Dr Oliver Mathematics
Advance Level Further Mathematics
Further Pure Mathematics 1: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. Use Simpson's rule with 4 intervals to estimate (5)

$$\int_{0.4}^{2} \mathrm{e}^{x^2}\,\mathrm{d}x.$$

Solution

$$\frac{2-0.4}{4} = 0.4.$$

x	0.4	0.8	1.2	1.6	2
e^{x^2}	$\mathrm{e}^{0.16}$	$\mathrm{e}^{0.64}$	$\mathrm{e}^{1.44}$	$\mathrm{e}^{2.56}$	e^{4}

$$\begin{aligned}\int_{0.4}^{2} \mathrm{e}^{x^2}\,\mathrm{d}x &\approx \tfrac{1}{3} \times 0.4 \times \left[\mathrm{e}^{0.16} + 4\mathrm{e}^{0.64} + 2\mathrm{e}^{1.44} + 4\mathrm{e}^{2.56} + \mathrm{e}^{4}\right] \\ &= 16.472\,299\,38 \text{ (FCD)} \\ &= \underline{\underline{16.5 \text{ (3 sf)}}}.\end{aligned}$$

2. Given that k is a real non-zero constant and that (4)

$$y = x^3 \sin kx,$$

use Leibnitz's theorem to show that

$$\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} = (k^2x^2 + A)k^3 x \cos kx + B(k^2x^2 + C)k^2 \sin kx,$$

where A, B, and C are integers to be determined.

Solution

$u = x^3$	$v = \sin kx$
$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$	$\dfrac{\mathrm{d}v}{\mathrm{d}x} = k\cos kx$
$\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} = 6x$	$\dfrac{\mathrm{d}^2v}{\mathrm{d}x^2} = -k^2\sin kx$
$\dfrac{\mathrm{d}^3u}{\mathrm{d}x^3} = 6$	$\dfrac{\mathrm{d}^3v}{\mathrm{d}x^3} = -k^3\cos kx$
$\dfrac{\mathrm{d}^4u}{\mathrm{d}x^4} = 0$	$\dfrac{\mathrm{d}^4v}{\mathrm{d}x^4} = k^4\sin kx$
$\dfrac{\mathrm{d}^5u}{\mathrm{d}x^5} = 0$	$\dfrac{\mathrm{d}^5v}{\mathrm{d}x^5} = k^5\cos kx$

Finally,

$$\begin{aligned}\frac{\mathrm{d}^5y}{\mathrm{d}x^5} &= u\frac{\mathrm{d}^5v}{\mathrm{d}x^5} + 5\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}^4v}{\mathrm{d}x^4} + 10\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\frac{\mathrm{d}^3v}{\mathrm{d}x^3} + 10\frac{\mathrm{d}^3u}{\mathrm{d}x^3}\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 5\frac{\mathrm{d}^4u}{\mathrm{d}x^4}\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}^5u}{\mathrm{d}x^5}v \\ &= k^5x^3\cos kx + 15k^4x^2\sin kx - 60k^3x\cos kx - 60k^2\sin kx \\ &= \underline{\underline{(k^2x^2-60)k^3x\cos kx + 15(k^2x^2-4)k^2\sin kx}};\end{aligned}$$

hence, $\underline{\underline{A = -60}}$, $\underline{\underline{B = 15}}$, and $\underline{\underline{C = -4}}$.

3.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x - y^2 \quad \text{(I)}.$$

(a) Show that (4)

$$\frac{\mathrm{d}^5y}{\mathrm{d}x^5} = ay\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + b\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + c\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2,$$

where a, b, and c are integers to be determined.

Solution

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = x - y^2 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 1 - 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} \\
\Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= -2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2y\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\
\Rightarrow \frac{\mathrm{d}^4y}{\mathrm{d}x^4} &= -4\frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y\,\frac{\mathrm{d}^3y}{\mathrm{d}x^3} \\
\Rightarrow \frac{\mathrm{d}^4y}{\mathrm{d}x^4} &= -6\frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y\,\frac{\mathrm{d}^3y}{\mathrm{d}x^3} \\
\Rightarrow \frac{\mathrm{d}^5y}{\mathrm{d}x^5} &= -6\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right) - 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + y\,\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right) \\
\Rightarrow \underline{\underline{\frac{\mathrm{d}^5y}{\mathrm{d}x^5} = -6\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2 - 8\frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 2y\,\frac{\mathrm{d}^4y}{\mathrm{d}x^4}}};
\end{aligned}
$$

hence, $\underline{\underline{a = -2}}$, $\underline{\underline{b = -8}}$, and $\underline{\underline{c = -6}}$.

