Dr Oliver Mathematics
Understanding Derivatives

Contents

1 Introduction

The purpose of these notes is to allow you to check that you fully understand what the first and second derivatives of a function tell you about the original function. It is not enough just to be able to calculate derivatives: you need to understand their graphical implications. In each of the examples the original function will be drawn in black, the first derivative will be drawn in blue, and the second derivative will be drawn in red; the three graphs will be drawn one above the other so that you can see how they relate to each other.

2 Critical Points

A function $y = f(x)$ has a *critical point* when $x = a$ if either $f'(a) = 0$ or $f'(a)$ is undefined; in that case, $x = a$ is a *critical value*. When trying to establish the extrema on a closed interval it is vital to consider both types of critical points as the following example shows.

Suppose that

$$y = 2x - 3x^{\frac{2}{3}}.$$

Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - 2x^{-\frac{1}{3}}.$$

If we only solve for

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

then we only get one critical value, namely $x = 1$. But, if we consider the values for which the first derivative is undefined then we get a second critical value, namely $x = 0$.

Note that any critical value must be in the function's domain: the derivative of the function $f(x) = \frac{1}{x}$ is not defined when $x = 0$ but this is not a critical value as 0 is not in the function's domain.

3 The Extreme Value Theorem

Theorem 3.1. Let $f(x)$ be a continuous function on the closed interval $[a, b]$, i.e., $a \leqslant x \leqslant b$. Then $f(x)$ has both a maximum value and a minimum value on the interval.

Neither the maximum nor the minimum value need occur at only one place: they could be attained on multiple occasions. There is a simply procedure for finding the maximum and minimum values on a closed interval:

1. Find the critical values of $f(x)$ on the open interval (a, b), i.e., on $a < x < b$.
2. Evaluate $f(x)$ at each critical value.
3. Evaluate $f(a)$ and $f(b)$.
4. The greatest value that you have found is the maximum value and the smallest number you have found is the minimum value.

Thus, in the example of $y = 2x - 3x^{\frac{2}{3}}$, if we want to find the maximum and minimum values that the function takes in the interval $[-1, 3]$ then we need to evaluate the function at two

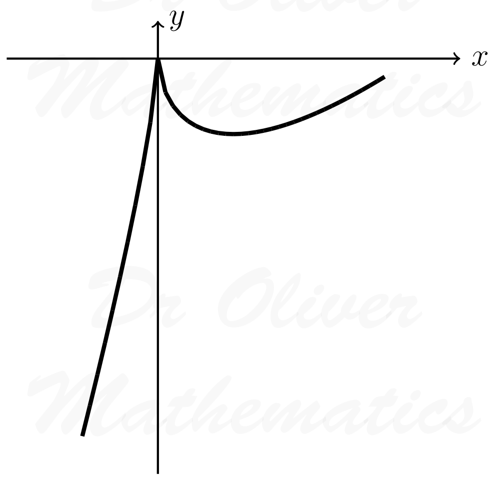

Figure 1: the graph of $y = 2x - 3x^{\frac{2}{3}}$ on the interval $[-1, 3]$

critical values and the endpoints of the interval.

$$\begin{aligned} f(-1) &= -5 \\ f(0) &= 0 \\ f(1) &= -1 \\ f(3) &= -0.240 \text{ (3 sf)}. \end{aligned}$$

So the maximum value in the interval is 0 and the minimum is -5.

4 Increasing and Decreasing Functions, Stationary Points, and Points of Inflexion

A function $y = f(x)$ is

(a) *increasing* at the point $x = a$ if $f'(a) > 0$; in other words, as you move along the graph to the right the graph is moving up the page,

(b) *decreasing* at the point $x = a$ if $f'(a) < 0$; in other words, as you move along the graph to the right the graph is moving down the page,

(c) *stationary* at the point $x = a$ if $f'(a) = 0$; in other words, the tangent to the curve is horizontal.

When $f'(x) = 0$ we have a point that is a *candidate* point to be either a local maximum or minimum. But note that the fact that $f'(x) = 0$ on its own tells us nothing other than the gradient of the tangent to the curve is zero at that point.

(d) If $f'(x) = 0$ and $f''(x) < 0$ then we have a *local maximum*,

(e) if $f'(x) = 0$ and $f''(x) > 0$ then we have a *local minimum*,

(f) if $f'(x) = 0$ and $f''(x) = 0$ then we do not have any new information about the behaviour of the graph when $x = a$.

We can, however, use the second derivative to classify *points of inflexion*. Informally, these are points on a curve where, if you were trying to sketch it with your hand moving inside the curve, you would want to turn the page around; examples of such points would be the origin on $y = x^3$ and the points where the graphs of $y = \sin x$ and $y = \cos x$ intersect the x-axis.

(g) If $f''(x)$ changes sign across the point $x = a$ then we have a *point of inflexion* at $x = a$.

A point of inflexion can be stationary (as in $y = x^3$) or not (as in $y = \sin x$). Moreover, the requirement that the second derivative changes sign does not mean that it has to be zero; the second derivate of $y = \sqrt[3]{x}$ is not defined when $x = 0$ but it has a different sign on either side of the origin and so it is a point of inflexion.

5 $y = x^2$

If $y = x^2$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2.$$

The three graphs have been plotted, one above the other, in Figure 2 on page 5.

