Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2013 November Paper 2 Variant 3: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Find the coordinates of the stationary points on the curve (5)

$$y = x^3 - 6x^2 - 36x + 16.$$

Solution

Well,

$$y = x^3 - 6x^2 - 36x + 16 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x - 36$$

and

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3x^2 - 12x - 36 = 0 \\
&\Rightarrow 3(x^2 - 4x - 12) = 0
\end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & -12 \end{array}\right\} -6,\ +2$$

$$\begin{aligned}
&\Rightarrow 3(x^2 - 6x + 2x - 12) = 0 \\
&\Rightarrow 3[x(x-6) + 2(x-6)] = 0 \\
&\Rightarrow 3(x+2)(x-6) = 0 \\
&\Rightarrow x + 2 = 0 \text{ or } x - 6 = 0 \\
&\Rightarrow x = -2 \text{ or } x = 6 \\
&\Rightarrow y = 56 \text{ or } y = -200;
\end{aligned}$$

hence, the coordinates of the stationary points are $\underline{\underline{(-2, 56)}}$ and $\underline{\underline{(6, -200)}}$.

2. (a) Find how many different numbers can be formed using 4 of the digits 1, 2, 3, 4, 5, 6, and 7 if no digit is repeated. (1)

Solution

$$7 \times 6 \times 5 \times 4 = \underline{\underline{840}}.$$

Find how many of these 4-digit numbers are

(b) odd, (1)

Solution

$$4 \times 6 \times 5 \times 4 = \underline{\underline{480}}.$$

(c) odd and less than 3 000. (3)

Solution

Well,

$$\begin{aligned} \text{odd and less than } 3\,000 &= \text{begin 1, less than } 3\,000 + \text{begin 2, less than } 3\,000 \\ &= (1 \times 5 \times 4 \times 3) + (1 \times 5 \times 4 \times 4) \\ &= 60 + 80 \\ &= \underline{\underline{140}}. \end{aligned}$$

3. Find the set of values of k for which the line (6)

$$y = 3x - k$$

does not meet the curve

$$y = kx^2 + 11x - 6.$$

Solution

Now,

$$kx^2 + 11x - 6 = 3x - k \Rightarrow kx^2 + 8x + (k - 6) = 0$$

and

$$\begin{aligned} b^2 - 4ac < 0 &\Rightarrow 8^2 - 4(k)(k-6) < 0 \\ &\Rightarrow 64 < 4(k)(k-6) \\ &\Rightarrow 16 < k^2 - 6k \\ &\Rightarrow k^2 - 6k - 16 > 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -6 \\ \text{multiply to:} & -16 \end{array}\right\} -8,\ +2$$

$$\Rightarrow (k-8)(k+2) > 0.$$

We need a 'table of signs':

	$k < -2$	$k = -2$	$-2 < k < 8$	$k = 8$	$k > 8$
$(k+2)$	$-$	0	$+$	$+$	$+$
$(k-8)$	$-$	$-$	$-$	0	$+$
$(k+2)(k-8)$	$+$	0	$-$	0	$+$

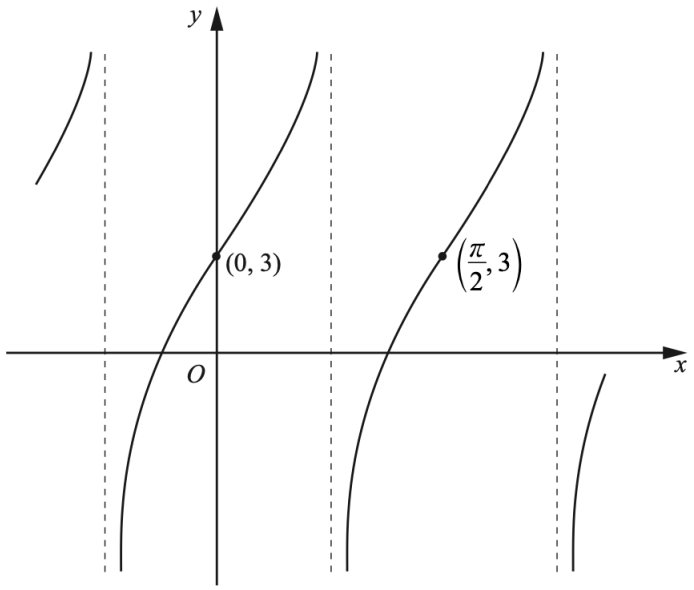
So

$$\underline{\underline{k < -2 \text{ or } k > 8}}.$$

4. The diagram shows the graph of

$$y = A + C\tan(Bx)$$

passing through the points $(0, 3)$ and $(\frac{1}{2}\pi, 3)$.