(b) Hence find a series solution, in ascending powers of x as far as the term in x^5, of the differential equation (I), given that $y = 1$ at $x = 0$. (5)

Solution

$$
\begin{aligned}
x = 0, y = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -1 \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 3 \\
\Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= -8 \\
\Rightarrow \frac{\mathrm{d}^4y}{\mathrm{d}x^4} &= 34 \\
\Rightarrow \frac{\mathrm{d}^5y}{\mathrm{d}x^5} &= -186.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
y &= 1 + (-1)x + \tfrac{1}{2!}(3)x^2 + \tfrac{1}{3!}(-8)x^3 + \tfrac{1}{4!}(34)x^4 + \tfrac{1}{5!}(-186)x^5 + \dots \\
&= \underline{\underline{1 - x + \tfrac{3}{2}x^2 - \tfrac{4}{3}x^3 + \tfrac{17}{12}x^4 - \tfrac{31}{20}x^5 + \dots}}.
\end{aligned}
$$

4. The parabola C has equation (8)

$$y^2 = 16x.$$

The distinct points $P(p^2, 4p)$ and $Q(q^2, 4q)$ lie on C, where $p \neq q$.

The tangent to C at P and the tangent to C at Q meet at the point $R(-28, 6)$.

Show that the area of triangle PQR is $1\,331$.

Solution

Now, the tangent to C at P:

$$\begin{aligned} y^2 = 16x \Rightarrow 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 16 \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{16}{2y} = \frac{8}{y} \end{aligned}$$

and, at the point $P(p^2, 4p)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8}{4p} = \frac{2}{p}.$$

Now,

$$\begin{aligned} y - 4p = \frac{2}{p}(x - p^2) &\Rightarrow y - 4p = \frac{2}{p}x - 2p \\ &\Rightarrow y = \frac{2}{p}x + 2p \end{aligned}$$

and, similarly, the tangent to C at Q:

$$y = \frac{2}{q}x + 2q.$$

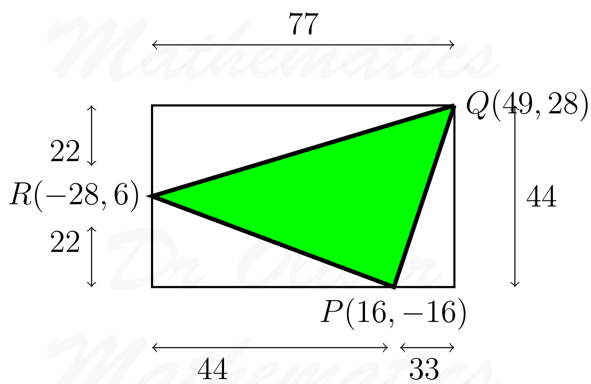
Next,

$$\begin{aligned} x = -28, y = 6 &\Rightarrow 6 = -\frac{56}{p}x + 2p \\ &\Rightarrow 6p = -56 + 2p^2 \\ &\Rightarrow 2p^2 - 6p - 56 = 0 \\ &\Rightarrow p^2 - 3p - 28 = 0 \\ &\Rightarrow (p - 7)(p + 4) = 0 \\ &\Rightarrow p = -4 \text{ or } p = 7; \end{aligned}$$

and, hence,

$$q = -4 \text{ or } q = 7.$$

So, the points are $P(16, -16)$ and $Q(49, 28)$. Now, let us look at a picture:



Hence,

$$\begin{aligned} \text{area} &= (77 \times 44) - (\tfrac{1}{2} \times 44 \times 33) - (\tfrac{1}{2} \times 44 \times 22) - (\tfrac{1}{2} \times 77 \times 22) \\ &= 3\,388 - 726 - 484 - 847 \\ &= \underline{\underline{1\,331}}, \end{aligned}$$

as required.