(a) The function is decreasing for all $x < 0$ and this is why the graph of the the first derivative is negative for all $x < 0$.

(b) The function is increasing for all $x > 0$ and this is why the graph of the the first derivative is negative for all $x > 0$.

(c) The function has a stationary point when $x = 0$ and this is why the graph of the first derivative is zero when $x = 0$. (Note that the converse of this statement, i.e., "the first derivative is zero and so there is a stationary point", is false.)

(d) The first derivative is increasing for all values of x. This means that the gradient of the tangent to the curve is increasing for all values of x.

(e) The first derivative is increasing at a uniform rate precisely because the second derivative is a constant function.

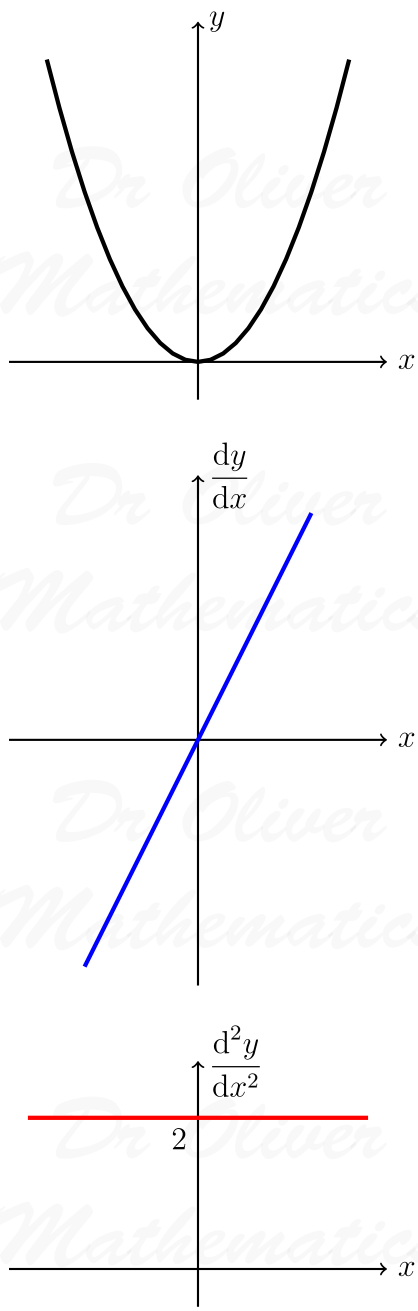

Figure 2: the graphs of $y = x^2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$

6 $y = x(x-2)^2$

If $y = x(x-2)^2$ then we can multiply out the bracket to get $y = x^3 - 4x^2 + 4x$ and then we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 8x + 4 \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 8.$$

The three graphs have been plotted, one above the other, in Figure 3 on page 7.

(a) The function is increasing for $x < \frac{2}{3}$ and for $x > 2$ since the first derivative is positive for these values.

(b) The function is decreasing for $\frac{2}{3} < x < 2$ since the first derivative is negative for these values.

(c) Every cubic graph has precisely one point of inflexion. That is because if

$$y = ax^3 + bx^2 + cx + d$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 2bx + c$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6ax + 2b$$

and so the second derivative changes sign across $x = -\dfrac{b}{3a}$.

(d) The function has a local maximum at $x = \frac{2}{3}$ and this is confirmed by the fact that the first derivative is zero and the second derivative is negative.

(e) The function has a non-stationary point of inflexion at $x = \frac{4}{3}$: this is confirmed by the fact that the first derivative has a non-zero local minimum here (and so the gradient of the tangent to the curves achieves its most negative value) and the second derivative changes sign across this point.

(f) The function has a local minimum at $x = 2$ and this is confirmed by the fact that the first derivative is zero and the second derivative is positive.

7 $y = \sin x$

If $y = \sin x$ (and recall that the absence of a degree symbol means that we are working in radians) then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\sin x.$$

