Dr Oliver Mathematics
Further Mathematics: Further Pure Mathematics 2 (Paper 4A)
November 2021: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. Without performing any division, explain why (4)

$$n = 20\,210\,520$$

is divisible by 66.

Solution

Well, we make sure that is product is divisible by 2, 3, and 11:

Divisible by 2?: It is, as the last digit is an even number.

Divisible by 3?:

$$\begin{aligned} 2 + 0 + 2 + 1 + 0 + 5 + 2 &= 12 \\ &= 4 \times 3, \end{aligned}$$

which means it divisible by 3.

Divisible by 11?:

$$\begin{aligned} (2 + 2 + 0 + 2) - (0 + 1 + 5 + 0) &= 6 - 6 \\ &= 0 \\ &= 0 \times 11, \end{aligned}$$

which means it divisible by 11.

Hence, $20\,210\,520$ is divisible by $\underline{\underline{66}}$.

2. A binary operation $\star$ on the set of non-negative integers, $\mathbb{Z}_0^+$, is defined by

$$m \star n = |m - n|,\ m\,, n \in \mathbb{Z}_0^+.$$

(a) Explain why $\mathbb{Z}_0^+$ is closed under the operation $\star$. (1)

Solution
For all such m and n, the difference of any two integers is an integer so

$$(m - n) \in \mathbb{Z}.$$

In particular,

$$|m - n| \in \mathbb{Z}_0^+,$$

and so it closed under $\star$.

(b) Show that 0 is an identity for $(\mathbb{Z}_0^+, \star)$. (2)

Solution
For $m \in \mathbb{Z}_0^+$,

$$\begin{aligned} m \star 0 &= |m - 0| \\ &= |m| \\ &= m \end{aligned}$$

and

$$\begin{aligned} 0 \star m &= |0 - m| \\ &= |-m| \\ &= m; \end{aligned}$$

hence, 0 is an identity for $(\mathbb{Z}_0^+, \star)$.

(c) Show that all elements of $\mathbb{Z}_0^+$ have an inverse under $\star$. (2)

Solution
Well,

$$|m - m| = |0| = 0,$$

so, for $m \in \mathbb{Z}_0^+$, m is self-inverse.

(d) Determine if $\mathbb{Z}_0^+$ forms a group under $\star$, giving clear justification for your answer. (3)

Solution

Well,

$$\begin{aligned}(1 \star 2) \star 3 &= |1-2| \star 3 \\ &= |-1| \star 3 \\ &= 1 \star 3 \\ &= |1-3| \\ &= |-2| \\ &= 2\end{aligned}$$

and

$$\begin{aligned}1 \star (2 \star 3) &= 1 \star |2-3| \\ &= 1 \star |-1| \\ &= 1 \star 1 \\ &= |1-1| \\ &= |0| \\ &= 0;\end{aligned}$$

so the operation $\star$ is not associative. Hence, $\mathbb{Z}_0^+$ does not form a group.

3. (a) Use the Euclidean Algorithm to find integers a and b such that (5)

$$125a + 87b = 1.$$

Solution

Well,

$$\begin{aligned}125 &= 1 \times 87 + 38 \\ 87 &= 2 \times 38 + 11 \\ 38 &= 3 \times 11 + 5 \\ 11 &= 2 \times 5 + 1,\end{aligned}$$

and

$$\begin{aligned}1 &= 11 - 2 \times 5 \\ &= 11 - 2(38 - 3 \times 11) \\ &= 7 \times 11 - 2 \times 38 \\ &= 7(87 - 2 \times 38) - 2 \times 38 \\ &= 7 \times 87 - 16 \times 38 \\ &= 7 \times 87 - 16(125 - 1 \times 87) \\ &= \underline{\underline{-16 \times 125 + 23 \times 87}};\end{aligned}$$

hence, $\underline{\underline{a = -16}}$ and $\underline{\underline{b = 23}}$.

(b) Hence write down a multiplicative inverse of 87 modulo 125. (1)

Solution
Now,

$$23 \times 87 \equiv 1 \pmod{125}$$

so a multiplicative inverse of 87 is $\underline{\underline{23}}$.

