# Dr Oliver Mathematics
# Mathematics
# Numerical Methods
# Past Examination Questions

This booklet consists of 23 questions across a variety of examination topics.
The total number of marks available is 197.

1.

$$f(x) = 3e^x - \tfrac{1}{2}\ln x - 2, x > 0.$$

(a) Differentiate to find $f'(x)$. (3)

The curve with equation $y = f(x)$ has a turning point at $P$. The $x$-coordinate of $P$ is $\alpha$.

(b) Show that $\alpha = \frac{1}{6}e^{-\alpha}$. (2)

The iterative formula

$$x_{n+1} = \tfrac{1}{6}e^{-x_n},\ x_0 = 1$$

is used to find an approximate value for $\alpha$.

(c) Calculate the values of $x_1$, $x_2$, $x_3$, and $x_4$, giving your answers to 4 decimal places. (2)

(d) By considering the change of sign of $f'(x)$ in a suitable interval, prove that $\alpha = 0.1443$, correct to 4 decimal places. (2)

2.

$$f(x) = 2x^3 - x - 4.$$

(a) Show that the equation $f(x) = 0$ can be written as (3)

$$x = \sqrt{\frac{2}{x} + \frac{1}{2}}.$$

The equation $2x^3 - x - 4 = 0$ has a root between 1.35 and 1.4.

(b) Use the iteration formula (3)

$$x_{n+1} = \sqrt{\frac{2}{x_n} + \frac{1}{2}},$$

with $x_0 = 1.35$, to find, to 2 decimal places, the values of $x_1$, $x_2$, and $x_3$.

The only real root of $f(x) = 0$ is $\alpha$.

(c) By choosing a suitable interval, prove that $\alpha = 1.392$, to 3 decimal places. (3)

3.

$$y = (2x-1)\tan 2x,\ 0 \leqslant x < \tfrac{\pi}{4}.$$

The curve has a minimum at the point $P$. The $x$-coordinate of $P$ is $k$.

(a) Show that $k$ satisfies the equation (6)

$$4k + \sin 4k - 2 = 0.$$

The iterative formula

$$x_{n+1} = \tfrac{1}{4}(2 - \sin 4x_n),\ x_0 = 0.3,$$

is used to find an approximate value for $k$.

(b) Calculate the values of $x_1$, $x_2$, $x_3$, and $x_4$, giving your answers to 4 decimal places. (3)

(c) Show that $k = 0.277$, correct to 3 significant figures. (2)

4. The iterative formula

$$x_{n+1} = -\tfrac{1}{2}e^{x_n},\ x_0 = -0.3,$$

is used to find an approximate value for $k$.

(a) Calculate the values of $x_1$ and $x_2$, giving your answers to 4 decimal places. (2)

(b) Find the value of $k$ to 3 decimal places. (2)

5.

$$f(x) = -x^3 + 3x^2 - 1.$$

(a) Show that the equation $f(x) = 0$ can be rewritten as (2)

$$x = \sqrt{\frac{1}{3-x}}.$$

(b) Starting with $x_1 = 0.6$, use the iteration (2)

$$x_{n+1} = \sqrt{\frac{1}{3-x_n}}$$

to calculate the values of $x_2$, $x_3$, and $x_4$, giving all your answers to 4 decimal places.

(c) Show that $x = 0.653$ is a root of $f(x) = 0$ correct to 3 decimal places. (3)

6.

$$f(x) = \ln(x+2) - x + 1,\ x > -2,\ x \in \mathbb{R}.$$

(a) Show that there is a root of $f(x) = 0$ in the interval $2 < x < 3$. (2)

(b) Use the iterative formula (3)

$$x_{n+1} = \ln(x_n + 2) + 1,\ x_0 = 2.5,$$

to calculate the values of $x_1$, $x_2$, and $x_3$ giving your answers to 5 decimal places.

