Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2016 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

Section A

1. Solve the inequality (3)

$$1 - 2(x - 3) > 4x.$$

Solution

$$\begin{aligned} 1 - 2(x-3) > 4x &\Rightarrow 7 - 2x > 4x \\ &\Rightarrow 7 > 6x \\ &\Rightarrow \underline{\underline{x < 1\tfrac{1}{6}}}. \end{aligned}$$

2. The gradient function of a curve is given by (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x + 2.$$

Find the equation of the curve, given that it passes through the point $(1, 3)$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x + 2 \Rightarrow y = x^3 - 2x^2 + 2x + c,$$

for some constant c. Now,

$$3 = 1 - 1 + 1 + c \Rightarrow c = 2$$

and so the equation is

$$\underline{\underline{y = x^3 - 2x^2 + 2x + 2}}.$$

3. Find all the values of x in the range $0° < x < 360°$ that satisfy (4)

$$3\sin x = 4\cos x.$$

Solution

$$\begin{aligned} 3\sin x = 4\cos x \Rightarrow \tan x &= \tfrac{4}{3} \\ \Rightarrow x &= 53.130\,102\,35,\ 233.130\,102\,35\ \text{(FCD)} \\ \Rightarrow \underline{\underline{x = 53.1,\ 233\ \text{(3 sf)}}}. \end{aligned}$$

4. You are given that

$$f(x) = x^3 - x^2 + x - 6.$$

Show that

(a) $(x - 2)$ is a factor of $f(x)$, (1)

Solution

We use synthetic division:

2	1	−1	1	6
	↓	2	−2	−2
	1	1	3	0

As the remainder equals 0, $(x - 2)$ is a factor of $f(x)$

(b) the equation $f(x) = 0$ has only one real root. (4)

Solution

$$x^3 - x^2 + x - 6 = (x - 2)(x^2 + x + 3).$$

Now, $a = 1$, $b = 1$, and $c = 3$:

$$\begin{aligned} b^2 - 4ac &= 1^2 - 4 \times 1 \times 3 \\ &= -11 \\ &< 0 \end{aligned}$$

and so the cubic has only one real root.

5. John draws a triangle ABC with sides $AB = 12$ cm, $BC = 16$ cm and $AC = 20$ cm. However, he can only measure the sides to the nearest centimetre.

(a) State the smallest possible length of AB in John's drawing. (1)

Solution
11.5 cm.

(b) Hence calculate the largest possible value of the angle B in John's drawing. (3)

Solution
Make $AB = 11.5$ cm, $BC = 15.5$ cm and $AC = 20.5$ cm:

$$\begin{aligned}\cos ABC = \frac{11.5^2 + 15.5^2 - 20.5^2}{2 \times 11.5 \times 15.5} &\Rightarrow \cos ABC = -\tfrac{191}{1\,426} \\ &\Rightarrow \angle ABC = 97.697\,392\,95 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\angle ABC = 97.7^\circ \text{ (3 sf)}}}.\end{aligned}$$

6. Two cars are initially at rest facing in the same direction on a straight road.
Car A is 100 m ahead of car B.
The two cars start from rest at the same moment.
Car A moves with constant acceleration of 1.5 ms^{-2} and car B moves with constant acceleration of 2 ms^{-2}

Find

(a) the distance that car B travels before it overtakes car A, (4)

Solution
Car A: $s = 100$, $u = 0$, $v = ?$, $a = 1.5$, and $t = ?$
Car B: $s = 0$, $u = 0$, $v = ?$, $a = 2$, and $t = ?$

$$\begin{aligned}100 + 0 + \tfrac{1}{2} \times 1.5 \times t^2 = 0 + \tfrac{1}{2} \times 2 \times t^2 &\Rightarrow 100 + 0.75t^2 = t^2 \\ &\Rightarrow 100 = 0.25t^2 \\ &\Rightarrow t^2 = 400 \\ &\Rightarrow t = 20 \text{ s}.\end{aligned}$$