5. (8)

$$I = \int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x,\ 0 < x < \tfrac{1}{4}\pi.$$

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \tfrac{1}{5}\ln\left(\frac{2 + \tan\left(\frac{x}{2}\right)}{1 - 2\tan\left(\frac{x}{2}\right)}\right) + k,$$

where k is an arbitrary constant.

Solution

Recall:

$$\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ \text{and } \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t.$$

Now,

$$
\begin{aligned}
I &= \int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x \\
&= \int \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{2}{4(1-t^2) - 6t}\,\mathrm{d}t \\
&= \int \frac{2}{4 - 6t - 4t^2}\,\mathrm{d}t \\
&= \int \frac{2}{2(2 - 3t - 2t^2)}\,\mathrm{d}t \\
&= \int \frac{1}{2 - 3t - 2t^2}\,\mathrm{d}t \\
&= \int \frac{1}{(2+t)(1-2t)}\,\mathrm{d}t.
\end{aligned}
$$

Partial fractions:

$$
\begin{aligned}
\frac{1}{(2+t)(1-2t)} &\equiv \frac{A}{2+t} + \frac{B}{1-2t} \\
&\equiv \frac{A(1-2t) + B(2+t)}{(2+t)(1-2t)}
\end{aligned}
$$

which means

$$1 \equiv A(1-2t) + B(2+t).$$

$\underline{x = \frac{1}{2}}$: $1 = \frac{5}{2}B \Rightarrow B = \frac{2}{5}$.
$\underline{x = -2}$: $1 = 5A \Rightarrow A = \frac{1}{5}$.
So

$$\frac{1}{(2+t)(1-2t)} \equiv \frac{\frac{1}{5}}{2+t} + \frac{\frac{2}{5}}{1-2t}.$$

Next,

$$\begin{aligned} I &= \int \left(\frac{\frac{1}{5}}{2+t} + \frac{\frac{2}{5}}{1-2t} \right) \mathrm{d}t \\ &= \tfrac{1}{5} \int \left(\frac{1}{2+t} + \frac{2}{1-2t} \right) \mathrm{d}t \\ &= \tfrac{1}{5} \left(\ln(2+t) - \ln(1-2t) \right) + k \\ &= \tfrac{1}{5} \ln \left(\frac{2+t}{1-2t} \right) + k \\ &= \underline{\underline{\tfrac{1}{5} \ln \left(\frac{2+\tan\left(\frac{x}{2}\right)}{1-2\tan\left(\frac{x}{2}\right)} \right) + k}}, \end{aligned}$$

as required.

6. The concentration of a drug in the bloodstream of a patient, t hours after the drug has been administered, where $t \leqslant 6$, is modelled by the differential equation

$$t^2 \frac{\mathrm{d}^2 C}{\mathrm{d}t^2} - 5t \frac{\mathrm{d}C}{\mathrm{d}t} + 8C = t^3 \quad \text{(I)},$$

where C is measured in micrograms per litre.

(a) Show that the transformation $t = \mathrm{e}^x$ transforms equation (I) into the equation (5)

$$\frac{\mathrm{d}^2 C}{\mathrm{d}x^2} - 6 \frac{\mathrm{d}C}{\mathrm{d}x} + 8C = \mathrm{e}^{3x} \quad \text{(II)}.$$

Solution

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}x} &= \frac{\mathrm{d}C}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{\mathrm{d}C}{\mathrm{d}t} \times \mathrm{e}^x \\ &= t \frac{\mathrm{d}C}{\mathrm{d}t} \end{aligned}$$

