# Dr Oliver Mathematics

Matrix Transformations

Dr Oliver Mathematics

Monday 14 July 2014

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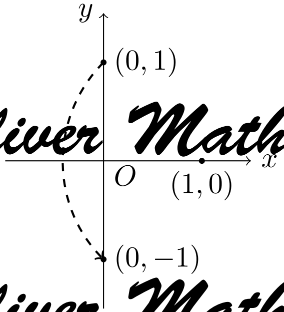






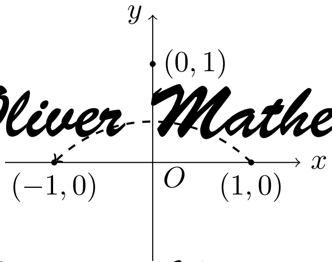

Hence

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

so the required matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

# Reflection in the $y$-axis






Hence



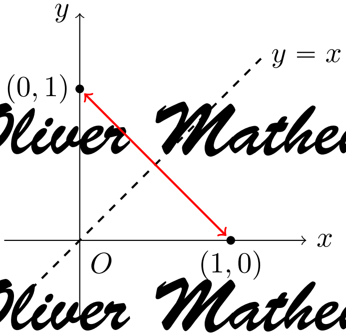
$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}-1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix}$$

so the required matrix is



$$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}.$$

# Reflection in the line $y = x$



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Hence

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so the required matrix is

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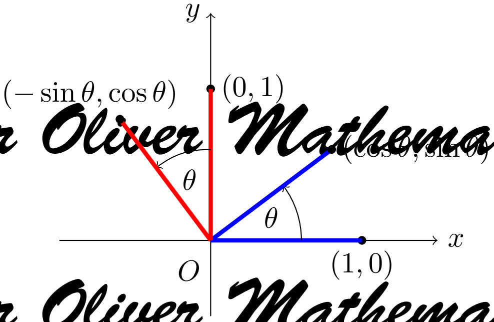
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$