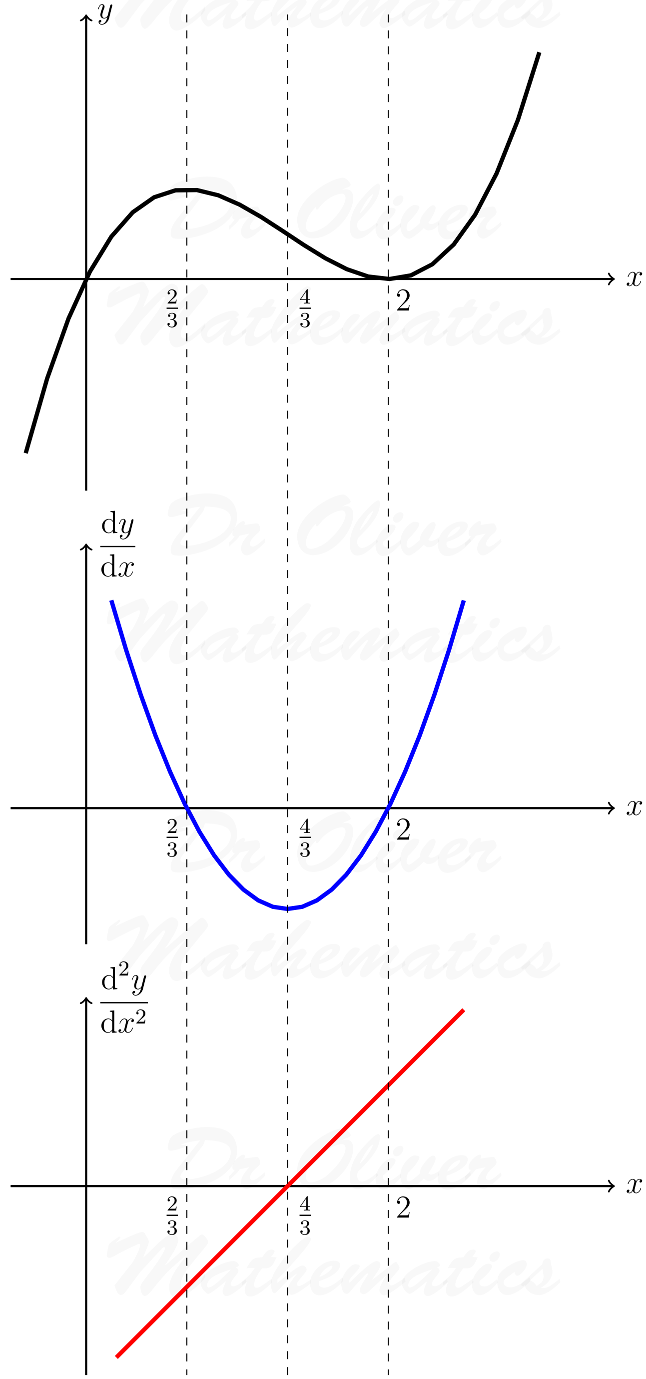

Figure 3: the graphs of $y = x^3 - 4x^2 + 4x$, $\dfrac{dy}{dx} = 3x^2 - 8x + 4$, and $\dfrac{d^2y}{dx^2} = 6x - 8$

The three functions are shown, one above the other, in Figure 4 on page 9: we have restricted our attention to the domain $[-\pi, \pi]$.

(a) The function is increasing between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ and this explains why the first derivative is positive between these values.

(b) The function is decreasing to the left of $x = -\frac{\pi}{2}$ and to the right of $x = \frac{\pi}{2}$ and this explains why the first derivative is positive between these values.

(c) The function clearly has a local maximum when $x = \frac{\pi}{2}$ and this corresponds to the first derivative being zero and the second derivative being negative.

(d) The function clearly has a local minimum when $x = -\frac{\pi}{2}$ and this corresponds to the first derivative being zero and the second derivative being positive.

(e) Since the second derivative is simply the negative of the original function the second derivative changes sign precisely when the function does and so we have points of inflexion where the sine curve cuts the x-axis.

(f) At the points of inflexion the curve has its most positive (negative) slope and the first derivative has a local maximum (minimum) at these points.

(g) Since the sine function is odd we could deduce the behaviour of the function for values $x < 0$ simply by knowing what happens when $x > 0$.

8 $y = e^{-\frac{1}{2}x^2}$

Let

$$f(x) = e^{-\frac{1}{2}x^2}.$$

Then, using the chain rule, we have

$$f'(x) = -xe^{-\frac{1}{2}x^2}.$$

A second application of the chain rule, together with an application of the product rule, gives

$$\begin{aligned} f''(x) &= (-x) \times \left(-xe^{-\frac{1}{2}x^2}\right) + (-1) \times \left(-xe^{-\frac{1}{2}x^2}\right) \\ &= x^2 e^{-\frac{1}{2}x^2} - e^{-\frac{1}{2}x^2} \\ &= \left(x^2 - 1\right) e^{-\frac{1}{2}x^2}. \end{aligned}$$

We now plot these three graphs on above the other to see the relationship between the function and its first two derivatives (see Figure 5 on page 11).

