Dr Oliver Mathematics
Mathematics: Advanced Higher
2022 Paper 1: Non-Calculator
1 hour

The total number of marks available is 36.
You must write down all the stages in your working.

1. (a) Given (3)

$$y = \frac{1-3x}{x^2+4},$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
Simplify your answer.

Solution

$$u = 1 - 3x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -3$$
$$v = x^2 + 4 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x$$

Now,

$$\begin{aligned} y = \frac{1-3x}{x^2+4} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x^2+4)(-3)-(1-3x)(2x)}{(x^2+4)^2} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(-3x^2-12)-(2x-6x^2)}{(x^2+4)^2} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2-2x-12}{(x^2+4)^2}. \end{aligned}$$

[NB: it does not factorise - $(-2)^2 - 4 \times 3 \times (-12) = 148$]

(b) Given (2)

$$\mathrm{f}(x) = \operatorname{cosec} 5x,$$

find $\mathrm{f}'(x)$.

Solution

$$\mathrm{f}(x) = \operatorname{cosec} 5x \Rightarrow \mathrm{f}'(x) = -5 \operatorname{cosec} 5x \cot 5x.$$

2. Use Gaussian elimination to solve the following system of equations: (4)

$$\begin{aligned} x - 2y + z &= 4 \\ 2x + y - 3z &= 3 \\ x - 7y - 4z &= 9. \end{aligned}$$

Solution

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 2 & 1 & -3 & 3 \\ 1 & -7 & -4 & 9 \end{array}\right)$$

Do $R_2 - 2R_1$ and $R_3 - R_1$:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -5 & -5 \\ 0 & -5 & -5 & 5 \end{array}\right)$$

Do $R_3 + R_2$:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -5 & -5 \\ 0 & 0 & -10 & 0 \end{array}\right)$$

Hence,

$$\underline{\underline{x = 2,\ y = -1,\ z = 0}}.$$

3. Given that (3)

$$z_1 = 5 + 3\mathrm{i} \text{ and } z_2 = 6 + 2\mathrm{i},$$

express $z_1\bar{z}_2$ in the form $a + \mathrm{i}b$, where a and b are real numbers.

Solution

×	5	+3i
6	30	+18i
−2i	−10i	+6

Hence,

$$z_1\bar{z}_2 = \underline{\underline{36 + 8\mathrm{i}}};$$

so $\underline{\underline{a = 36}}$ and $\underline{\underline{b = 8}}$.

4. A curve is defined by the equation

$$y^3 + 4y = 2xy + 1.$$

(a) Use implicit differentiation to find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. (3)

Solution

$$\begin{aligned} y^3 + 4y = 2xy + 1 &\Rightarrow 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 4\frac{\mathrm{d}y}{\mathrm{d}x} = 2y + 2x\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Rightarrow 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 4\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 2y \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}(3y^2 + 4 - 2x) = 2y \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{3y^2 + 4 - 2x}}}. \end{aligned}$$

(b) Find the gradient of the tangent to the curve when $y = -1$. (1)

Solution

$$\begin{aligned} y = -1 &\Rightarrow (-1)^3 + 4(-1) = 2x(-1) + 1 \\ &\Rightarrow -1 - 4 = -2x + 1 \\ &\Rightarrow -6 = -2x \\ &\Rightarrow x = 3 \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2(-1)}{3(-1)^2 + 4 - 2(3)} \\ &= \frac{-2}{3 + 4 - 6} \\ &= \underline{\underline{-2}}. \end{aligned}$$

(c) Show that the curve has no stationary point. (2)

Solution

Stationary points occur when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Now,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \frac{2y}{3y^2 + 4 - 2x} = 0 \\ &\Rightarrow y = 0; \end{aligned}$$

substitute that and we get

$$0 + 0 = 0 + 1$$

which is a contradiction! Hence, the curve has no stationary point.

5. (a) Find, and simplify, the Maclaurin expansion for e^{-4x}, up to and including the term in x^3. (2)

Solution

Let $\mathrm{f}(x) = \mathrm{e}^{-4x}$. Then

$$\begin{aligned} \mathrm{f}(x) &= \mathrm{e}^{-4x} \Rightarrow \mathrm{f}(0) = 1 \\ \mathrm{f}'(x) &= -4\mathrm{e}^{-4x} \Rightarrow \mathrm{f}'(0) = -4 \\ \mathrm{f}''(x) &= 16\mathrm{e}^{-4x} \Rightarrow \mathrm{f}''(0) = 16 \\ \mathrm{f}'''(x) &= -64\mathrm{e}^{-4x} \Rightarrow \mathrm{f}'''(0) = -64 \end{aligned}$$

and

$$\begin{aligned} \mathrm{f}(x) &= 1 + \tfrac{1}{1!}\mathrm{f}'(0)x + \tfrac{1}{2!}\mathrm{f}''(0)x^2 + \tfrac{1}{3!}\mathrm{f}'''(0)x^3 \ldots \\ &= 1 + (-4)x + \tfrac{1}{2}(16)x^2 + \tfrac{1}{6}(-64)x^3 + \ldots \\ &= \underline{\underline{1 - 4x + 8x^2 + \tfrac{32}{3}x^3 + \ldots}} \end{aligned}$$