(c) Solve the linear congruence (2)

$$87x \equiv 16 \pmod{125}.$$

Solution

$$\begin{aligned}87x \equiv 16 \pmod{125} &\Rightarrow x \equiv 23 \times 16 \pmod{125} \\ &\Rightarrow x \equiv 368 \pmod{125} \\ &\Rightarrow x \equiv 243 \pmod{125} \\ &\Rightarrow \underline{\underline{x \equiv 118 \pmod{125}}}.\end{aligned}$$

4. Let G be a group of order $46^{46} + 47^{47}$. (7)

Using Fermat's Little Theorem and explaining your reasoning, determine which of the following are possible orders for a subgroup of G

(a) 11,

Solution

Well, the order of a subgroup must divide the order of a group by Lagrange's Theorem.

So, we need to check if 11 and 21 divides $46^{46} + 47^{47}$.

We will need Fermat's Little Theorem:

$$a^{10} \equiv 1 \pmod{11}.$$

Now,

$$46 = 4 \times 11 + 2 \text{ and } 47 = 4 \times 11 + 3$$

Modulo 11,

$$\begin{aligned} 46^{46} + 47^{47} &\equiv (4 \times 11 + 2)^{46} + (4 \times 11 + 3)^{47} \\ &\equiv 2^{10 \times 4 + 6} + 3^{10 \times 4 + 7} \\ &\equiv 2^6 + 3^7 \\ &\equiv 64 + 2\,187 \\ &\equiv 9 + 9 \\ &\equiv 18 \\ &\equiv 7; \end{aligned}$$

so, 11 is not a divisor of $46^{46} + 47^{47}$.

Hence, 11 is not a possible order for a subgroup.

(b) 21.

Solution

$$21 = 3 \times 7$$

and we need to check that

$$a^2 \equiv 1 \pmod{3} \textbf{ and } a^6 \equiv 1 \pmod{7}.$$

Modulo 3,

$$\begin{aligned} 46^{46} + 47^{47} &\equiv (2 \times 23)^{46} + (2 \times 23 + 1)^{47} \\ &\equiv 1^{46} + 2^{47} \\ &\equiv 1 + 2^{2 \times 23 + 1} \\ &\equiv 1 + 2 \\ &\equiv 0 \end{aligned}$$

and, modulo 7,

$$\begin{aligned}46^{46} + 47^{47} &\equiv (6 \times 7 + 4)^{46} + (6 \times 7 + 5)^{47} \\ &\equiv 4^{46} + 5^{47} \\ &\equiv 4^{6\times7+4} + 5^{6\times7+5} \\ &\equiv 4^4 + 5^5 \\ &\equiv 256 + 3\,125 \\ &\equiv 3\,381 \\ &\equiv 0;\end{aligned}$$

so, each of 3 and 7 is a divisor of $46^{46} + 47^{47}$.
Hence, 21 is a possible order for a subgroup.

5. The point P in the complex plane represents a complex number z such that

$$|z + 9| = 4|z - 12\mathrm{i}|.$$

Given that, as z varies, the locus of P is a circle,

(a) determine the centre and radius of this circle. (6)