Hence, the distance that car B travels before it overtakes car A is

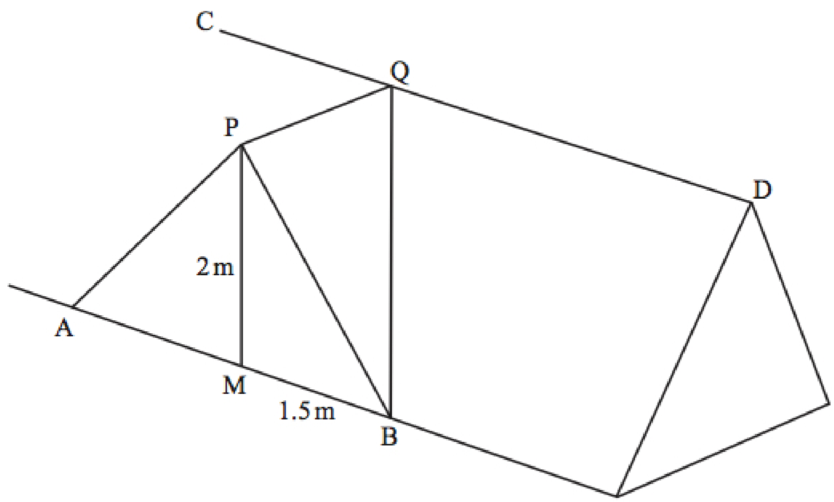
$$s = 0 + \tfrac{1}{2} \times 2 \times 20^2$$
$$= \underline{\underline{400 \text{ m}}}.$$

(b) the speed of car B at the moment when it overtakes car A. (2)

Solution

$$v = 0 + 2 \times 20 = \underline{\underline{40 \text{ ms}^{-1}}}.$$

7. An extension to the roof of a house is shown in the diagram below.



The ridge, CD, and the lines AB and PQ are horizontal.
PQ is perpendicular to CD.
M is the midpoint of AB.
The line PM is vertical.
APB is an isosceles triangle with height 2 metres and base length 3 metres.
Angle PQM is 45°.

Find

(a) the length of PQ, (1)

Solution
$\underline{\underline{PQ = 2 \text{ m}}}$.

(b) the angle PBQ. (4)

Solution

$$\begin{aligned} PB &= \sqrt{2^2 + 1.5^2} \\ &= 2.5 \end{aligned}$$

and

$$\begin{aligned} \tan PBQ = \frac{2}{2.5} &\Rightarrow \angle PBQ = 38.659\,808\,25 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\angle PBQ = 38.7^\circ \text{ (3 sf)}}}. \end{aligned}$$

8. (a) Write down the binomial expansion of (2)

$$(1 + \delta)^3.$$

Solution

$$(1 + \delta)^3 = \underline{\underline{1 + 3\delta + 3\delta^2 + \delta^3}}.$$

(b) Hence explain why, if δ is small, (1)

$$(1 + \delta)^3 \approx 1 + 3\delta.$$

Solution
If δ is small, then that means δ^2 and δ^3 are really small so

$$(1 + \delta)^3 \approx \underline{\underline{1 + 3\delta}}.$$

You are given that the equation

$$x^3 - 0.9x - 0.206 = 0$$

has a root very close to $x = 1$.

(c) Substitute $x = 1 + \delta$ into the equation and use the approximation in part (b) to find an estimate of this root, correct to 3 significant figures. Show all your working. (4)

Solution

$$\begin{aligned} x^3 - 0.9x - 0.206 = 0 &\Rightarrow (1+\delta)^3 - 0.9(1+\delta) - 0.206 = 0 \\ &\approx (1+3\delta) - 0.9(1+\delta) - 0.206 = 0 \\ &\Rightarrow 2.1\delta = 0.106 \\ &\Rightarrow \delta = \tfrac{53}{1\,050} \end{aligned}$$

and the root is

$$x = \underline{\underline{1\tfrac{53}{1\,050} \text{ or } 1.05 \text{ (3 sf)}}}.$$

9. A curve has equation

$$y = x^3 - 3x^2 - 3x + 4.$$

Points P and Q lie on the curve. The coordinates of P are $(3, -5)$.