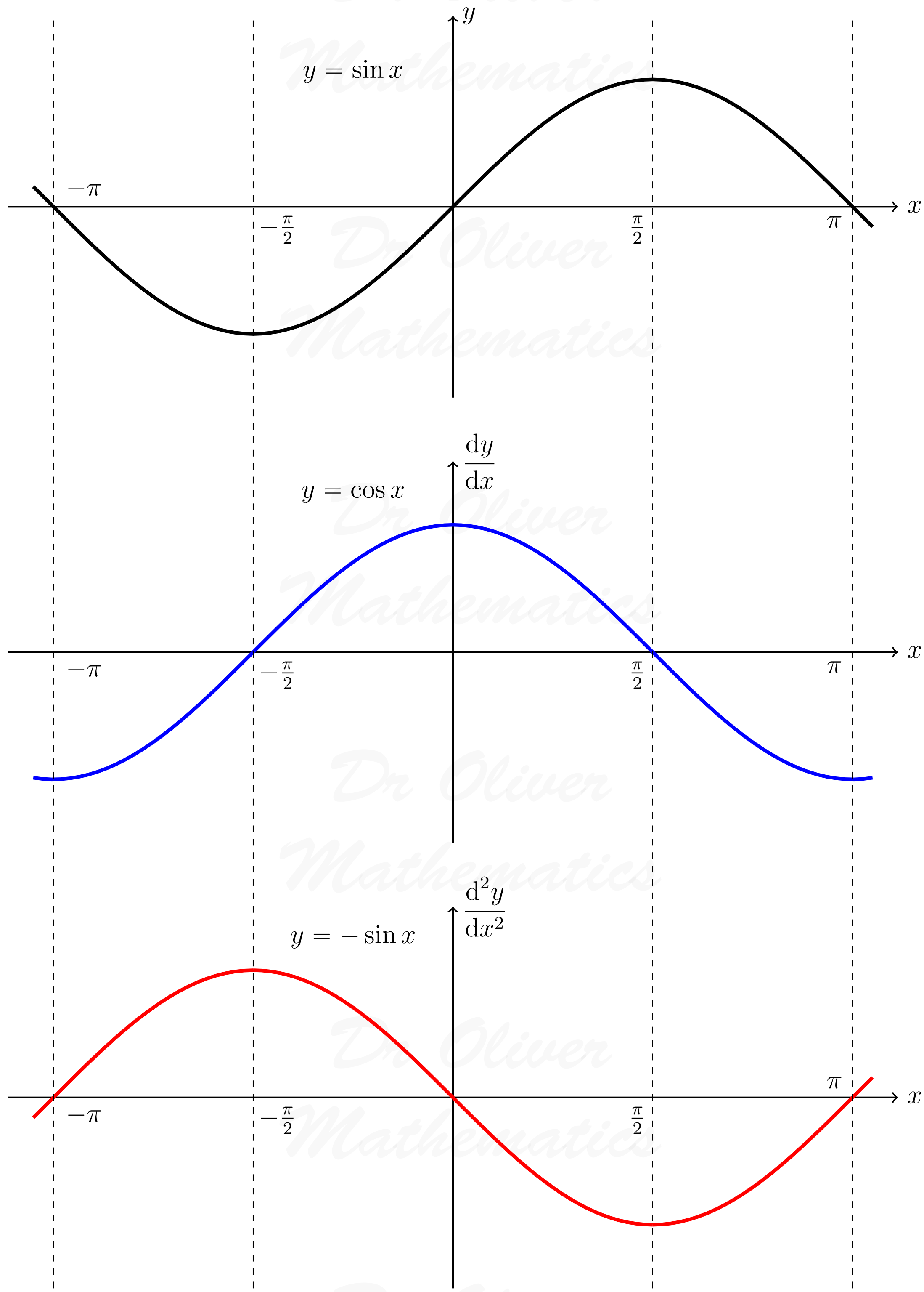

Figure 4: the graphs of $y = \sin x$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos x$, and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\sin x$

(a) The function is clearly decreasing for all $x > 0$ and this is why the graph of the first derivative is always below the x-axis for $x > 0$.

(b) The function is clearly increasing for all $x > 0$ and this is why the graph of the first derivative is always above the x-axis for $x > 0$.

(c) We can identify three immediate points of interest from the graph: a local maximum at $(0, 1)$ and two non-stationary points of inflexion.

(d) When $x = 0$ we see that the graph of the first derivative is zero, supporting the idea that we have a stationary point. This is confirmed when we look at the graph of the second derivative: the second derivative is negative when $x = 0$ and this confirms that we do, in fact, have a local maximum.

(e) On the graph of the original function we have a non-stationary point to the right of the y-axis. Here, the graph has a point where the gradient of the tangent to the curve achieves a local minimum, i.e., it attains its most negative value. This is supported by the fact that the first derivative achieves a local minimum for this value of x. Finally, the second derivative changes sign across this x-value, confirming that we genuinely have a point of inflexion at this point. It is only now that we can solve the equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ and establish that the point of inflexion occurs precisely when $x = 1$.

(f) In the same way, it appears that there is a point of inflexion to the left of the y-axis. Here the graph has a point where the gradient of the curve achieves a local maximum, i.e., it attains its most positive value. This is supported by the fact that the first derivative achieves a local maximum for this value of x. Finally, the second derivative changes sign across this x-value, confirming that we genuinely have a point of inflexion at this point. It is only now that we can solve the equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ and establish that the point of inflexion occurs precisely when $x = -1$.

(g) Of course, in this case we could exploit the fact that the function is even (and so is symmetrical about the y-axis) in order to confirm the existence of the second point of inflexion.

