1}{2}\cos 2x + \tan x + c$$
$$\Rightarrow \underline{\underline{y = \cos x(-\tfrac{1}{2}\cos 2x + \tan x + c)}}.$$

Given that $y = 5\sqrt{2}$ when $x = \frac{\pi}{4}$,

(b) find the value of y when $x = \frac{\pi}{6}$, giving your answer in the form $a + b\sqrt{3}$, where a and b are rational numbers to be found. (3)

Solution

$$5\sqrt{2} = \cos\tfrac{\pi}{4}(-\tfrac{1}{2}\cos\tfrac{\pi}{2} + \tan\tfrac{\pi}{4} + c)$$
$$\Rightarrow \quad 5\sqrt{2} = \tfrac{\sqrt{2}}{2}(0 + 1 + c)$$
$$\Rightarrow \quad 10 = 1 + c$$
$$\Rightarrow \quad c = 9$$
$$\Rightarrow \quad y = \cos x(-\tfrac{1}{2}\cos 2x + \tan x + 9)$$

and

$$x = \tfrac{\pi}{6} \Rightarrow y = \cos\tfrac{\pi}{6}(-\tfrac{1}{2}\cos\tfrac{\pi}{3} + \tan\tfrac{\pi}{6} + 9)$$
$$\Rightarrow y = \tfrac{\sqrt{3}}{2}(-\tfrac{1}{4} + \tfrac{\sqrt{3}}{3} + 9)$$
$$\Rightarrow y = -\tfrac{\sqrt{3}}{8} + \tfrac{1}{2} + \tfrac{9\sqrt{3}}{2}$$
$$\Rightarrow \underline{\underline{y = \tfrac{1}{2} + \tfrac{35}{8}\sqrt{3}}}.$$