(a) Find the equation of the tangent to the curve at P. (4)

Solution

$$y = x^3 - 3x^2 - 3x + 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x - 3$$

and

$$x = 3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 27 - 18 - 3 = 6.$$

Finally, the equation is

$$\begin{aligned} y + 5 = 6(x-3) &\Rightarrow y + 5 = 6x - 18 \\ &\Rightarrow \underline{\underline{y = 6x - 23}}. \end{aligned}$$

The tangent to the curve at Q is parallel to the tangent to the curve at P.

(b) Find the coordinates of Q. (3)

Solution

$$\begin{aligned} 3x^2 - 6x - 3 = 6 &\Rightarrow 3x^2 - 6x - 9 = 0 \\ &\Rightarrow 3(x^2 - 2x - 3) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -2 \\ \text{multiply to:} & -3 \end{array}\right\} -3,\ +1$$

$$\begin{aligned} &\Rightarrow 3(x-3)(x+1) = 0 \\ &\Rightarrow x = 3 \text{ or } x = -1; \end{aligned}$$

now,

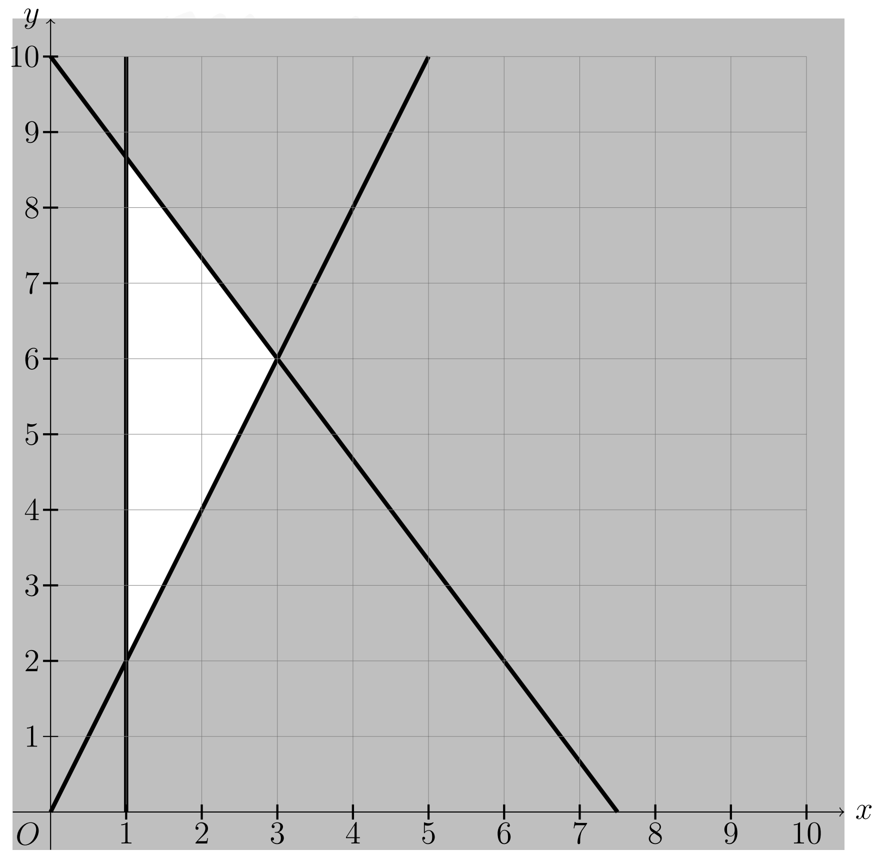
$$x = -1 \Rightarrow y = -1 - 3 - 3 + 4 = -3$$

and hence the point is $\underline{\underline{Q(-1, -3)}}$.