and

$$
\begin{aligned}
\frac{\mathrm{d}^2C}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}C}{\mathrm{d}x}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(t\,\frac{\mathrm{d}C}{\mathrm{d}t}\right) \\
&= t\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}C}{\mathrm{d}t}\right) + \frac{\mathrm{d}t}{\mathrm{d}x}\,\frac{\mathrm{d}C}{\mathrm{d}t} \\
&= t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}C}{\mathrm{d}t}\right)\frac{\mathrm{d}t}{\mathrm{d}x} + t\left(\frac{1}{t}\frac{\mathrm{d}C}{\mathrm{d}x}\right) \\
&= t^2\frac{\mathrm{d}^2C}{\mathrm{d}t^2} + \frac{\mathrm{d}C}{\mathrm{d}x}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& t^2\,\frac{\mathrm{d}^2C}{\mathrm{d}t^2} - 5t\,\frac{\mathrm{d}C}{\mathrm{d}t} + 8C = (\mathrm{e}^x)^3 \\
\Rightarrow\quad & \left(\frac{\mathrm{d}^2C}{\mathrm{d}x^2} - \frac{\mathrm{d}C}{\mathrm{d}x}\right) - 5t\left(\frac{1}{t}\frac{\mathrm{d}C}{\mathrm{d}x}\right) + 8C = \mathrm{e}^{3x} \\
\Rightarrow\quad & \frac{\mathrm{d}^2C}{\mathrm{d}x^2} - \frac{\mathrm{d}C}{\mathrm{d}x} - 5\frac{\mathrm{d}C}{\mathrm{d}x} + 8C = \mathrm{e}^{3x} \\
\Rightarrow\quad & \underline{\underline{\frac{\mathrm{d}^2C}{\mathrm{d}x^2} - 6\frac{\mathrm{d}C}{\mathrm{d}x} + 8C = \mathrm{e}^{3x}}},
\end{aligned}
$$

as required.

(b) Hence find the general solution for the concentration C at time t hours. (7)

Solution

Complementary function:

$$m^2 - 6m + 8 = 0 \Rightarrow (m-2)(m-4) = 0 \Rightarrow m = 2 \text{ or } m = 4$$

and hence the complementary function is

$$C = A\mathrm{e}^{2x} + B\mathrm{e}^{4x}.$$

Particular integral: try

$$C = D\mathrm{e}^{3x} \Rightarrow \frac{\mathrm{d}C}{\mathrm{d}x} = 3D\mathrm{e}^{3x} \Rightarrow \frac{\mathrm{d}^2C}{\mathrm{d}x^2} = 9D\mathrm{e}^{3x}.$$

Substitute into the differential equation:

$$9D - 18D + 8D = 1 \Rightarrow D = -1.$$

Hence, the general solution is

$$\begin{aligned} & C = Ae^{2x} + Be^{4x} - e^{3x} \\ \Rightarrow \quad & C = A(e^{x})^2 + B(e^{x})^4 - (e^{x})^3 \\ \Rightarrow \quad & \underline{\underline{C = At^2 + Bt^4 - t^3}}. \end{aligned}$$

Given that when $t = 6$, $C = 0$ and $\dfrac{dC}{dt} = -36$,

(c) find the maximum concentration of the drug in the bloodstream of the patient. (5)

Solution

$$\begin{aligned} t = 6, C = 0 \Rightarrow 0 &= 36A + 1\,296B - 216 \\ \Rightarrow A + 36B &= 6 \quad (1). \end{aligned}$$

Now,

$$\frac{dC}{dt} = 2At + 4Bt^3 - 3t^2$$

and

$$\begin{aligned} t = 6, \frac{dC}{dt} = -36 \Rightarrow -36 &= 12A + 864B - 108 \\ \Rightarrow 12A + 864B &= 72 \\ \Rightarrow A + 72B &= 6 \quad (2). \end{aligned}$$

Do (2) − (1):

$$\begin{aligned} 36B = 0 \Rightarrow B &= 0 \\ \Rightarrow A &= 6 \end{aligned}$$

and

$$C = 6t^2 - t^3.$$

Next,

$$\begin{aligned} \frac{dC}{dt} = 0 \Rightarrow 12t - 3t^2 &= 0 \\ \Rightarrow 3t(4 - t) &= 0 \\ \Rightarrow t = 0 \text{ or } t &= 4. \end{aligned}$$

Finally, the maximum concentration of the drug in the bloodstream of the patient is

$$C = 6(4^2) - (4^3) = \underline{\underline{32 \text{ micrograms per litre}}}.$$

7. With respect to a fixed origin O, the points A, B, and C have coordinates $(3, 4, 5)$, $(10, -1, 5)$, and $(4, 7, -9)$. respectively.

The plane Π has equation

$$4x - 8y + z = 2.$$

The line segment AB meets the plane Π at the point P and the line segment BC meets the plane Π at the point Q.