(c) Show that $x = 2.505$ is a root of $\mathrm{f}(x) = 0$ correct to 3 decimal places. (2)

7.

$$\mathrm{f}(x) = 3x^3 - 2x - 6.$$

(a) Show that $\mathrm{f}(x) = 0$ has a root, $\alpha$, between $x = 1.4$ and $x = 1.45$. (2)

(b) Show that the equation $\mathrm{f}(x) = 0$ can be written as (3)

$$x = \sqrt{\frac{2}{x} + \frac{2}{3}},\ x \neq 0.$$

(c) Starting with $x_0 = 1.43$, use the iteration (3)

$$x_{n+1} = \sqrt{\frac{2}{x_n} + \frac{2}{3}}$$

to calculate the values of $x_1$, $x_2$, and $x_3$, giving your answers to 4 decimal places.

(d) By choosing a suitable interval, show that $\alpha = 1.435$ is correct to 3 decimal places. (3)

8.

$$\mathrm{f}(x) = 3x\mathrm{e}^x - 1.$$

The curve with equation $y = \mathrm{f}(x)$ has a turning point $P$.

(a) Find the exact coordinates of $P$. (5)

The equation $\mathrm{f}(x) = 0$ has a root between $x = 0.25$ and $x = 0.3$.

(b) Use the iterative formula (3)

$$x_{n+1} = \tfrac{1}{3}\mathrm{e}^{-x_n},$$

with $x_0 = 0.25$ to find, to 4 decimal places, the values of $x_1$, $x_2$, and $x_3$.

(c) By choosing a suitable interval, show that a root of $\mathrm{f}(x) = 0$ is $x = 0.2576$ correct to 4 decimal places. (3)

9.

$$y = -x^3 + 2x^2 + 2,$$

which intersects the $x$-axis at the point $A$ where $x = \alpha$. To find an approximation to $\alpha$, the iterative formula

$$x_{n+1} = \frac{2}{(x_n)^2} + 2$$

is used.

(a) Taking $x_0 = 2.5$, find the values of $x_1$, $x_2$, $x_3$, and $x_4$. Give your answers to 3 decimal places where appropriate. (3)

(b) Show that $\alpha = 2.359$ correct to 3 decimal places. (3)

10.

$$f(x) = x^3 + 2x^2 - 3x - 11.$$

(a) Show that $f(x) = 0$ can be rearranged as (2)

$$x = \sqrt{\frac{3x + 11}{x + 2}}, \ x \neq -2.$$

The equation $f(x) = 0$ has one positive root $\alpha$. The iterative formula

$$x_{n+1} = \sqrt{\frac{3x_n + 11}{x_n + 2}}$$

is used to find a approximation to $\alpha$.

(b) Taking $x_1 = 0$, find, to 3 decimal places, the values of $x_2$, $x_3$, and $x_4$. (3)

(c) Show that $\alpha = 2.057$ correct to 3 decimal places. (3)

11.

$$f(x) = 4 \operatorname{cosec} x - 4x + 1,$$

where $x$ is in radians.

(a) Show that there is a root $\alpha$ of $f(x) = 0$ in the interval $[1.2, 1.3]$. (2)

(b) Show that the equation $f(x) = 0$ can be written in the form (2)

$$x = \frac{1}{\sin x} + \frac{1}{4}.$$

(c) Use the iterative formula (3)

$$x_{n+1} = \frac{1}{\sin x_n} + \frac{1}{4}, \ x_0 = 1.25,$$

to calculate the values of $x_1$, $x_2$, and $x_3$, giving your answers to 4 decimal places.

(d) By considering the change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 1.291$ correct to 3 decimal places. (2)

12.

$$f(x) = (8 - x) \ln x, \ x > 0.$$

The curve has a maximum turning point at $Q$.

(a) Find $f'(x)$. (3)

(b) Show that the $x$-coordinate of $Q$ lies between 3.5 and 3.6. (2)

(c) Show that the $x$-coordinate of $Q$ is the solution of (3)

$$x = \frac{8}{1 + \ln x}.$$

To find an approximation for the $x$-coordinate of $Q$, the iteration formula

$$x_{n+1} = \frac{8}{1 + \ln x_n}$$

is used.