10. (a) Indicate the region for which the following inequalities hold. You should shade the region that is **not** satisfied by the inequalities. (5)

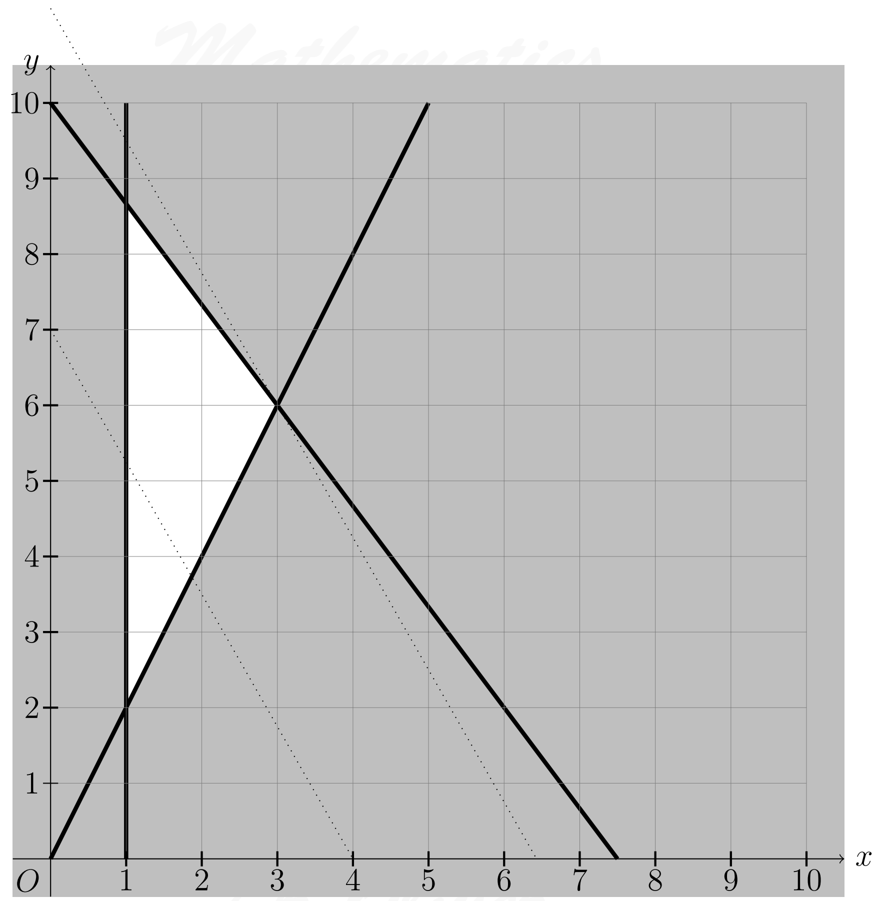
$$\begin{aligned} 4x + 3y &\leqslant 30 \\ y &\geqslant 2x \\ x &\geqslant 1. \end{aligned}$$

Solution



(b) Find the maximum value of $7x + 4y$ subject to these conditions. (2)

Solution

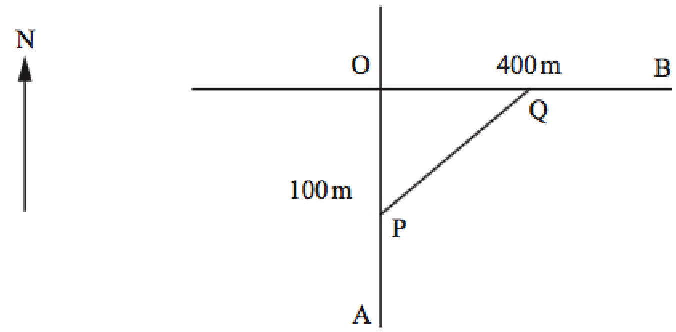


Hence, $x = 3$ and $y = 6$:

$$(7 \times 3) + (4 \times 6) = \underline{\underline{45}}.$$

Section B

11. A railway track runs due east-west and is crossed at O by a road running due south-north, as shown below. The crossing has no barriers.