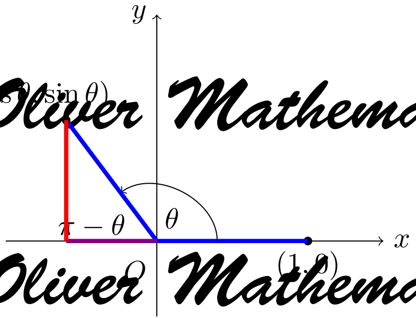

Hence

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

so the required matrix is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

# But what if $\theta$ is obtuse?



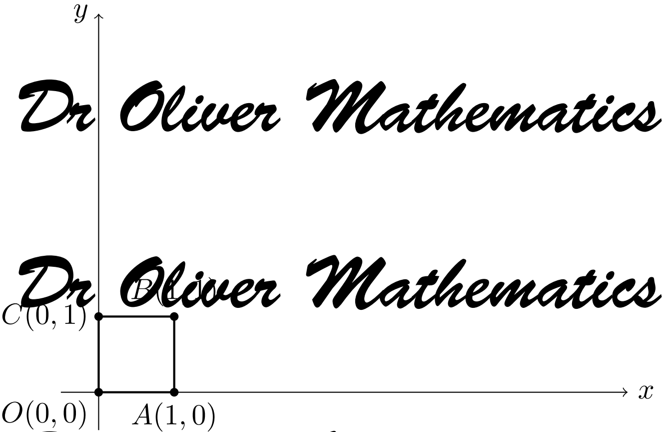



Now

$$\cos(\pi - \theta) = \cos\pi\cos\theta + \sin\pi\sin\theta = \cos\theta$$

and

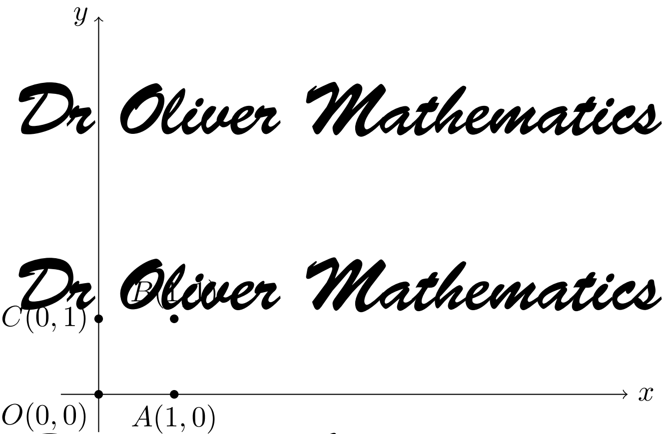
$$\sin(\pi - \theta) = \sin\pi\cos\theta - \sin\theta\cos\pi = \sin\theta$$




$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a & b & a+b \\ 0 & c & d & c+d \end{pmatrix}$$

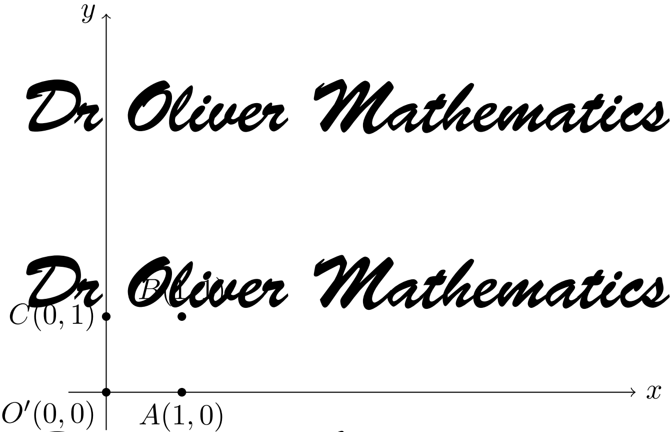
# The Area Scale Factor



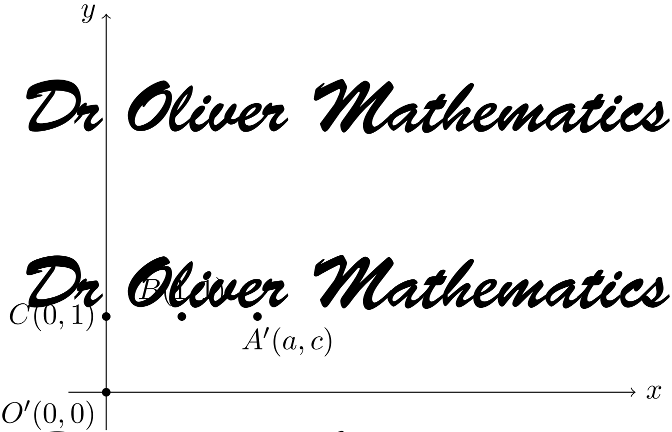





$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a & b & a+b \\ 0 & c & d & c+d \end{pmatrix}$$



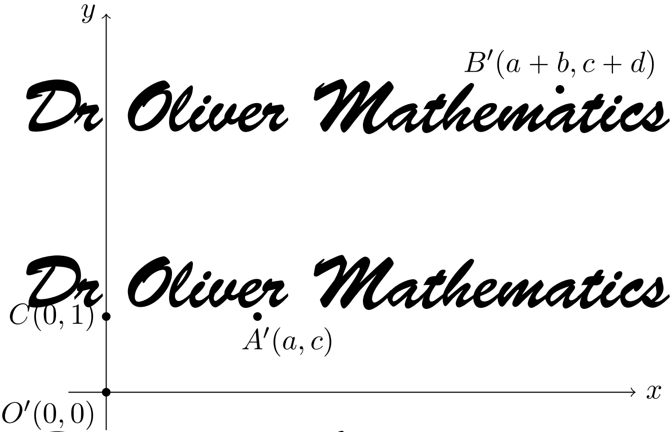

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a & b & a+b \\ 0 & c & d & c+d \end{pmatrix}$$