(a) (i) Find the value of A and of B. (1)

Solution
$\underline{\underline{A = 3}}$ and $\underline{\underline{B = 2}}$.

(ii) Given that the point $(\frac{1}{8}\pi, 7)$ also lies on the graph, find the value of C. (2)

Solution
Well,

$$x = \tfrac{1}{8}\pi, y = 7 \Rightarrow 7 = 3 + C\tan(\tfrac{1}{4}\pi)$$
$$\Rightarrow \underline{\underline{C = 4}}.$$

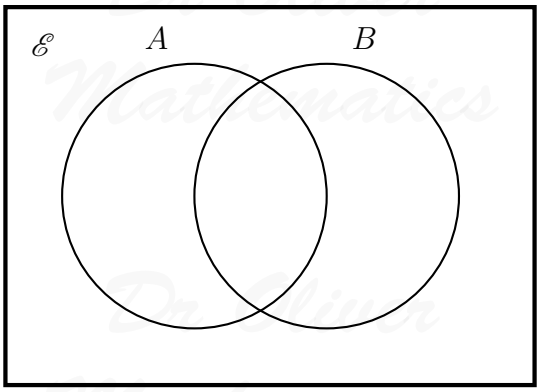
(b) Given that (2)

$$\mathrm{f}(x) = 8 - 5\cos 3x,$$

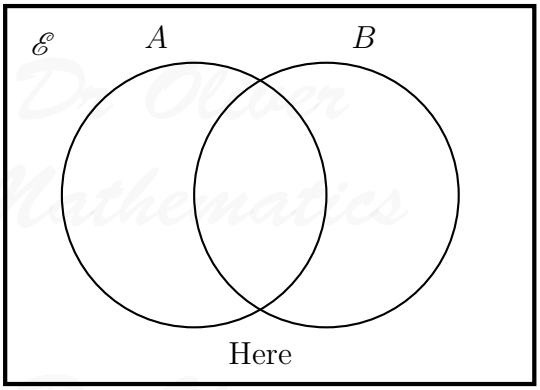
state the period and the amplitude of f.

Solution
The period is $\underline{\underline{\frac{2}{3}\pi}}$ and the amplitude is $\underline{\underline{5}}$.

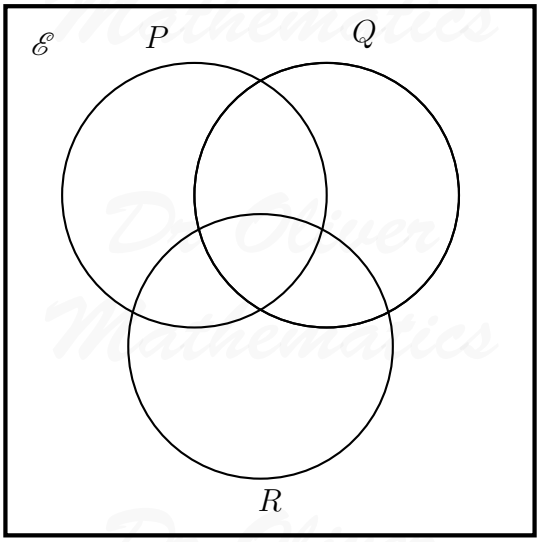
5. (a) (i) In the Venn diagram below shade the region that represents $(A \cup B)'$. (1)



Solution



(ii) In the Venn diagram below shade the region that represents $P \cap Q \cap R'$. (1)