Solution

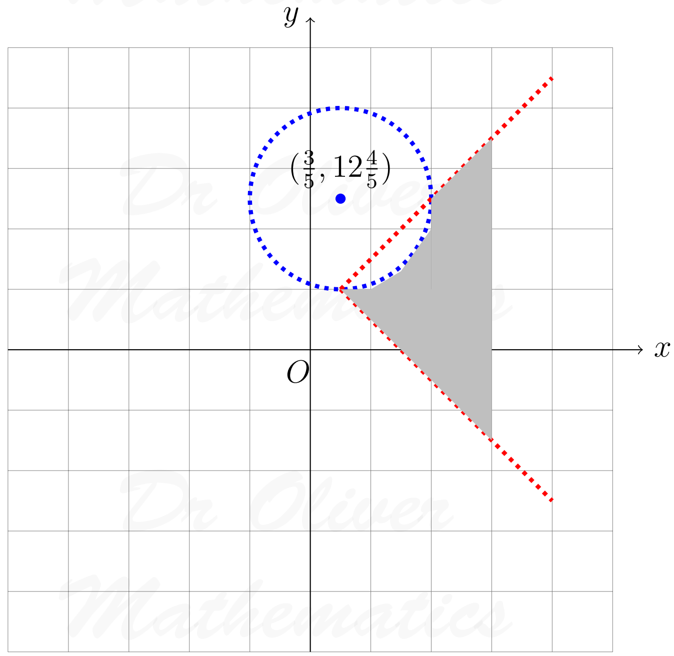
$$\begin{aligned} & |z+9| = 4|z-12\mathrm{i}| \\ \Rightarrow\ & |z+9|^2 = 16|z-12\mathrm{i}|^2 \\ \Rightarrow\ & |(x+9)+\mathrm{i}y|^2 = 16|x+(y-12)\mathrm{i}|^2 \\ \Rightarrow\ & (x+9)^2 + y^2 = 16[x^2 + (y-12)^2] \\ \Rightarrow\ & (x^2 + 18x + 81) + y^2 = 16x^2 + 16(y^2 - 24y + 144) \\ \Rightarrow\ & x^2 + 18x + 81 + y^2 = 16x^2 + 16y^2 - 384y + 2\,304 \\ \Rightarrow\ & 15x^2 - 18x + 15y^2 - 384y = -2\,223 \\ \Rightarrow\ & x^2 - \tfrac{6}{5}x + y^2 - \tfrac{128}{5}y = -\tfrac{741}{5} \\ \Rightarrow\ & (x^2 - \tfrac{6}{5}x + \tfrac{9}{25}) + (y^2 - \tfrac{128}{5}y + \tfrac{4096}{25}) = -\tfrac{741}{5} + \tfrac{9}{25} + \tfrac{4\,096}{25} \\ \Rightarrow\ & (x - \tfrac{3}{5})^2 + (y - \tfrac{64}{5})^2 = 16; \end{aligned}$$

hence, the centre is $(\frac{3}{5}, 12\frac{4}{5})$ and radius is 4.

(b) Shade on an Argand diagram the region defined by the set (4)

$$\{z \in \mathbb{C} : |z+9| < 4|z-12\mathrm{i}|\} \cap \left\{z \in \mathbb{C} : -\tfrac{1}{4}\pi < \arg(z - \tfrac{1}{5}(3+44\mathrm{i})) < \tfrac{1}{4}\pi\right\}.$$

Solution



6. A recurrence system is defined by (6)

$$
\begin{aligned}
u_{n+2} &= 9(n+1)^2 u_n - 3u_{n+1},\ n \geqslant 1, \\
u_1 &= -3,\ \text{and} \\
u_2 &= 18.
\end{aligned}
$$

Prove by induction that, for $n \in \mathbb{N}$,

$$u_n = (-3)^n\, n!.$$

Solution

$\underline{n = 1}$: LHS $= (-3)^1\, 1! = -3 =$ RHS.
$\underline{n = 2}$: LHS $= (-3)^2\, 2! = 3^2 \times 2 = 18 =$ RHS.
So, the result is true for $n = 1$ and $n = 2$.

Suppose that the result for some $n = k$ and $n = k + 1$, i.e.,

$$u_k = (-3)^k\, k! \text{ and } u_{k+1} = (-3)^{k+1}\, (k+1)!.$$

Now,

$$\begin{aligned}
u_{k+2} &= 9(k+1)^2 u_k - 3u_{k+1} \\
&= 9(k+1)^2[(-3)^k\,k!] - 3[(-3)^{k+1}\,(k+1)!] \\
&= (-3)^k\,k!\left[9(k+1)^2 - 3(-3)(k+1)\right] \\
&= (-3)^k\,k!\left[9(k^2+2k+1) + 9(k+1)\right] \\
&= (-3)^k\,k!\left(9k^2 + 18k + 9 + 9k + 9\right) \\
&= (-3)^k\,k!\left(9k^2 + 27k + 18\right) \\
&= (-3)^k\,k!\left[9(k^2 + 3k + 2)\right] \\
&= (-3)^k\,k!(-3)^2(k+1)(k+2) \\
&= (-3)^{k+2}\,(k+2)!,
\end{aligned}$$

and it is true for $n = k + 2$.

By mathematical induction, therefore, it is true for all $n \in \mathbb{N}$.