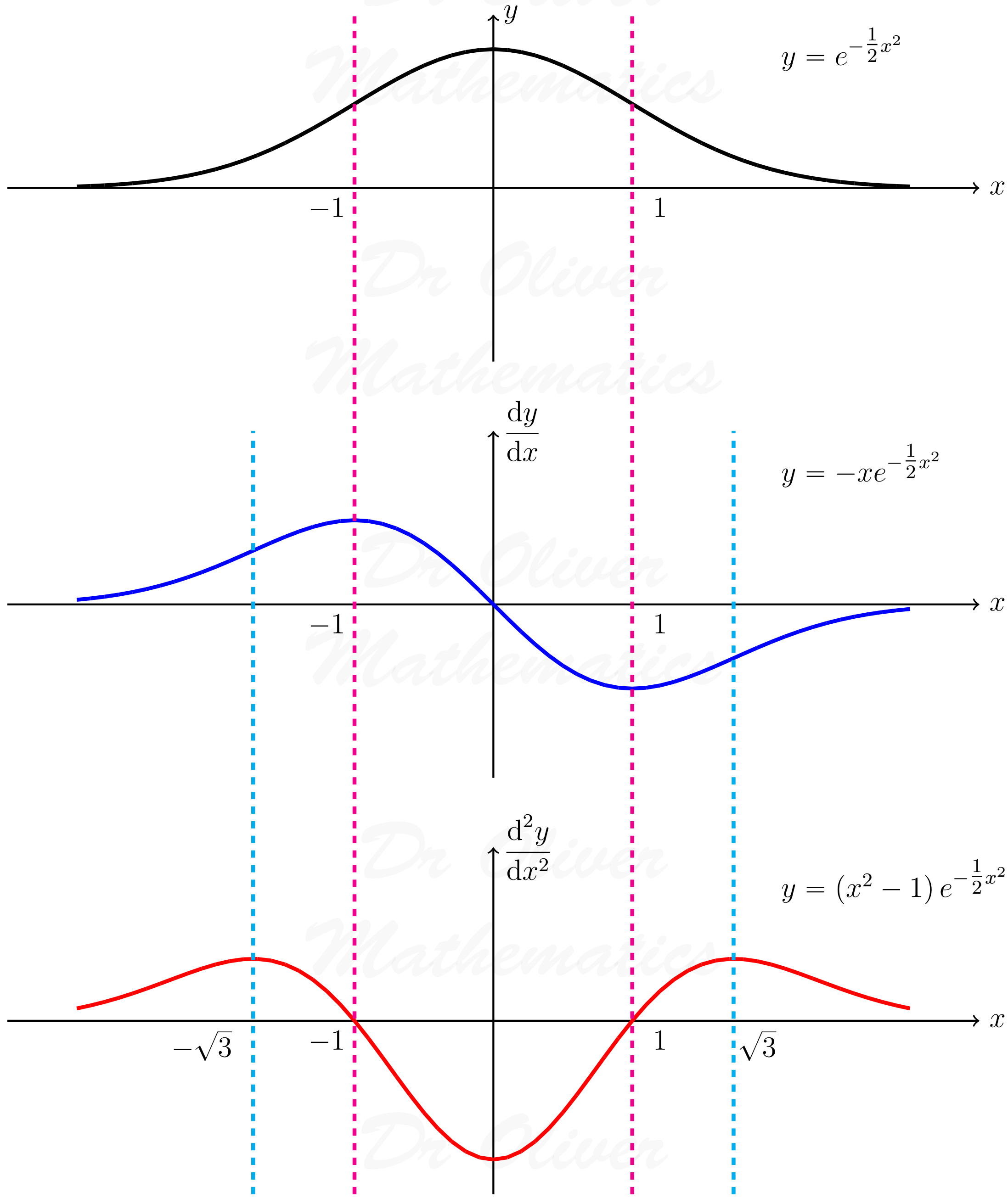

Figure 5: the graphs of $y = e^{-\frac{1}{2}x^2}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -xe^{-\frac{1}{2}x^2}$, and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(x^2-1\right)e^{-\frac{1}{2}x^2}$

9 $y = \dfrac{16x}{x^2+16}$

This is, because of its shape, known as the serpentine curve. Let $y = \frac{16x}{x^2+16}$. Then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{16(x^2+16) - 2x \times 16x}{(x^2+16)^2} \\ &= \frac{16(x^2+16) - 32x^2}{(x^2+16)^2} \\ &= \frac{256 - 16x^2}{(x^2+16)^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{-32x(x^2+16)^2 - 4x(x^2+16)(256-16x^2)}{(x^2+16)^4} \\ &= \frac{-32x\left[(x^2+16) + 2(16-x^2)\right]}{(x^2+16)^3} \\ &= \frac{-32x(48-x^2)}{(x^2+16)^3}.\end{aligned}$$