(d) Taking $x_0 = 3.55$, find the values of $x_1$, $x_2$, and $x_3$. Give your answers to 3 decimal places. (3)

13.

$$f(x) = 2\sin(x^2) + x - 2,\ 0 \leqslant x < 2\pi.$$

(a) Show that $f(x) = 0$ has a root $\alpha$ between $x = 0.75$ and $x = 0.85$. (2)

The equation $f(x) = 0$ can be written as

$$x = [\arcsin(1 - 0.5x)]^{\frac{1}{2}}.$$

(b) Use the iterative formula (3)

$$x_{n+1} = [\arcsin(1 - 0.5x_n)]^{\frac{1}{2}},\ x_0 = 0.8,$$

to find the values of $x_1$, $x_2$, and $x_3$, giving your answers to 5 decimal places.

(c) Show that $\alpha = 0.80157$ is correct to 5 decimal places. (3)

14.

$$f(x) = x^2 - 3x + 2\cos(\tfrac{1}{2}x),\ 0 \leqslant x \leqslant \pi.$$

(a) Show that the equation $f(x) = 0$ has a solution in the interval $0.8 < x < 0.9$ (2)

The curve with equation $y = f(x)$ has a minimum point $P$.

(b) Show that the $x$-coordinate of $P$ is the solution of the equation (4)

$$x = \frac{3 + \sin(\frac{1}{2}x)}{2}.$$

(c) Using the iteration formula (3)

$$x_{n+1} = \frac{3 + \sin(\frac{1}{2}x_n)}{2},\ x_0 = 2,$$

find the values of $x_1$, $x_2$, and $x_3$ , giving your answers to 3 decimal places.

(d) By choosing a suitable interval, show that the $x$-coordinate of $P$ is 1.9078 correct to 4 decimal places. (3)

15.

$$\mathrm{f}(x) = x^3 + 3x^2 + 4x - 12.$$

(a) Show that the equation $\mathrm{f}(x) = 0$ can be written as (3)

$$x = \sqrt{\frac{4(3-x)}{3+x}},\ x \neq -3.$$

The equation $x^3 + 3x^2 + 4x - 12 = 0$ has a single root which is between 1 and 2.

(b) Use the iteration formula (3)

$$x_{n+1} = \sqrt{\frac{4(3-x_n)}{3+x_n}},\ n \geqslant 0,$$

with $x_0 = 1$ to find, to 2 decimal places, the value of $x_1$, $x_2$, and $x_3$.

The root of $\mathrm{f}(x) = 0$ is $\alpha$.

(c) By choosing a suitable interval, prove that $\alpha = 1.272$ to 3 decimal places. (3)

16.

$$\mathrm{g}(x) = \mathrm{e}^{x-1} + x - 6.$$

(a) Show that the equation $\mathrm{g}(x) = 0$ can be written as (3)

$$x = \ln(6 - x) + 1,\ x < 6.$$

The root of $\mathrm{g}(x) = 0$ is $\alpha$. The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1,\ x_0 = 2,$$

is used to find an approximate value for $\alpha$.

(b) Calculate the values of $x_1$, $x_2$, and $x_3$ to 4 decimal places. (3)

(c) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places. (3)

17.

$$\mathrm{f}(x) = 25x^2\mathrm{e}^{2x} - 16,\ x \in \mathbb{R}.$$

(a) Show that the equation $f(x) = 0$ can be written as (1)

$$x = \pm\tfrac{4}{5}e^{-x}.$$

The equation $f(x) = 0$ has a root $\alpha$, where $\alpha = 0.5$ to 1 decimal place.

(b) Starting with $x_0 = 0.5$, use the iteration formula (3)

$$x_{n+1} = \tfrac{4}{5}e^{-x_n}$$

to calculate the values of $x_1$, $x_2$, and $x_3$, giving your answers to 3 decimal places.

(c) Give an accurate estimate for $\alpha$ to 2 decimal places, and justify your answer. (2)

18.

$$f(x) = (x^2 + 3x + 1)e^{x^2}.$$

The curve has a minimum turning point at the point $P$.