7.

$$I_n = \int t^n \sqrt{4 + 5t^2}\,\mathrm{d}t.$$

(a) Show that, for $n > 1$, (5)

$$I_n = \frac{t^{n-1}}{5(n+2)}(4+5t^2)^{\frac{3}{2}} - \frac{4(n-1)}{5(n+2)} I_{n-2}.$$

Solution

Integration by parts:

$$\begin{aligned}
u = t^{n-1} &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)t^{n-2} \\
\frac{\mathrm{d}v}{\mathrm{d}x} = t(4+5t^2)^{\frac{1}{2}} &\Rightarrow v = \tfrac{1}{15}(4+5t^2)^{\frac{3}{2}}
\end{aligned}$$

and

$$\begin{aligned}
I_n &= \int t^n \sqrt{4+5t^2}\,\mathrm{d}t \\
&= \int t^{n-1}\, t(4+5t^2)^{\frac{1}{2}}\,\mathrm{d}t \\
&= \tfrac{1}{15}t^{n-1}(4+5t^2)^{\frac{3}{2}} - \int \tfrac{1}{15}(n-1)t^{n-2}(4+5t^2)^{\frac{3}{2}}\,\mathrm{d}t \\
&= \tfrac{1}{15}t^{n-1}(4+5t^2)^{\frac{3}{2}} - \tfrac{1}{15}(n-1)\int t^{n-2}(4+5t^2)(4+5t^2)^{\frac{1}{2}}\,\mathrm{d}t \\
&= \tfrac{1}{15}t^{n-1}(4+5t^2)^{\frac{3}{2}} - \tfrac{1}{15}(n-1)\int (4t^{n-2}+5t^n)(4+5t^2)^{\frac{1}{2}}\,\mathrm{d}t \\
&= \tfrac{1}{15}t^{n-1}(4+5t^2)^{\frac{3}{2}} - \tfrac{4}{15}(n-1)I_{n-2} - \tfrac{1}{3}(n-1)I_n
\end{aligned}$$

and

$$\begin{aligned}
& 15I_n = t^{n-1}(4+5t^2)^{\frac{3}{2}} - 4(n-1)I_{n-2} - 5(n-1)I_n \\
\Rightarrow\quad & 15I_n + 5(n-1)I_n = t^{n-1}(4+5t^2)^{\frac{3}{2}} - 4(n-1)I_{n-2} \\
\Rightarrow\quad & 5I_n[3+(n-1)] = t^{n-1}(4+5t^2)^{\frac{3}{2}} - 4(n-1)I_{n-2} \\
\Rightarrow\quad & 5(n+2)I_n = t^{n-1}(4+5t^2)^{\frac{3}{2}} - 4(n-1)I_{n-2} \\
\Rightarrow\quad & \underline{\underline{I_n = \frac{t^{n-1}}{5(n+2)}(4+5t^2)^{\frac{3}{2}} - \frac{4(n-1)}{5(n+2)}I_{n-2}}},
\end{aligned}$$

as required.

The curve shown in Figure 1 is defined by the parametric equations

$$x = \tfrac{1}{\sqrt{5}}t^5,\ y = \tfrac{1}{2}t^4,\ 0 \leqslant t \leqslant 1.$$

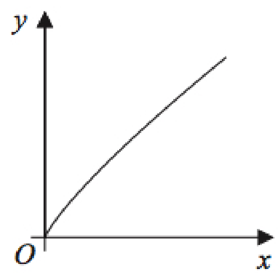


Figure 1: $x = \frac{1}{\sqrt{5}}t^5$, $y = \frac{1}{2}t^4$

This curve is rotated through 2π radians about the x-axis to form a hollow open shell.

(b) Show that the external surface area of the shell is given by (5)

$$\pi \int_0^1 t^7\sqrt{4 + 5t^2}\,\mathrm{d}t.$$

Solution

Well,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{5}t^4 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = 2t^3.$$

Now, using

$$2\pi \int y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t,$$

$$\begin{aligned}
\text{surface area} &= 2\pi \int_0^1 (\tfrac{1}{2}t^4)\sqrt{(\sqrt{5}t^4)^2 + (2t^3)^2}\,\mathrm{d}t \\
&= \pi \int_0^1 t^4\sqrt{5t^8 + 4t^6}\,\mathrm{d}t \\
&= \pi \int_0^1 t^4\sqrt{t^6(4 + 5t^2)}\,\mathrm{d}t \\
&= \underline{\underline{\pi \int_0^1 t^7\sqrt{4 + 5t^2}\,\mathrm{d}t}},
\end{aligned}$$

as required.