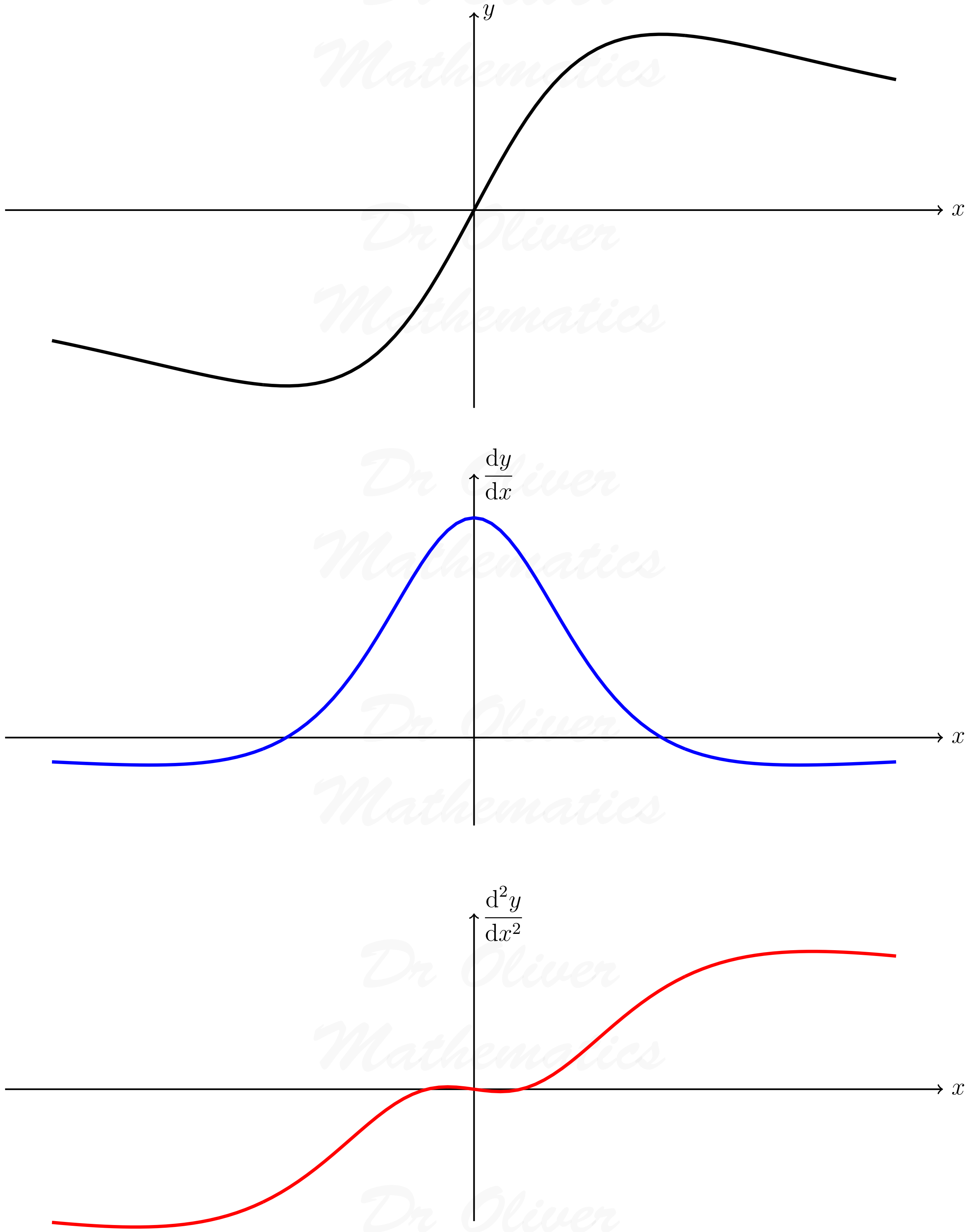

Figure 6: the graphs of $y = \dfrac{16x}{x^2+16}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{256-16x^2}{(x^2+16)^2}$, and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{-32x(48-x^2)}{(x^2+16)^3}$

10 $y = 2\cos x + \sin 2x$

Let $y = 2\cos x + \sin 2x$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\sin x + 2\cos 2x \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\cos x - 4\sin 2x.$$

We now plot these three graphs on above the other to see the relationship between the function and its first two derivatives (see Figure 7 on page 15). Note the function has a period of 2π and so we have drawn the portion of the graph with domain $[0, 2\pi]$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow -2\sin x + 2\cos 2x = 0 \\
&\Rightarrow -2\sin x + 2\left(1 - 2\sin^2 x\right) = 0 \\
&\Rightarrow -2\sin x + 2 - 4\sin^2 x = 0 \\
&\Rightarrow -2\left(2\sin^2 x + \sin x - 1\right) \\
&\Rightarrow -2(2\sin x - 1)(\sin x + 1) = 0 \\
&\Rightarrow \sin x = \tfrac{1}{2} \text{ or } \sin x = -1 \\
&\Rightarrow x = \tfrac{\pi}{6}, \tfrac{5\pi}{6}, \tfrac{3\pi}{2}.
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 &\Rightarrow -2\cos x - 4\sin 2x = 0 \\
&\Rightarrow -2\cos x - 8\sin x\cos x = 0 \\
&\Rightarrow -2\cos x(1 + 4\sin x) = 0 \\
&\Rightarrow \cos x = 0 \text{ or } \sin x = -\tfrac{1}{4} \\
&\Rightarrow x = \tfrac{\pi}{2}, 3.39 \text{ (3 sf)}, \tfrac{3\pi}{2}, 6.03 \text{ (3 sf)}
\end{aligned}$$