(a) Find $f'(x)$. (3)

(b) Show that the $x$-coordinate of $P$ is the solution of (3)

$$x = -\frac{3(2x^2 + 1)}{2(x^2 + 2)}.$$

(c) Use the iteration formula (3)

$$x_{n+1} = -\frac{3(2x_n^2 + 1)}{2(x_n^2 + 2)},\ x_0 = -2.4,$$

to calculate the values of $x_1$, $x_2$, and $x_3$, giving your answers to 3 decimal places.

The $x$-coordinate of $P$ is $\alpha$.

(d) By choosing a suitable interval, prove that $\alpha = -2.43$ correct to 2 decimal places. (2)

19.

$$y = 2\cos(\tfrac{1}{2}x^2) + x^3 - 3x - 2.$$

The curve crosses the $x$-axis at the point $Q$ and has a minimum turning point at $R$.

(a) Show that the $x$-coordinate of $Q$ lies between 2.1 and 2.2. (2)

(b) Show that the $x$-coordinate of $R$ is a solution of the equation (4)

$$x = \sqrt{1 + \tfrac{2}{3}x\sin(\tfrac{1}{2}x^2)}.$$

Using the iterative formula

$$x_{n+1} = \sqrt{1 + \tfrac{2}{3}x_n\sin(\tfrac{1}{2}x_n^2)},\ x_0 = 1.3,$$

(c) find the values of $x_1$ and $x_2$ to 3 decimal places. (2)

20. A curve $C$ has equation

$$y = \mathrm{e}^{4x} + x^4 + 8x + 5.$$

(a) Show that the $x$-coordinate of any turning point of $C$ satisfies the equation (3)

$$x^3 = -2 - \mathrm{e}^{4x}.$$

The iteration formula

$$x_{n+1} = (-2 - \mathrm{e}^{4x_n})^{\frac{1}{3}},\ x_0 = -1,$$

can be used to find an approximate value for this root.

(b) Calculate the values of $x_1$ and $x_2$, giving your answers to 5 decimal places. (2)

(c) Hence deduce the coordinates, to 2 decimal places, of the turning point of the curve $C$. (2)

21. Figure 1 is a sketch showed part of the curve with equation $y = 2^{x+1} - 3$ and part of the line with equation $y = 17 - x$.

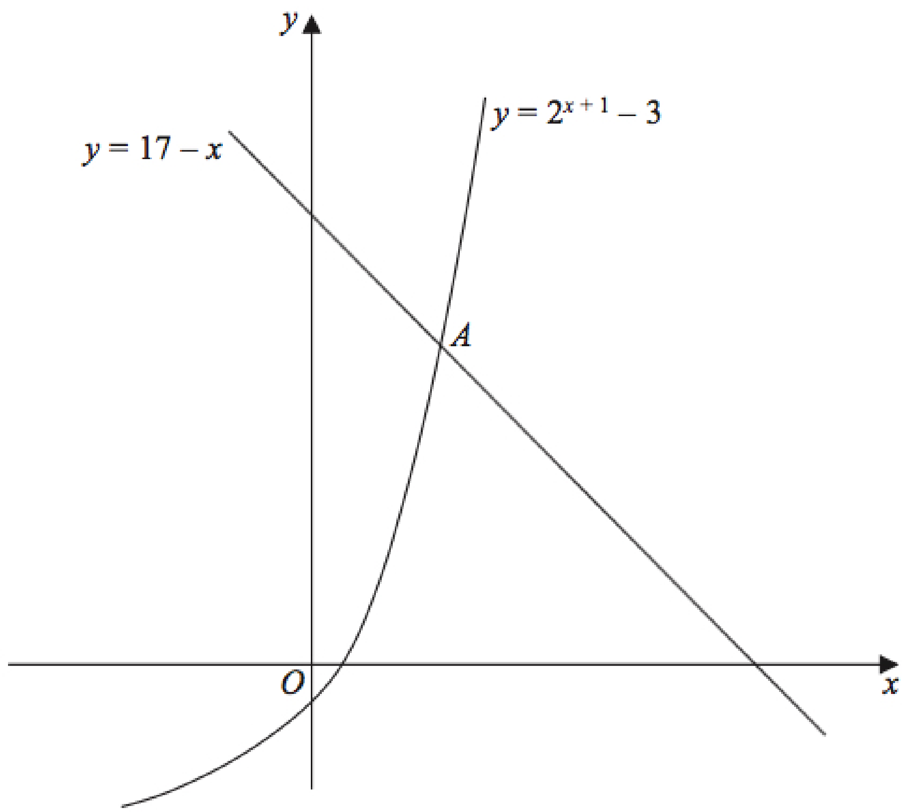


Figure 1: $y = 2^{x+1} - 3$ and $y = 17 - x$

The curve and the line intersect at the point $A$.