Initially a train is at point B, 400 m from O, and a car is at point A, 100 m from O. The train is travelling at a constant speed of 25 ms^{-1} towards O and the car is travelling at a constant speed of 20 ms^{-1} towards O.

At time t seconds the train is at point Q and the car is at point P.

(a) Find expressions for the distances OP and OQ as functions of t. (2)

Solution

$$OP = \underline{\underline{(100 - 20t)\ \text{ms}^{-1}}} \text{ and } OQ = \underline{\underline{(400 - 25t)\ \text{ms}^{-1}}}.$$

The distance between the car and the train at time t s is x m.

(b) Find a formula for x^2 in terms of t. Give your formula in the form (3)

$$x^2 = a + bt + ct^2,$$

where a, b, and c are to be determined.

Solution

$\times$	100	$-20t$
100	$10\,000$	$-2\,000t$
$-20t$	$-2\,000t$	$+400t^2$

$\times$	400	$-25t$
400	$16\,000$	$-10\,000t$
$-25t$	$-10\,000t$	$+625t^2$

Hence,

$$\begin{aligned} x^2 &= (100 - 20t)^2 + (400 - 25t)^2 \\ &= (10\,000 - 4\,000t + 400t^2) + (16\,000 - 20\,000t + 625^2) \\ &= \underline{\underline{(17\,000 - 24\,000t + 1\,025t^2)\text{ m}^2}}. \end{aligned}$$

(c) Differentiate this formula with respect to t and find the time at which x^2 is a minimum. Hence find the shortest distance between the car and the train. (6)

Solution

$$x^2 = 17\,000 - 24\,000t + 1\,025t^2 \Rightarrow \underline{\underline{\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = -24\,000 + 2\,050t}}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 0 &\Rightarrow -24\,000 + 2\,050t = 0 \\ &\Rightarrow 2\,050t = 24\,000 \\ &\Rightarrow t = 11\tfrac{29}{41}\text{ s} \\ &\Rightarrow x^2 = 29\,512\tfrac{8}{41}\text{ m}^2 \\ &\Rightarrow x = 171.791\,138\,1\text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 172\text{ m (3 sf)}}}. \end{aligned}$$

(d) Show that the car passes point O before the train. (1)

Solution

The car takes

$$\frac{100}{20} = 5\text{ s}$$

and the train takes

$$\frac{400}{25} = 16\text{ s}$$

so the car passes point O before the train.

12. The line L_1 has equation

$$3x - y = 1$$

and the point P has coordinates $(8, 3)$.

(a) Find the equation of the line L_2 which passes through P and is perpendicular to line L_1. (3)

Solution

$$3x - y = 1 \Rightarrow y = 3x - 1$$

and the gradient of the line which is perpendicular is $-\frac{1}{3}$. Hence, the equation of the line is

$$y - 3 = -\tfrac{1}{3}(x - 8) \Rightarrow y - 3 = -\tfrac{1}{3}x + \tfrac{8}{3}$$
$$\Rightarrow \underline{\underline{y = -\tfrac{1}{3}x + \tfrac{17}{3}}}.$$

(b) Find the coordinates of the point Q where L_1 and L_2 intersect. (3)

Solution

$$3x - 1 = -\tfrac{1}{3}x + \tfrac{17}{3} \Rightarrow \tfrac{10}{3}x = \tfrac{20}{3}$$
$$\Rightarrow x = 2$$
$$\Rightarrow y = 5;$$

hence, the coordinates are $\underline{\underline{Q(2, 5)}}$.