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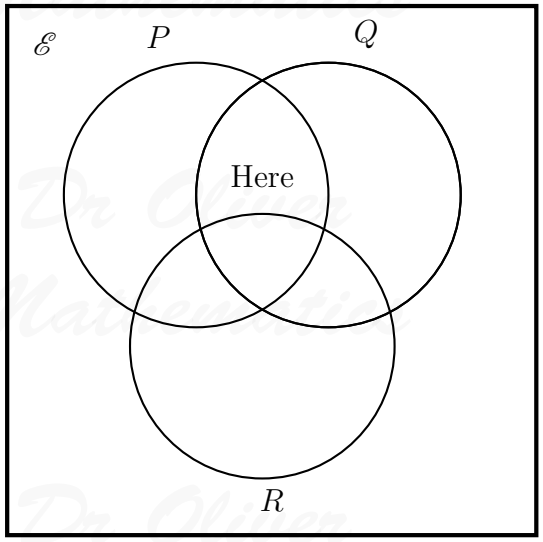
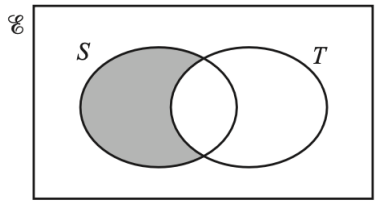
Solution



(b) Express, in set notation, the set represented by the shaded region. (1)



Solution

$S \cap T'$.

(c) The universal set $\mathscr{E}$ and the sets V and W are such that (3)

- $\mathrm{n}(\mathscr{E}) = 40$,
- $\mathrm{n}(V) = 18$, and
- $\mathrm{n}(W) = 14$.

Given that

$$\mathrm{n}(V \cap W) = x \text{ and } \mathrm{n}[(V \cup W)'] = 3x,$$

find the value of x.
You may use the Venn diagram below to help you.

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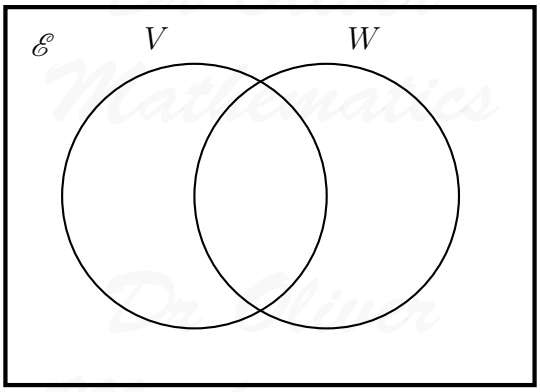
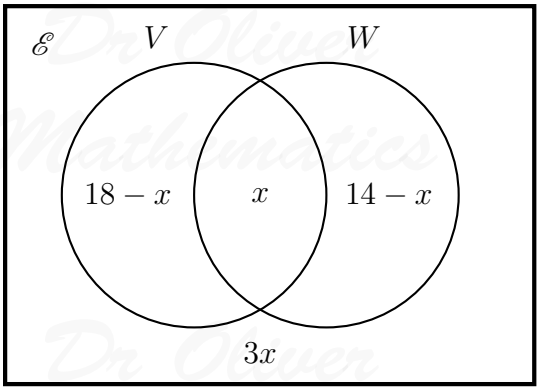


Solution
Well,



and

$$\begin{aligned}(18-x)+x+(14-x)+3x=40 &\Rightarrow 32+2x=40\\ &\Rightarrow 2x=8\\ &\Rightarrow \underline{\underline{x=4}}.\end{aligned}$$

6. The expression

$$2x^3+ax^2+bx+21$$

has a factor $(x+3)$ and leaves a remainder of 65 when divided by $(x-2)$.

(a) Find the value of a and of b. (5)

Solution
We use synthetic division twice:

-3	2	a	b	21
	$\downarrow$	-6	$-3a+18$	$9a-3b-54$
	2	$a-6$	$-3a+b+18$	$9a-3b-33$

so

$$9a-3b-33=0 \Rightarrow 9a-3b=33$$
$$\Rightarrow 3a-b=11 \quad (1).$$

2	2	a	b	21
	$\downarrow$	2	$2a+8$	$4a+2b+16$
	2	$a+4$	$2a+b+8$	$4a+2b+37$

so

$$4a+2b+37=65 \Rightarrow 4a+2b=28$$
$$\Rightarrow 2a+b=14 \quad (2).$$

Do (1) + (2):

$$5a=25 \Rightarrow \underline{\underline{a=5}}$$
$$\Rightarrow \underline{\underline{b=4}}.$$

(b) Hence find the value of the remainder when the expression is divided by $(2x+1)$. (2)

Solution

The expression is

$$\mathrm{f}(x)=2x^3+5x^2+4x+21$$

and

$$\mathrm{f}(-\tfrac{1}{2})=-\tfrac{1}{4}+\tfrac{5}{4}-2+21=\underline{\underline{20}}.$$

7. The line (8)

$$4x+y=16$$

intersects the curve

$$\frac{4}{x}-\frac{8}{y}=1$$

at the points A and B.