(a) Show that the $x$coordinate of $A$ satisfies the equation (3)

$$x = \frac{\ln(20 - x)}{\ln 2} - 1.$$

(b) Use the iterative formula (3)

$$x_{n+1} = \frac{\ln(20 - x_n)}{\ln 2} - 1,\ x_0 = 3,$$

to calculate the values of $x_1$, $x_2$, and $x_3$, giving your answers to 3 decimal places.

(c) Use your answer to part (b) to deduce the coordinates of the point $A$, giving your answers to one decimal place. (2)

22.

$$\mathrm{f}(x) = 4\mathrm{e}^{2x} - 25 \text{ and } \mathrm{g}(x) = 2x + 43.$$

The equation $\mathrm{f}(x) = \mathrm{g}(x)$ has a positive root at $x = \alpha$.

(a) Show that $\alpha$ is a solution of (2)

$$x = \tfrac{1}{2}\ln(\tfrac{1}{2}x + 17).$$

The iteration formula

$$x_{n+1} = \tfrac{1}{2}\ln(\tfrac{1}{2}x_n + 17)$$

can be used to find an approximation for $\alpha$.

(b) Taking $x_0 = 1.4$, find $x_1$ and $x_2$. Give each answer to 4 decimal places. (2)

(c) By choosing a suitable interval, show that $\alpha = 1.437$ to 3 decimal places. (2)

23. Figure 2 shows a sketch of part of the curve $C$ with equation

$$y = 2\ln(2x + 5) - \tfrac{3}{2}x,\ x > -2.5.$$

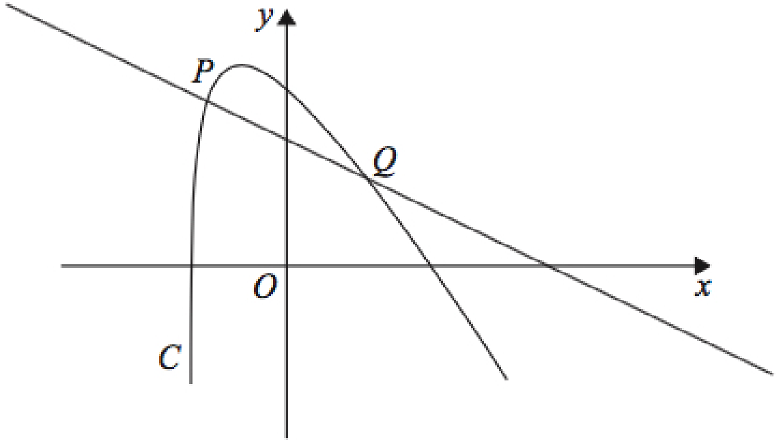


Figure 2: $y = 2\ln(2x+5) - \frac{3}{2}x$

The point $P$ with $x$-coordinate $-2$ lies on $C$. The normal to $C$ at $P$ cuts the curve again at the point $Q$.

(a) Find an equation of the normal to $C$ at $P$. Write your answer in the form $ax+by=c$, where $a$, $b$, and $c$ are integers. (5)

(b) Show that the $x$-coordinate of $Q$ is a solution of the equation (3)

$$x = \frac{20}{11}\ln(2x+5) - 2.$$

The iteration formula

$$x_{n+1} = \frac{20}{11}\ln(2x_n+5) - 2$$

can be used to find an approximation for the $x$-coordinate of $Q$.

(c) Taking $x_1 = 2$, find the values of $x_2$ and $x_3$, giving each answer to 4 decimal places. (2)