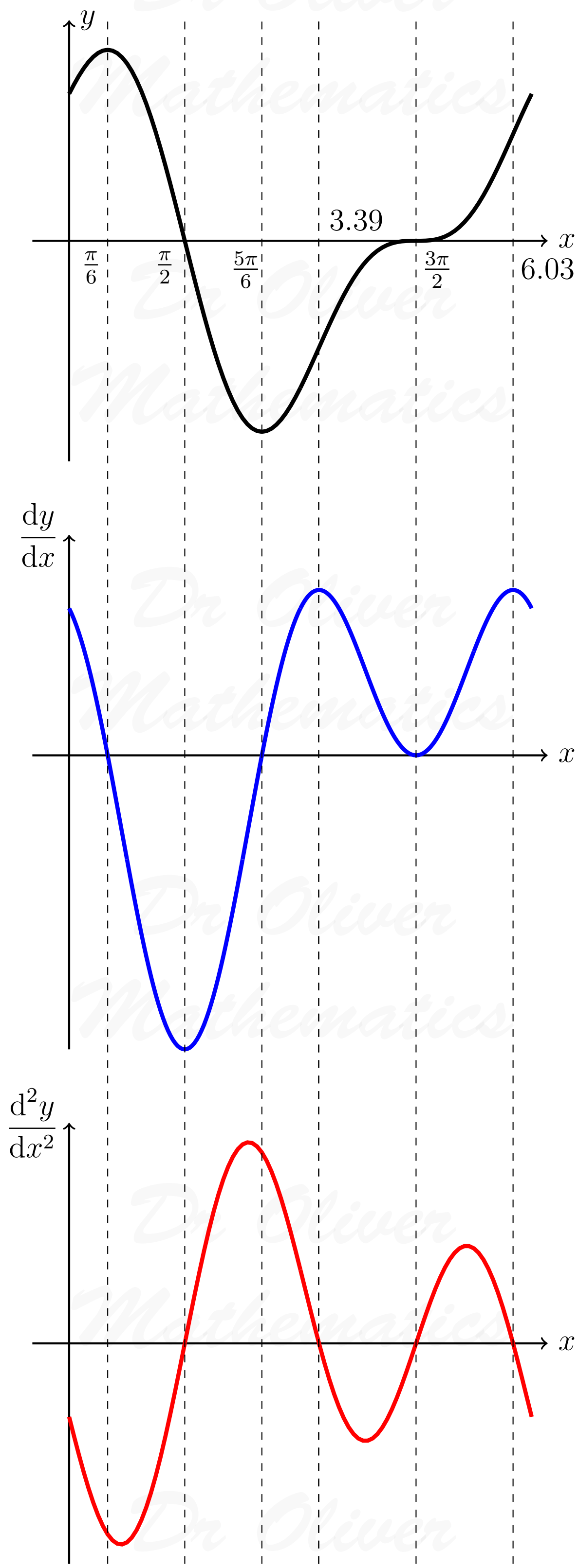

Figure 7: $y = 2\cos x + \sin 2x$, $\dfrac{dy}{dx} = -2\sin x + 2\cos 2x$, and $\dfrac{d^2y}{dx^2} = -2\cos x - 4\sin 2x$

11 $y = \mathrm{f}'(x)$

Suppose we *do not* tell you $y = \mathrm{f}(x)$ but we *do* tell you $y = \mathrm{f}'(x)$.

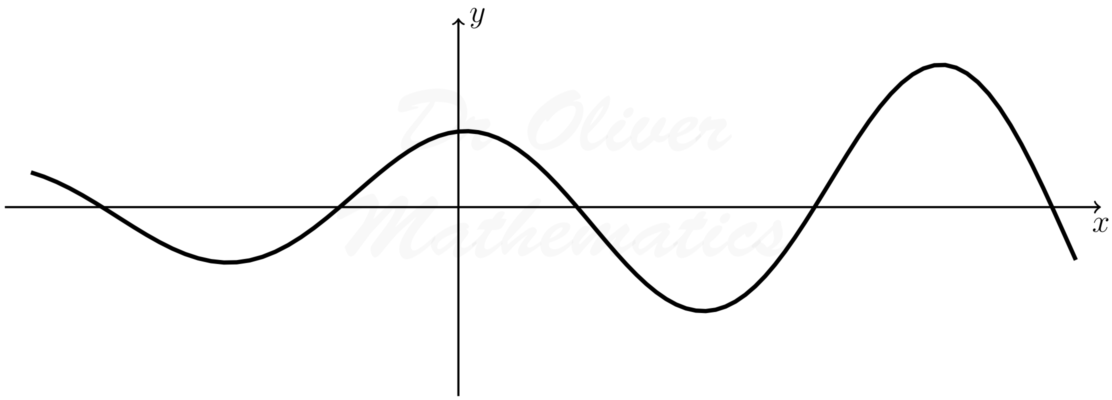

Figure 8: $y = \mathrm{f}'(x)$

So, what do we do? Well, we look to see what are the critical values.

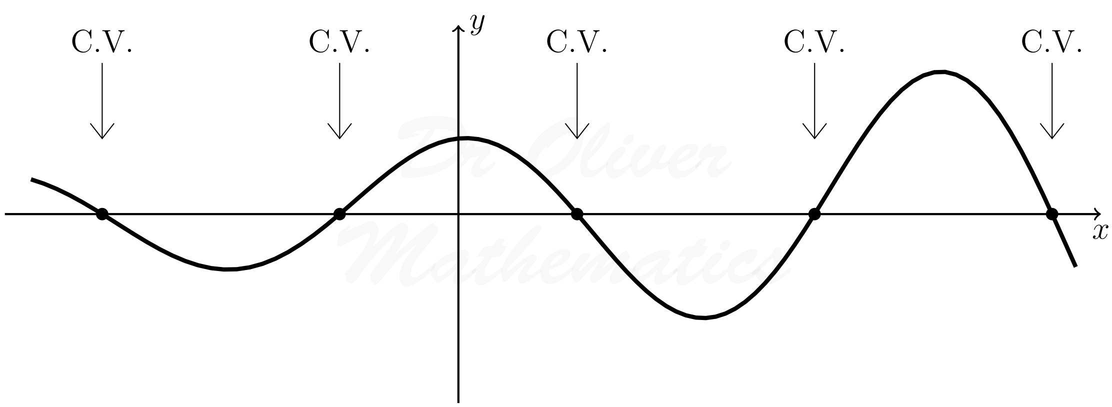

Figure 9: critical values of $y = \mathrm{f}'(x)$

When is $y = \mathrm{f}'(x)$ increasing and when is it decreasing? And what are its stationary points?