(c) Find the length PQ. (2)

Solution

$$\begin{aligned} PQ &= \sqrt{(8-2)^2 + (3-5)^2} \\ &= \sqrt{36 + 4} \\ &= \underline{\underline{2\sqrt{10}}}. \end{aligned}$$

(d) Write down the equation of the circle that has centre P and line L_1 as a tangent. (1)

Solution

$$\underline{\underline{(x-8)^2 + (y-3)^2 = 40}}.$$

(e) Find the equation of the other line that is a tangent to the circle and is parallel to line L_1. (3)

Solution

$$\begin{pmatrix} 8 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

and

$$\begin{pmatrix} 8 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 14 \\ 1 \end{pmatrix}.$$

Now, $y = 3x + c$ is satisfied by $(14, 1)$:

$$1 = (3 \times 14) + c \Rightarrow c = 41$$

and hence the line is

$$\underline{\underline{y = 3x - 41}}.$$

13. The cost of a packet of buns in a local supermarket is x pence and the cost of a loaf of bread is $(x + 75)$ pence.

(a) Write an expression for the number of packets of buns that can be bought for £5.40 and an expression for the number of loaves that can be bought for £5.40. (2)

Solution

Number of packets of buns $= \underline{\underline{\dfrac{540}{x}}}$ and number of packets of bread $= \underline{\underline{\dfrac{540}{x+75}}}$.

The number of packets of buns that can be bought for £5.40 is 5 more than the number of loaves that can be bought for £5.40.

(b) Using this information and your answer to part (a), derive an equation in x and show that it simplifies to (5)

$$x^2 + 75x - 8\,100 = 0.$$

Solution

$$\begin{aligned} \frac{540}{x} - \frac{540}{x+75} = 5 &\Rightarrow 540(x + 75) - 540x = 5x(x + 75) \\ &\Rightarrow 108(x + 75) - 108x = x(x + 75) \\ &\Rightarrow 108x + 8\,100 - 108x = x^2 + 75x \\ &\Rightarrow \underline{\underline{x^2 + 75x - 8\,100 = 0}}, \end{aligned}$$

as required.

(c) Solve this equation to find the cost of a packet of buns and the cost of a loaf of bread. (5)

Solution

$$\left.\begin{array}{ll}\text{add to:} & +75 \\ \text{multiply to:} & -8\,100\end{array}\right\} -60,\ +135$$

$$\begin{aligned} x^2 + 75x - 8\,100 = 0 &\Rightarrow (x-60)(x+135) = 0 \\ &\Rightarrow x = 60 \text{ or } x = -135. \end{aligned}$$

Hence, the cost of packets of buns is $\underline{\underline{60\text{ p}}}$ and the cost of packets of bread is $\underline{\underline{£1.35}}$.

14. The equation of a curve is given by

$$y = x^3 + ax^2 + bx + 1.$$

The points $P(-3,7)$ and $Q(1,3)$ lie on the curve.

(a) Form two equations in a and b. Solve these equations to show that $a = 3$ and $b = -2$. (4)

Solution

$$\begin{aligned} (-3,7): &\quad 7 = -27 + 9a - 3b + 1 \Rightarrow 9a - 3b = 33 \quad (1) \\ (1,3): &\quad 3 = 1 + a - b + 1 \Rightarrow a + b = 1 \quad (2). \end{aligned}$$

Do $3 \times (2)$:

$$3a + 3b = 3 \quad (3)$$

and do $(1) + (3)$:

$$\begin{aligned} 12a = 36 &\Rightarrow \underline{\underline{a = 3}} \\ &\Rightarrow 1 + b = 3 \\ &\Rightarrow \underline{\underline{b = -2}}, \end{aligned}$$

as required.