(c) determine the value of the external surface area of the shell, giving your answer to 3 significant figures. (5)

Solution

$$\begin{aligned}
I_7 &= \left[\frac{t^6}{5\times 9}(4+5t^2)^{\frac{3}{2}}\right]_{t=0}^{1} - \frac{4\times 6}{5\times 9}I_5 \\
&= \left(\tfrac{1}{45}(27)-0\right) - \tfrac{8}{15}I_5 \\
&= \tfrac{3}{5} - \tfrac{8}{15}I_5 \\
&= \tfrac{3}{5} - \tfrac{8}{15}\left\{\left[\frac{t^4}{5\times 7}(4+5t^2)^{\frac{3}{2}}\right]_{t=0}^{1} - \frac{4\times 4}{5\times 7}I_3\right\} \\
&= \tfrac{3}{5} - \tfrac{8}{15}\left\{\left(\tfrac{1}{35}(27)-0\right) - \tfrac{16}{35}I_3\right\} \\
&= \tfrac{3}{5} - \tfrac{72}{175} + \tfrac{128}{525}I_3 \\
&= \tfrac{33}{175} + \tfrac{128}{525}I_3 \\
&= \tfrac{33}{175} + \tfrac{128}{525}\left\{\left[\frac{t^2}{5\times 5}(4+5t^2)^{\frac{3}{2}}\right]_{t=0}^{1} - \frac{4\times 2}{5\times 5}I_1\right\} \\
&= \tfrac{33}{175} + \tfrac{128}{525}\left(\tfrac{1}{25}(27)-0\right) - \tfrac{8}{25}I_1) \\
&= \tfrac{33}{175} + \tfrac{1\,152}{4\,375} - \tfrac{1\,024}{13\,125}I_1 \\
&= \tfrac{1\,977}{4\,375} - \tfrac{1\,024}{13\,125}I_1
\end{aligned}$$

Now,

$$\begin{aligned}
I_1 &= \int_0^1 t\sqrt{4+5t^2}\,\mathrm{d}t \\
&= \left[\tfrac{1}{15}(4+5t^2)^{\frac{3}{2}}\right]_{t=0}^{1} \\
&= \tfrac{1}{15}(27) - \tfrac{1}{15}(4) \\
&= \tfrac{19}{15},
\end{aligned}$$

and so

$$\begin{aligned}
I_7 &= \tfrac{1\,977}{4\,375} - \tfrac{1\,024}{13\,125}\times\tfrac{19}{15} \\
&= 0.353\,061\,587\,3 \text{ (FCD)}.
\end{aligned}$$

But we need to multiply by π! Hence,

$$\begin{aligned}
\text{external shell} &= 0.353\ldots\times\pi \\
&= 1.109\,175\,689 \text{ (FCD)} \\
&= \underline{\underline{1.11 \text{ (3 sf)}}}.
\end{aligned}$$

8.
$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & p \\ -6 & 6 & -4 \end{pmatrix}.$$

Given that
$$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$
is an eigenvector for **A**,

(a) (i) determine the eigenvalue corresponding to this eigenvector, (1)

Solution
Well,
$$\begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & p \\ -6 & 6 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3-2p \\ 2 \end{pmatrix};$$
hence, the eigenvalue is $\underline{\underline{-1}}$.

(ii) hence show that $p = 2$, (2)

Solution
Consider the second row:
$$\begin{aligned} 3 - 2p = -1 &\Rightarrow -2p = -4 \\ &\Rightarrow \underline{\underline{p = 2}}, \end{aligned}$$
as required.

(iii) determine the remaining eigenvalues and corresponding eigenvectors of **A**. (7)

Solution
$$\begin{aligned} & \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \\ \Rightarrow & \begin{vmatrix} 5-\lambda & -2 & 5 \\ 0 & 3-\lambda & 2 \\ -6 & 6 & -4-\lambda \end{vmatrix} = 0 \\ \Rightarrow & (5-\lambda)\left[(3-\lambda)(-4-\lambda) - 12\right] + 2(0+12) + 5\left[0 + 6(3-\lambda)\right] = 0 \\ \Rightarrow & (5-\lambda)\left[(-12+\lambda+\lambda^2) - 12\right] + 24 + 5\left(18 - 6\lambda\right) = 0 \\ \Rightarrow & (5-\lambda)\left(-24+\lambda+\lambda^2\right) + 24 + 90 - 30\lambda = 0 \end{aligned}$$