The x-coordinate of A is less than the x-coordinate of B.

Given that the point C lies on the line AB such that

$$AC : CB = 1 : 2,$$

find the coordinates of C.

Solution

Well,

$$4x + y = 16 \Rightarrow y = 16 - 4x \quad (1)$$

and

$$\begin{aligned}\frac{4}{x} - \frac{8}{y} = 1 &\Rightarrow \frac{4y}{xy} - \frac{8x}{xy} = 1 \\ &\Rightarrow \frac{4y - 8x}{xy} = 1 \\ &\Rightarrow 4y - 8x = xy \quad (2).\end{aligned}$$

Insert (1) into (2):

$$\begin{aligned}4(16 - 4x) - 8x = x(16 - 4x) &\Rightarrow 64 - 16x - 8x = 16x - 4x^2 \\ &\Rightarrow 4x^2 - 40x + 64 = 0 \\ &\Rightarrow 4(x^2 - 10x + 16) = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -10 \\ \text{multiply to:} & +16\end{array}\right\} -8, -2$$

$$\begin{aligned}&\Rightarrow 4(x - 8)(x - 2) = 0 \\ &\Rightarrow x - 8 = 0 \text{ or } x - 2 = 0 \\ &\Rightarrow x = 8 \text{ or } x = 2 \\ &\Rightarrow y = -16 \text{ or } y = 8;\end{aligned}$$

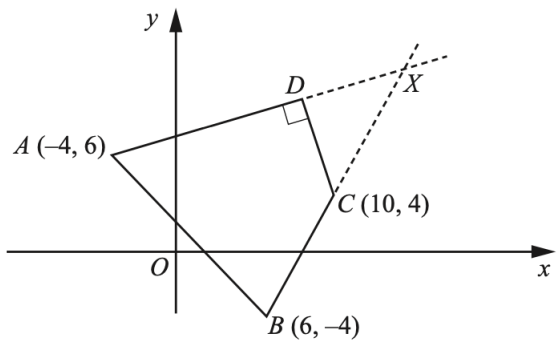
hence, $A(2, 8)$ and $B(8, -16)$.

Now,

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + \tfrac{1}{3}\overrightarrow{AB} \\ &= \begin{pmatrix} 2 \\ 8 \end{pmatrix} + \tfrac{1}{3}\begin{pmatrix} 8-2 \\ -16-8 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 8 \end{pmatrix} + \tfrac{1}{3}\begin{pmatrix} 6 \\ -24 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 8 \end{pmatrix} + \begin{pmatrix} 2 \\ -8 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 0 \end{pmatrix};\end{aligned}$$

so, $\underline{\underline{C(4,0)}}$.

8. **Solutions to this question by accurate drawing will not be accepted.**
The diagram shows a quadrilateral $ABCD$, with vertices $A(-4,6)$, $B(6,-4)$, $C(10,4)$, and D.



- The angle $ADC = 90°$.
- The lines BC and AD are extended to intersect at the point X.

(a) Given that C is the midpoint of BX, find the coordinates of D. (7)

Solution

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Well,

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OB} + \overrightarrow{BX} \\ &= \overrightarrow{OB} + 2\overrightarrow{BC} \\ &= \begin{pmatrix} 6 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 10-6 \\ 4-(-4) \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 8 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -4 \end{pmatrix} + \begin{pmatrix} 8 \\ 16 \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ 12 \end{pmatrix};\end{aligned}$$

so, $X(14, 12)$.

Now,

$$\begin{aligned}m_{AX} &= \frac{12-6}{14-(-4)} \\ &= \frac{6}{18} \\ &= \tfrac{1}{3}\end{aligned}$$

when the equation of AX is

$$\begin{aligned}y - 6 = \tfrac{1}{3}(x+4) &\Rightarrow y - 6 = \tfrac{1}{3}x + \tfrac{4}{3} \\ &\Rightarrow y = \tfrac{1}{3}x + \tfrac{22}{3} \quad (1).\end{aligned}$$

Next,

$$m_{CD} = -3$$

and the equation of CD is

$$\begin{aligned}y - 4 = -3(x-10) &\Rightarrow y - 4 = -3x + 30 \\ &\Rightarrow y = -3x + 34 \quad (2).\end{aligned}$$

Finally, do (1) = (2):

$$\begin{aligned}\tfrac{1}{3}x + \tfrac{22}{3} = -3x + 34 &\Rightarrow \tfrac{10}{3}x = \tfrac{80}{3} \\ &\Rightarrow x = 8 \\ &\Rightarrow y = 10;\end{aligned}$$

hence, $\underline{\underline{D(8, 10)}}$.