(b) Find the midpoint, R, of the line PQ and show that R lies on the curve. (2)

Solution

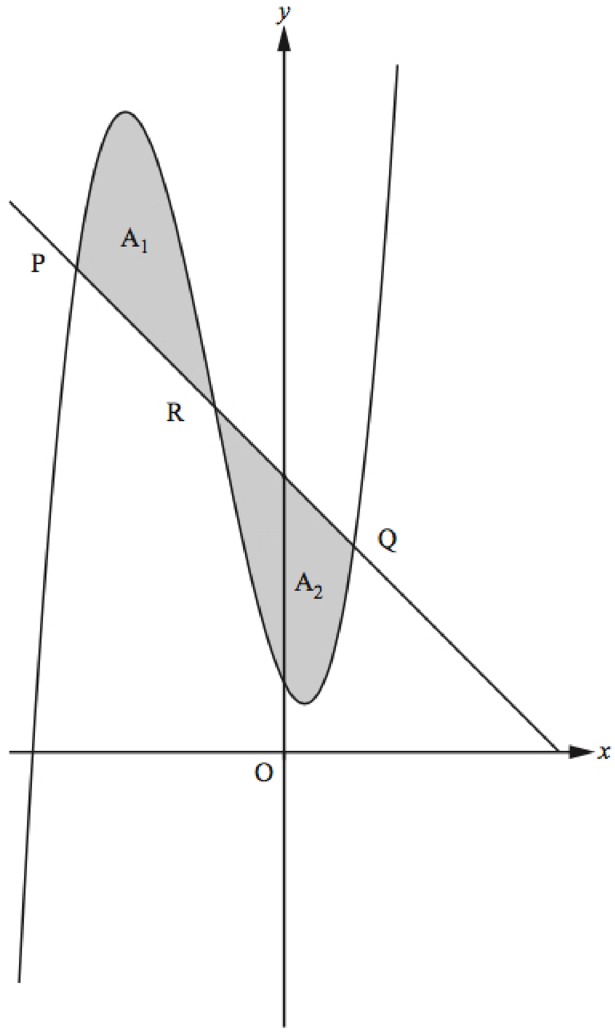
$$\left(\frac{-3+1}{2}, \frac{7+3}{2}\right) = \underline{\underline{R(-1,5)}}.$$

Now,

$$\begin{aligned} x = -1 \Rightarrow y &= -1 + 3 + 2 + 1 \\ \Rightarrow y &= 5; \end{aligned}$$

hence, R lies on the curve.

The diagram below shows the curve and the line PRQ.



The area between the curve and the line segment PR is A_1 and the area between the curve and the line segment RQ is A_2.

(c) Show that $A_1 = A_2$. (6)

Solution

The gradient of PQ is

$$\frac{3-7}{1-(-3)} = -1$$

and the equation of this line is

$$\begin{aligned} y - 3 = -(x-1) &\Rightarrow y - 3 = -x + 1 \\ &\Rightarrow y = -x + 4. \end{aligned}$$

Next,

$$\begin{aligned} A_1 &= \int_{-3}^{-1} \left[(x^3 + 3x^2 - 2x + 1) - (-x + 4) \right] \mathrm{d}x \\ &= \int_{-3}^{-1} (x^3 + 3x^2 - x - 3) \, \mathrm{d}x \\ &= \left[\tfrac{1}{4}x^4 + x^3 - \tfrac{1}{2}x^2 - 3x \right]_{x=-3}^{-1} \\ &= \left(\tfrac{1}{4} - 1 - \tfrac{1}{2} + 3 \right) - \left(20\tfrac{1}{4} - 27 - 4\tfrac{1}{2} + 9 \right) \\ &= 4 \end{aligned}$$

and

$$\begin{aligned} A_2 &= \int_{-1}^{1} \left[(-x + 4) - (x^3 + 3x^2 - 2x + 1) \right] \mathrm{d}x \\ &= \int_{-1}^{1} (-x^3 - 3x^2 + x + 3) \, \mathrm{d}x \\ &= \left[-\tfrac{1}{4}x^4 - x^3 + \tfrac{1}{2}x^2 + 3x \right]_{x=-1}^{1} \\ &= \left(-\tfrac{1}{4} + 1 + \tfrac{1}{2} + 3 \right) - \left(-\tfrac{1}{4} + 1 + \tfrac{1}{2} - 1 \right) \\ &= 4; \end{aligned}$$

hence, $\underline{\underline{A_1 = A_2}}$..