$\times$	-24	$+\lambda$	$+\lambda^2$
5	-120	$+5\lambda$	$+5\lambda^2$
$-\lambda$	$+24\lambda$	$-\lambda^2$	$-\lambda^3$

$$\begin{aligned}
&\Rightarrow \quad (-120 + 29\lambda + 4\lambda^2 - \lambda^3) + 114 - 30\lambda = 0 \\
&\Rightarrow \quad -6 - \lambda + 4\lambda^2 - \lambda^3 = 0 \\
&\Rightarrow \quad \lambda^3 - 4\lambda^2 + \lambda + 6 = 0.
\end{aligned}$$

Synthetic division:

-1	1	-4	1	6
	$\downarrow$	-1	5	-6
	1	-5	6	0

So,

$$\begin{aligned}
\lambda^3 - 4\lambda^2 + \lambda + 6 = 0 &\Rightarrow (\lambda + 1)(\lambda^2 - 5\lambda + 6) = 0 \\
&\Rightarrow (\lambda + 1)(\lambda - 2)(\lambda - 3) = 0 \\
&\Rightarrow \lambda = -1, \underline{\underline{\lambda = 2, \text{ or } \lambda = 3}}.
\end{aligned}$$

$\underline{\lambda = 2}$:

$$\begin{aligned}
\begin{pmatrix} 3 & -2 & 5 \\ 0 & 1 & 2 \\ -6 & 6 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\Rightarrow \quad \begin{pmatrix} 3x - 2y + 5z \\ y + 2z \\ -6x + 6y - 6z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};
\end{aligned}$$

so, an eigenvector is

$$\underline{\underline{\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}}}.$$

$\underline{\lambda = 3}$:

$$\begin{pmatrix} 4 & -2 & 5 \\ 0 & 0 & 2 \\ -6 & 6 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 2x - 2y + 5z \\ 2z \\ -6x + 6y - 7z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

so, an eigenvector is

$$\underline{\underline{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}}.$$

(b) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that (1)

$$\mathbf{A} = \mathbf{PDP}^{-1}.$$

Solution

E.g.,

$$\underline{\underline{\mathbf{P} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ -2 & -1 & 0 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}}}.$$

(c) (i) Solve the differential equation (2)

$$\dot{u} = ku,$$

where k is a constant.

Solution

$$\begin{aligned}\dot{u} = ku \Rightarrow \frac{\dot{u}}{u} &= k\\ \Rightarrow \int \frac{\dot{u}}{u}\,\mathrm{d}t &= \int k\,\mathrm{d}t\\ \Rightarrow \ln u &= kt + c\\ \Rightarrow u &= \mathrm{e}^{kt+c}\\ \Rightarrow u &= \mathrm{e}^{kt}\,\mathrm{e}^{c}\\ \Rightarrow \underline{\underline{u = A\mathrm{e}^{kt}}},\end{aligned}$$

where A is a constant.

With respect to a fixed origin O, the velocity of a particle moving through space is modelled by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

By considering

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

so that

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix},$$

(ii) determine a general solution for the displacement of the particle. (4)

Solution
We want a solution of

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{PDP}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Now,

$$\begin{aligned}
\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} &= \mathbf{PDP}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \\
&= \mathbf{PD} \left(\mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \right) \\
&= \mathbf{PD} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\
&= \mathbf{P} \left(\mathbf{D} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right) \\
&= \mathbf{P} \begin{pmatrix} -u \\ 2v \\ 3w \end{pmatrix}.
\end{aligned}$$

Integrate:

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \mathbf{P} \begin{pmatrix} A\mathrm{e}^{-t} \\ B\mathrm{e}^{2t} \\ C\mathrm{e}^{3t} \end{pmatrix} \\
&= \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} A\mathrm{e}^{-t} \\ B\mathrm{e}^{2t} \\ C\mathrm{e}^{3t} \end{pmatrix} \\
&= \underline{\underline{\begin{pmatrix} 2A\mathrm{e}^{-t} + 3B\mathrm{e}^{2t} + C\mathrm{e}^{3t} \\ A\mathrm{e}^{-t} + 2B\mathrm{e}^{2t} + C\mathrm{e}^{3t} \\ -2A\mathrm{e}^{-t} - B\mathrm{e}^{2t} \end{pmatrix}}}.
\end{aligned}$$