(b) Hence calculate the area of the quadrilateral $ABCD$. (2)

Solution

Imagine a box: $E(-4, 10)$, $F(10, 10)$, $G(6, -4)$, and $H(-4, -4)$ which envelopes the quadrilateral. Now,

$$\begin{aligned} & \text{quadrilateral} \\ = \ & \text{area of the rectangle} - \triangle AED - \triangle DFC - \triangle BGC - \triangle AHB \\ = \ & (14 \times 14) - (\tfrac{1}{2} \times 4 \times 12) - (\tfrac{1}{2} \times 2 \times 6) - (\tfrac{1}{2} \times 4 \times 8) - (\tfrac{1}{2} \times 10 \times 10) \\ = \ & 196 - 24 - 6 - 16 - 50 \\ = \ & \underline{\underline{100}}. \end{aligned}$$

9. A particle travels in a straight line so that, t s after passing through a fixed point O, its velocity, v ms^{-1}, is given by

$$v = 3 + 6\sin 2t.$$

(a) Find the velocity of the particle when $t = \frac{1}{4}\pi$. (1)

Solution

$$t = \tfrac{1}{4}\pi \Rightarrow \underline{\underline{v = 9 \text{ ms}^{-1}}}.$$

(b) Find the acceleration of the particle when $t = 2$. (3)

Solution

Well,

$$v = 3 + 6\sin 2t \Rightarrow a = 12\cos 2t$$

and

$$\begin{aligned} t = 2 &\Rightarrow a = -7.843\,723\,45 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{a = -7.84 \text{ ms}^{-1} \text{ (3 sf)}}}. \end{aligned}$$

The particle first comes to instantaneous rest at the point P.

(c) Find the distance OP. (5)

Solution

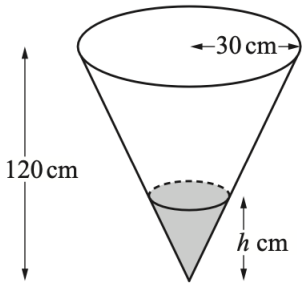
Now,

$$\begin{aligned} v = 0 &\Rightarrow 3 + 6\sin 2t = 0 \\ &\Rightarrow 6\sin 2t = -3 \\ &\Rightarrow \sin 2t = -\tfrac{1}{2} \\ &\Rightarrow 2t = \tfrac{7}{6}\pi \\ &\Rightarrow t = \tfrac{7}{12}\pi. \end{aligned}$$

Finally,

$$\begin{aligned} OP &= \int_0^{\frac{7}{12}\pi} (3 + 6\sin 2t)\,\mathrm{d}t \\ &= \left[3t - 3\cos 2t\right]_{t=0}^{\frac{7}{12}\pi} \\ &= \left(\tfrac{21}{12}\pi + \tfrac{3}{2}\sqrt{3}\right) - (0 - 3) \\ &= \underline{\underline{\tfrac{7}{4}\pi + \tfrac{3}{2}\sqrt{3} + 3}}. \end{aligned}$$

10. The diagram shows a container in the shape of a cone of height 120 cm and radius 30 cm.



The volume of a cone of height H and radius R is

$$\tfrac{1}{3}\pi R^2 H$$

Water is poured into the container at a rate of 20π cm^{3}s^{-1}.

(a) At the instant when the depth of water in the cone is h cm the volume of water in the cone is V cm^3. (3)

Show that

$$V = \frac{\pi h^3}{48}.$$

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Solution

Well,

$$\frac{r}{h} = \frac{30}{120} \Rightarrow r = \tfrac{1}{4}h$$

and

$$\begin{aligned} V &= \tfrac{1}{3}\pi r^2 h \\ &= \tfrac{1}{3}\pi(\tfrac{1}{4}h)^2 h \\ &= \underline{\underline{\frac{\pi h^3}{48}}}, \end{aligned}$$

as required.

