Dr Oliver Mathematics
Applied Mathematics: Mechanics or Statistics
Section B
2007 Paper
1 hour

The total number of marks available is 32.
You must write down all the stages in your working.

1. Find the exact value of (5)

$$\int_0^{\frac{1}{6}\pi} x \sin 3x \, \mathrm{d}x.$$

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 3x \Rightarrow v = -\tfrac{1}{3}\cos 3x$$

$$\begin{aligned}\int x \sin 3x \, \mathrm{d}x &= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}\int \cos 3x \, \mathrm{d}x \\ &= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x + c\end{aligned}$$

and

$$\begin{aligned}\int_0^{\frac{1}{6}\pi} x \sin 3x \, \mathrm{d}x &= \left[-\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x\right]_{x=0}^{\frac{1}{6}\pi} \\ &= \left(0 + \tfrac{1}{9}\right) - (0 + 0) \\ &= \underline{\underline{\tfrac{1}{9}}}.\end{aligned}$$

2. Use the binomial theorem to expand (4)

$$\left(x^3 - \frac{2}{x}\right)^4$$

and simplify your answer.

Solution

$$
\begin{aligned}
& \left(x^3 - \frac{2}{x}\right)^4 \\
= \;& (x^3)^4 + \binom{4}{1}(x^3)^3\left(-\frac{2}{x}\right) + \binom{4}{2}(x^3)^2\left(-\frac{2}{x}\right)^2 + \binom{4}{3}(x^3)\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4 \\
= \;& \underline{\underline{x^{12} - 8x^8 + 24x^4 - 32 + \frac{16}{x^4}}}.
\end{aligned}
$$

3. A curve is defined parametrically by (5)

$$x = \frac{t}{t^2+1} \text{ and } y = \frac{t-1}{t^2+1}.$$

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of t.

Solution

$$
\begin{aligned}
x = \frac{t}{t^2+1} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{(t^2+1)\cdot 1 - t\cdot(2t)}{(t^2+1)^2} \\
\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{t^2+1-2t^2}{(t^2+1)^2} \\
\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{1-t^2}{(t^2+1)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
y = \frac{t-1}{t^2+1} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{(t^2+1)\cdot 1 - (t-1)\cdot(2t)}{(t^2+1)^2} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{t^2+1-2t^2+2t}{(t^2+1)^2} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{1+2t-t^2}{(t^2+1)^2}.
\end{aligned}
$$

Finally,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{\frac{1+2t-t^2}{(t^2+1)^2}}{\frac{1-t^2}{(t^2+1)^2}} \\ &= \underline{\underline{\frac{1+2t-t^2}{1-t^2}}}.\end{aligned}$$

4. (a) For the matrix (3)

$$\mathbf{A} = \begin{pmatrix} \lambda & 2 \\ 2 & \lambda - 3 \end{pmatrix},$$

find the values of λ such that the matrix is singular.

Solution

$$\begin{aligned}\det \mathbf{A} = 0 &\Rightarrow \lambda(\lambda - 3) - 4 = 0 \\ &\Rightarrow \lambda^2 - 3\lambda - 4 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -3 \\ \text{multiply to:} & -4\end{array}\right\} -4,\ +1$$

$$\begin{aligned}&\Rightarrow (\lambda - 4)(\lambda + 1) = 0 \\ &\Rightarrow \underline{\underline{\lambda = 4 \text{ or } \lambda = -1}}.\end{aligned}$$

(b) Write down the matrix $\mathbf{A}^{-1}$ when $\lambda = 3$. (1)

Solution

$$\lambda = 3 \Rightarrow \det \mathbf{A} = 9 - 9 - 4 = -4$$

and

$$\mathbf{A}^{-1} = \underline{\underline{-\tfrac{1}{4}\begin{pmatrix} 0 & -2 \\ -2 & 3 \end{pmatrix}}}.$$

5. Obtain the solution of the differential equation (5)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^2\mathrm{e}^x,$$

for which $y = 2$ when $x = 1$.

Solution

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^2\mathrm{e}^x &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x\mathrm{e}^x \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \left(-\frac{1}{x}\right) y = x\mathrm{e}^x \end{aligned}$$

$$\begin{aligned} \mathrm{IF} &= \mathrm{e}^{\int -\frac{1}{x}\,\mathrm{d}x} \\ &= \mathrm{e}^{-\ln x} \\ &= \mathrm{e}^{\ln x^{-1}} \\ &= x^{-1} \end{aligned}$$

$$\begin{aligned} &\Rightarrow x^{-1}\frac{\mathrm{d}y}{\mathrm{d}x} - x^{-2}y = \mathrm{e}^x \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(x^{-1}y) = \mathrm{e}^x \\ &\Rightarrow x^{-1}y = \int \mathrm{e}^x\,\mathrm{d}x \\ &\Rightarrow x^{-1}y = \mathrm{e}^x + c \\ &\Rightarrow y = x(\mathrm{e}^x + c). \end{aligned}$$

Now,

$$x = 1, y = 2 \Rightarrow 2 = \mathrm{e} + c \Rightarrow c = 2 - \mathrm{e}$$

and

$$\underline{\underline{y = x(\mathrm{e}^x + 2 - \mathrm{e})}}.$$

6. (a) Express (4)

$$\frac{8}{x(x+2)(x+4)}$$

in partial fractions.

Solution

$$\frac{8}{x(x+2)(x+4)} \equiv \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+4}$$
$$\equiv \frac{A(x+2)(x+4) + Bx(x+4) + Cx(x+2)}{x(x+2)(x+4)}$$

and so

$$8 \equiv A(x+2)(x+4) + Bx(x+4) + Cx(x+2).$$

$\underline{x=0}$: $8 = 8A \Rightarrow A = 1$.
$\underline{x=-2}$: $8 = -4B \Rightarrow B = -2$.
$\underline{x=-4}$: $8 = 8C \Rightarrow B = 1$.
Hence,

$$\frac{8}{x(x+2)(x+4)} \equiv \underline{\underline{\frac{1}{x} - \frac{2}{x+2} + \frac{1}{x+4}}}.$$

(b) Calculate the area under the curve (5)

$$y = \frac{8}{x^3 + 6x^2 + 8x}$$

between $x = 1$ and $x = 2$.
Express your answer in the form $\ln \frac{a}{b}$, where a and b are positive integers.

Solution

$$\int_1^2 \frac{8}{x^3+6x^2+8x}\,\mathrm{d}x = \int_1^2 \frac{8}{x(x+2)(x+4)}\,\mathrm{d}x$$
$$= \int_1^2 \left(\frac{1}{x} - \frac{2}{x+2} + \frac{1}{x+4}\right)\mathrm{d}x$$
$$= \left[\ln|x| - 2\ln|x+2| + \ln|x+4|\right]_{x=1}^{2}$$
$$= (\ln 2 - 2\ln 4 + \ln 6) - (\ln 1 - 2\ln 3 + \ln 5)$$
$$= \ln 2 - \ln 4^2 + \ln 6 + \ln 3^2 - \ln 5$$
$$= \ln\left(\frac{2 \times 6 \times 9}{16 \times 5}\right)$$
$$= \ln\left(\frac{108}{80}\right)$$
$$= \underline{\underline{\ln\left(\tfrac{27}{20}\right)}};$$

hence, $\underline{\underline{a = 27}}$ and $\underline{\underline{b = 20}}$.