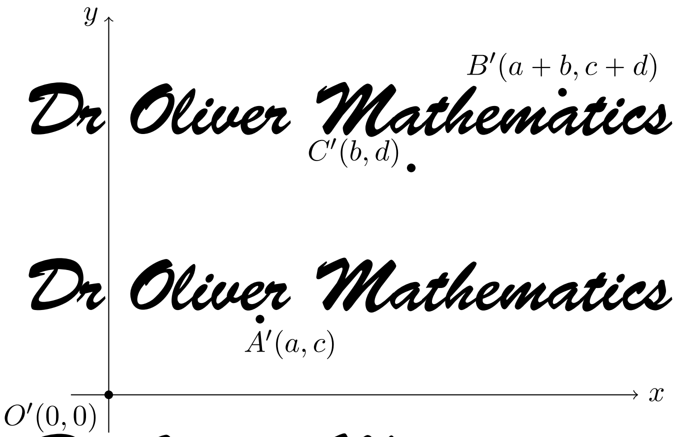

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a & b & a+b \\ 0 & c & d & c+d \end{pmatrix}$$



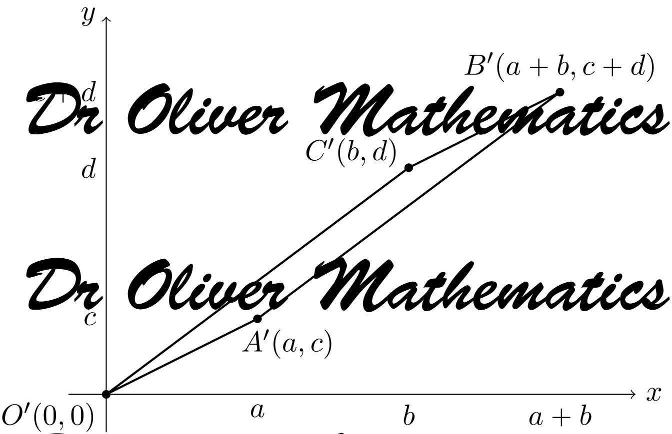



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a & b & a+b \\ 0 & c & d & c+d \end{pmatrix}$$

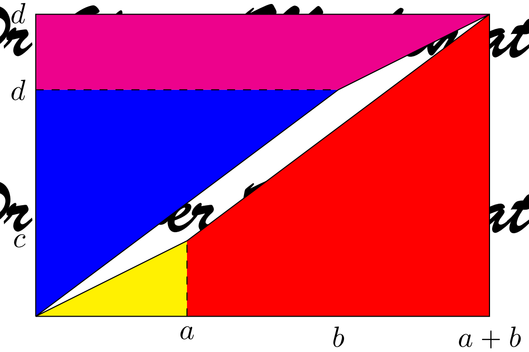
# The Area Scale Factor



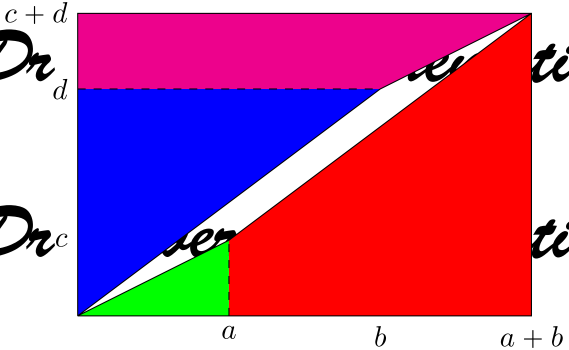

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a & b & a+b \\ 0 & c & d & c+d \end{pmatrix}$$