(b) Find the rate at which h is increasing when $h = 50$. (3)

Solution

Well,

$$V = \frac{\pi h^3}{48} \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi h^2}{16}$$

and

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t} &\Rightarrow 20\pi = \frac{\pi h^2}{16} \times \frac{\mathrm{d}h}{\mathrm{d}t} \\ &\Rightarrow \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{20\pi}{\frac{\pi h^2}{16}} \\ &\Rightarrow \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{320}{h^2}. \end{aligned}$$

Finally,

$$h = 50 \Rightarrow \underline{\underline{\frac{\mathrm{d}h}{\mathrm{d}t} = 0.128}}.$$

(c) Find the rate at which the circular area of the water's surface is increasing when $h = 50$. (4)

Solution

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Well,

$$A = \pi r^2 \Rightarrow A = \pi(\tfrac{1}{4}h)^2$$
$$\Rightarrow A = \tfrac{1}{16}\pi h^2$$
$$\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}h} = \tfrac{1}{8}\pi h.$$

Now,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = \tfrac{1}{8}\pi h \times 0.128$$
$$\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = (\tfrac{1}{8}\pi \times 50) \times 0.128$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}A}{\mathrm{d}t} = 0.8\pi}}.$$

11. In this question, **i** is a unit vector due east and **j** is a unit vector due north.

- At time $t = 0$, boat A leaves the origin O and travels with velocity $(2\mathbf{i} + 4\mathbf{j})$ kmh^{-1}.
- Also at time $t = 0$ boat B leaves the point with position vector $(-21\mathbf{i} + 22\mathbf{j})$ km and travels with velocity $(5\mathbf{i} + 3\mathbf{j})$ kmh^{-1}.

(a) Write down the position vectors of boats A and B after $t$hours. (2)

Solution

Boat A is at

$$\underline{\underline{(2t\mathbf{i} + 4t\mathbf{j})\text{ km}}}$$

and boat B is at

$$(-21\mathbf{i} + 22\mathbf{j}) + t(5\mathbf{i} + 3\mathbf{j}) = \underline{\underline{[(-21 + 5t)\mathbf{i} + (22 + 3t)\mathbf{j}]\text{ km}}}.$$

(b) Show that A and B are 25 km apart when $t = 2$. (3)

Solution

When $t = 2$, boat A is at

$$4\mathbf{i} + 8\mathbf{j}$$

and boat B is at

$$-11\mathbf{i} + 28\mathbf{j}.$$

Finally, A and B are

$$\begin{aligned}\sqrt{[4-(-11)]^2+(8-28)^2} &= \sqrt{15^2+(-20)^2}\\ &= \underline{\underline{25 \text{ km}}},\end{aligned}$$

as required.

(c) Find the length of time for which A and B are less than 25 km apart. (5)

Solution

Now,

$$\begin{aligned}& [(-21+5t)-2t]^2+[(22+3t)-4t]^2 < 25^2\\ \Rightarrow\quad & (-21+3t)^2+(22-t)^2 < 625\end{aligned}$$

$\times$	-21	$+3t$
-21	441	$-63t$
$+3t$	$-63t$	$+9t^2$

$\times$	22	$+t$
22	484	$-22t$
$+t$	$-22t$	$+t^2$

$$\begin{aligned}\Rightarrow\quad & (441-126t+9t^2)+(484-44t+t^2) < 625\\ \Rightarrow\quad & 10t^2-170t+300 < 0\\ \Rightarrow\quad & 10(t^2-17t+30) < 0\end{aligned}$$

add to: -17, multiply to: $+30$ $\Big\}$ $-15, -2$

$$\begin{aligned}\Rightarrow\quad & 10[t^2-15t-2t+30] < 0\\ \Rightarrow\quad & 10[t(t-15)-2(t-15)] < 0\\ \Rightarrow\quad & 10(t-2)(t-15) < 0.\end{aligned}$$

We need a 'table of signs':

	$t < 2$	$t = 2$	$2 < t < 15$	$t = 15$	$t > 15$
$(t-2)$	$-$	0	$+$	$+$	$+$
$(t-15)$	$-$	$-$	$-$	0	$+$
$(t-2)(t-15)$	$+$	0	$-$	0	$+$

Hence,

$$15 - 2 = \underline{\underline{13 \text{ hours}}}.$$

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