(a) Show that, to 3 significant figures, the area of quadrilateral $APQC$ is 38.5. (6)

Solution

Now, the line through AB is

$$\begin{aligned}\mathbf{r} &= 3\mathbf{i}+4\mathbf{j}+5\mathbf{k} + \lambda(7\mathbf{i} - 5\mathbf{j})\\ &= (3 + 7\lambda)\mathbf{i} + (4 - 5\lambda)\mathbf{j} + 5\mathbf{k}.\end{aligned}$$

Next,

$$\begin{aligned}4x - 8y + z = 2 &\Rightarrow 4(3 + 7\lambda) - 8(4 - 5\lambda) + 5 = 2\\ &\Rightarrow 12 + 28\lambda - 32 + 40\lambda + 5 = 2\\ &\Rightarrow 68\lambda = 17\\ &\Rightarrow \lambda = \tfrac{1}{4}\end{aligned}$$

which means that $P(4.75, 2.75, 5)$.

The line through BC is

$$\begin{aligned}\mathbf{r} &= 10\mathbf{i}-\mathbf{j}+5\mathbf{k} + \mu(-6\mathbf{i}+8\mathbf{j}-14\mathbf{k})\\ &= (10 - 6\mu)\mathbf{i} + (-1 + 8\mu)\mathbf{j} + (5 - 14\mu)\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}4x - 8y + z = 2 &\Rightarrow 4(10 - 6\mu) - 8(-1 + 8\mu) + (5 - 14\mu) = 2\\ &\Rightarrow 40 - 24\mu + 8 - 64\mu + 5 - 14\mu = 2\\ &\Rightarrow 102\mu = 51\\ &\Rightarrow \mu = \tfrac{1}{2}\end{aligned}$$

which means that $Q(7, 3, -2)$.

Finally,

$$\begin{aligned}\text{area of quadrilateral } APQC &= \tfrac{1}{2}\left|\overrightarrow{AQ} \times \overrightarrow{PC}\right| \\ &= \tfrac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & -7 \\ -0.75 & 4.25 & -14 \end{vmatrix} \\ &= \tfrac{1}{2}\left|43.75\mathbf{i} + 61.25\mathbf{j} + 16.5\mathbf{k}\right| \\ &= \tfrac{1}{2}\sqrt{43.75^2 + 61.25^2 + 16.5^2} \\ &= 38.528\,804\,16 \text{ (FCD)} \\ &= \underline{\underline{38.5 \text{ (3 sf)}}},\end{aligned}$$

as required.

The point D has coordinates $(k, 4, -1)$, where k is a constant.
Given that the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ form three edges of a parallelepiped of volume 226,

(b) find the possible values of the constant k. (4)

Solution
The volume of the parallelepiped is

$$\left|\overrightarrow{AB}.(\overrightarrow{AC} \times \overrightarrow{AD})\right|.$$

Now,

$$\begin{aligned} &\begin{vmatrix} 7 & -5 & 0 \\ 1 & 3 & -14 \\ k-3 & 0 & -6 \end{vmatrix} = 226 \\ \Rightarrow\quad &|7(-18-0) + 5[-6 + 14(k-3)] + 0| = 226 \\ \Rightarrow\quad &|-126 - 30 + 70k - 210| = 226 \\ \Rightarrow\quad &|70k - 366| = 226 \\ \Rightarrow\quad &70k - 366 = -226 \text{ or } 70k - 366 = 226 \\ \Rightarrow\quad &70k = 140 \text{ or } 70k = 592 \\ \Rightarrow\quad &\underline{\underline{k = 2 \text{ or } k = 8\tfrac{16}{35}}}.\end{aligned}$$

8. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

The line l_1 is the tangent to H at the point $P(4\cosh\theta, 3\sinh\theta)$.

The line l_1 meets the x-axis at the point A.

The line l_2 is the tangent to H at the point $(4, 0)$.

The lines l_1 and l_2 meet at the point B and the midpoint of AB is the point M.

(a) Show that, as θ varies, a Cartesian equation for the locus of M is (11)

$$y^2 = \frac{9(4-x)}{4x},\ p < x < q,$$

where p and q are values to be determined.