(b) Hence find the first four terms of the Maclaurin expansion of (2)

$$\frac{3 + 2x}{\mathrm{e}^{4x}}.$$

Solution

$$\begin{aligned} \frac{3 + 2x}{\mathrm{e}^{4x}} &= (3 + 2x)\mathrm{e}^{-4x} \\ &= (3 + 2x)(1 - 4x + 8x^2 + + \tfrac{32}{3}x^3 + \ldots) \end{aligned}$$

×	1	$-4x$	$+8x^2$	$-\frac{32}{3}x^3$
3	3	$-12x$	$+24x^2$	$-32x^3$
$+2x$	$+2x$	$-8x^2$	$+16x^3$	...

$$= \underline{\underline{3 - 10x + 16x^2 - 16x^3 + \ldots}}$$

6. (a) Consider the statement: (1)

For all odd numbers n, $n^2 + 4$ is prime.

Find a counterexample to show that the statement is false.

Solution

$$\begin{aligned} n = 1 &\Rightarrow 1^2 + 4 = 5, \text{ which is prime,} \\ n = 3 &\Rightarrow 3^2 + 4 = 13, \text{ which is prime,} \\ n = 5 &\Rightarrow 5^2 + 4 = 29, \text{ which is prime,} \\ n = 7 &\Rightarrow 7^2 + 4 = 53, \text{ which is prime,} \\ n = 9 &\Rightarrow 9^2 + 4 = 85 = 5 \times 17 \text{ which is } \underline{\underline{\text{not prime}}}. \end{aligned}$$

(b) Prove directly that the difference between the cubes of any two consecutive integers is not divisible by 3. (3)

Solution

Let $n \in \mathbb{N}$. Then

$$\begin{aligned} (n+1)^3 - n^3 &= (n^3 + 3n^2 + 3n + 1) - n^3 \\ &= 3n^2 + 3n + 1 \\ &= 3(n^2 + n) + 1 \\ &= 3 \times \text{some natural number} + 1; \end{aligned}$$

hence, the number is <u>not divisible</u> by 3.

7. (a) Use the substitution $u = y^2 + 1$, or otherwise, to find the exact value of (4)

$$\int_0^5 \frac{4y}{\sqrt{y^2+1}}\,\mathrm{d}y.$$

Solution

$$u = y^2 + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}y} = 2y$$
$$\Rightarrow \mathrm{d}u = 2y\,\mathrm{d}y$$

and

$$y = 0 \Rightarrow u = 1$$
$$y = 5 \Rightarrow u = 26.$$

Now,

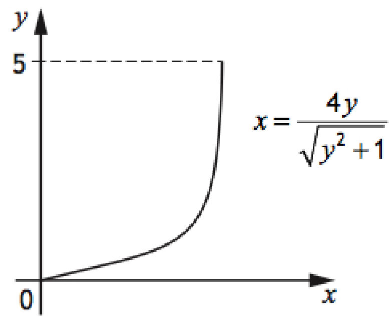
$$\int_0^5 \frac{4y}{\sqrt{y^2+1}}\,\mathrm{d}y = \int_1^{26} \frac{2}{\sqrt{u}}\,\mathrm{d}u$$
$$= \left[4u^{\frac{1}{2}}\right]_{u=1}^{26}$$
$$= \underline{\underline{4\sqrt{26} - 4}}.$$

Student engineers are using a 3D printer to make a model.

Relative to a suitable set of axes, the cross-section of the model is **symmetrical about the y-axis** and is represented **in the first quadrant** by the curve with equation

$$x = \frac{4y}{\sqrt{y^2+1}}, \; 0 \leqslant y \leqslant 5,$$

as shown in the diagram.



(b) State the area of the cross-section. (1)

Solution
$2(4\sqrt{26} - 4) = \underline{\underline{8\sqrt{26} - 8}}$.

(c) Express (1)

$$\frac{y^2}{y^2+1}$$

in the form

$$a + \frac{b}{y^2+1},$$

where a and b are real numbers.

Solution

$$\begin{aligned} \frac{y^2}{y^2+1} &= \frac{(y^2+1)-1}{y^2+1} \\ &= \frac{y^2+1}{y^2+1} - \frac{1}{y^2+1} \\ &= \underline{\underline{1 - \frac{1}{y^2+1}}}; \end{aligned}$$

hence, $\underline{\underline{a = 1}}$ and $\underline{\underline{b = -1}}$.

The curve

$$x = \frac{4y}{\sqrt{y^2 + 1}},\ 0 \leqslant y \leqslant 5,$$

will be rotated through 2π radians about the y-axis to make the model.

(d) Find the volume of the model. (4)

Solution

$$\begin{aligned}
\text{Volume} &= \int_0^5 \pi \left(\frac{4y}{\sqrt{y^2+1}} \right)^2 \, \mathrm{d}y \\
&= \pi \int_0^5 \frac{16y^2}{y^2+1} \, \mathrm{d}y \\
&= 16\pi \int_0^5 \left(1 - \frac{1}{y^2+1} \right) \, \mathrm{d}y \\
&= 16\pi \left[y - \arctan y \right]_{y=0}^{5} \\
&= 16\pi \left\{ (5 - \arctan 5) - (0 - 0) \right\} \\
&= \underline{\underline{16\pi(5 - \arctan 5)}}.
\end{aligned}$$