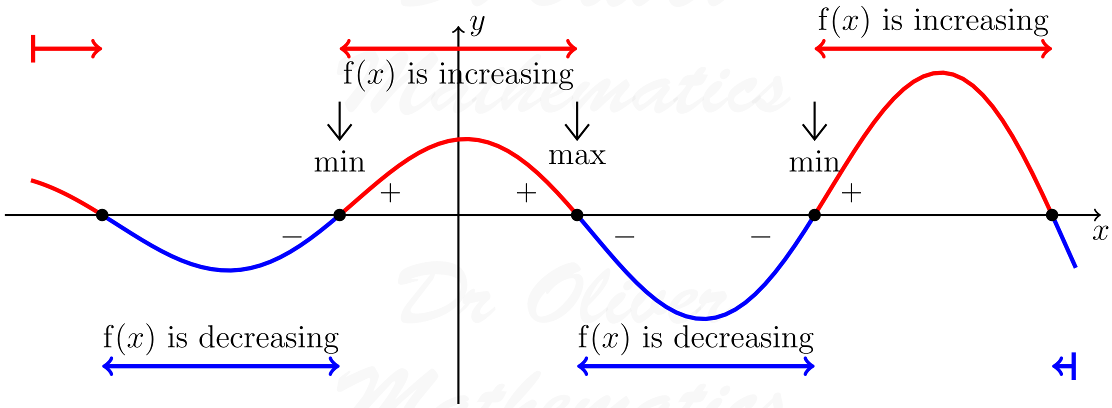

Figure 10: when is $y = \mathrm{f}'(x)$ increasing and when is it decreasing?

What are the inflexion points?

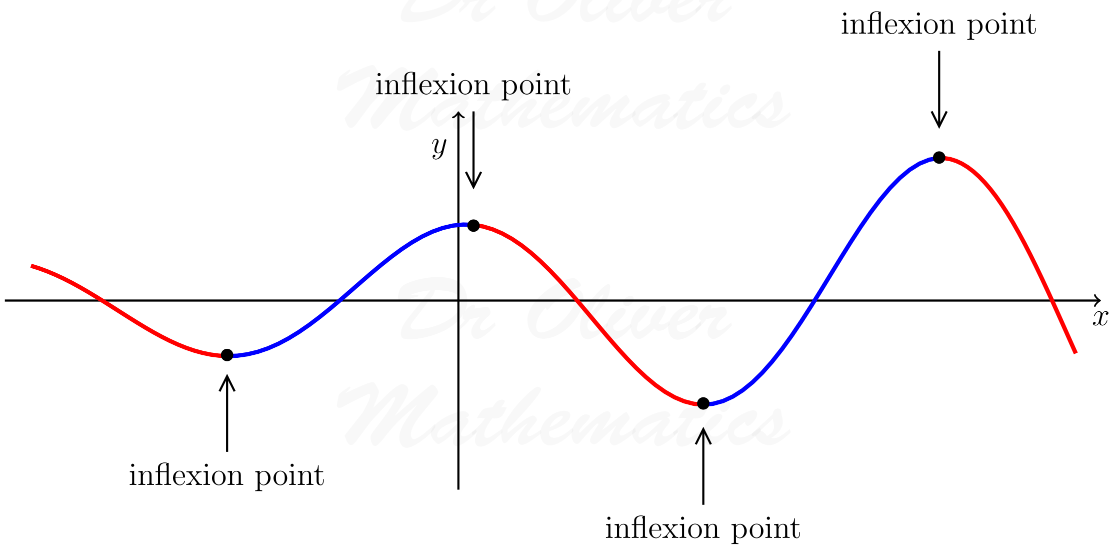

Figure 11: the inflexion points

What are the points of concavity?

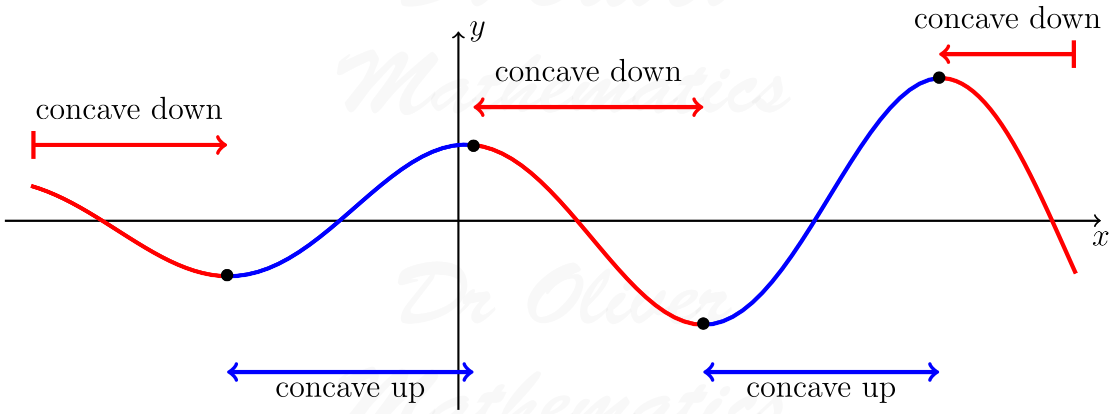

Figure 12: the points of concavity