Solution

Let $x = 4\cosh\theta$ and $y = 3\sinh\theta$. Then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4\sinh\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = 3\cosh\theta.$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}\\ &= \frac{3\cosh\theta}{4\sinh\theta}\end{aligned}$$

and the equation of the tangent to H is

$$y - 3\sinh\theta = \frac{3\cosh\theta}{4\sinh\theta}(x - 4\cosh\theta).$$

Next,

$$\begin{aligned}y = 0 &\Rightarrow -3\sinh\theta = \frac{3\cosh\theta}{4\sinh\theta}(x - 4\cosh\theta)\\ &\Rightarrow -12\sinh^2\theta = 3\cosh\theta(x - 4\cosh\theta)\\ &\Rightarrow -12\sinh^2\theta = 3x\cosh\theta - 12\cosh^2\theta\\ &\Rightarrow 12\cosh^2\theta - 12\sinh^2\theta = 3x\cosh\theta\\ &\Rightarrow 12(\cosh^2\theta - \sinh^2\theta) = 3x\cosh\theta\\ &\Rightarrow 12 = 3x\cosh\theta\\ &\Rightarrow x = 4\operatorname{sech}\theta\end{aligned}$$

and so $A(4\operatorname{sech}\theta, 0)$. Now, line l_2 has equation $x = 4$ (why?) and

$$\begin{aligned}
x = 4 \Rightarrow y - 3\sinh\theta &= \frac{3\cosh\theta}{4\sinh\theta}(4 - 4\cosh\theta) \\
\Rightarrow y &= 3\sinh\theta + \frac{12\cosh\theta - 12\cosh^2\theta}{4\sinh\theta} \\
\Rightarrow y &= \frac{12\cosh\theta + 12\sinh^2\theta - 12\cosh^2\theta}{4\sinh\theta} \\
\Rightarrow y &= \frac{12\cosh\theta - 12(\cosh^2\theta - \sinh^2\theta)}{4\sinh\theta} \\
\Rightarrow y &= \frac{12\cosh\theta - 12}{4\sinh\theta} \\
\Rightarrow y &= \frac{3\cosh\theta - 3}{\sinh\theta}.
\end{aligned}$$

The coordinates of M are

$$\left(\frac{4 + 4\operatorname{sech}\theta}{2}, \frac{3\cosh\theta - 3}{2\sinh\theta}\right).$$

Now,

$$\begin{aligned}
x = \frac{4 + 4\operatorname{sech}\theta}{2} &\Rightarrow x = 2 + 2\operatorname{sech}\theta \\
&\Rightarrow x - 2 = 2\operatorname{sech}\theta \\
&\Rightarrow \operatorname{sech}\theta = \frac{x - 2}{2} \\
&\Rightarrow \cosh\theta = \frac{2}{x - 2}
\end{aligned}$$

and, finally(!),

$$\begin{aligned} y^2 &= \left(\frac{3\cosh\theta - 3}{2\sinh\theta}\right)^2 \\ &= \frac{9(\cosh\theta - 1)^2}{4\sinh^2\theta} \\ &= \frac{9(\frac{2}{x-2} - 1)^2}{4\left(\left(\frac{2}{x-2}\right)^2 - 1\right)} \\ &= \frac{9(\frac{2}{x-2} - 1)^2}{4\left(\left(\frac{2}{x-2}\right) - 1\right)\left(\left(\frac{2}{x-2}\right) + 1\right)} \\ &= \frac{9(\frac{2}{x-2} - 1)}{4\left(\left(\frac{2}{x-2}\right) + 1\right)} \\ &= \frac{9(2 - (x-2))}{4\left(2 + (x-2)\right)} \\ &= \underline{\underline{\frac{9(4-x)}{4x}}}, \end{aligned}$$

as required. Moreover, $\underline{\underline{p = 2}}$ and $\underline{\underline{q = 4}}$.

Let S be the focus of H that lies on the positive x-axis.

(b) Show that the distance from M to S is greater than 1. (3)

Solution

$$\begin{aligned} b^2 = a^2(e^2 - 1) &\Rightarrow 9 = 16(e^2 - 1) \\ &\Rightarrow e^2 - 1 = \tfrac{9}{16} \\ &\Rightarrow e^2 = \tfrac{25}{16} \\ &\Rightarrow e = \tfrac{5}{4} \end{aligned}$$

and the focus of H is

$$x = ae = 5.$$

Now,

$$\begin{aligned} \text{distance of } M &> 5 - 4 \\ &= 1 \end{aligned}$$

and so the distance from M to S is greater than $\underline{\underline{1}}$.