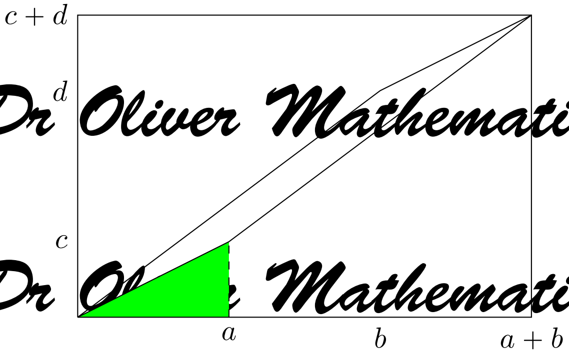

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a & b & a+b \\ 0 & c & d & c+d \end{pmatrix}$$



d
d
c
a
b
a + b

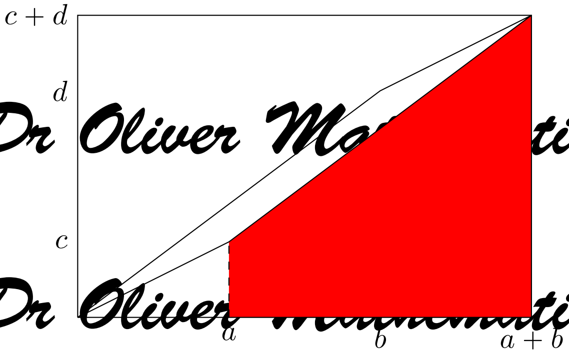



Area of the rectangle $= (a+b)(c+d)$.






$$\begin{aligned}\text{Green area} &= \text{area of a triangle} \\ &= \tfrac{1}{2} \times a \times c \\ &= \tfrac{1}{2}ac\end{aligned}$$

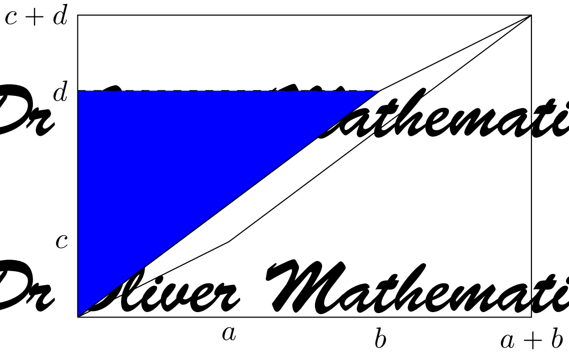






$$\begin{aligned}\text{Red area} &= \text{area of a trapezium}\\ &= \tfrac{1}{2} \times b \times [c + (c + d)]\\ &= \tfrac{1}{2} b\,(2c + d)\end{aligned}$$

# The Area Scale Factor

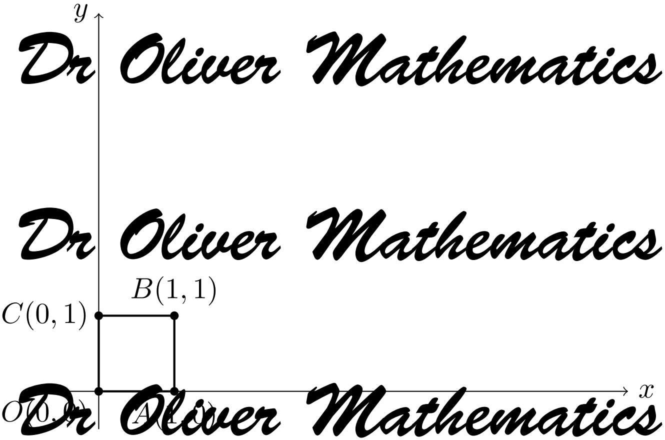



$$\text{Blue area} = \text{area of a triangle}$$
$$= \frac{1}{2} \times b \times d$$
$$= \frac{1}{2}bd$$




$$\begin{aligned}\text{Magenta area} &= \text{area of a trapezium}\\ &= \frac{1}{2} \times c \times [b + (a + b)]\\ &= \frac{1}{2}c\,(a + 2b)\end{aligned}$$

$$
\begin{aligned}
\text{Area of } OA'B'C' &= (a+b)(c+d) - \tfrac{1}{2}ac - \tfrac{1}{2}bd \\
&\quad - \tfrac{1}{2}c\,(a+2b) - \tfrac{1}{2}b\,(2c+d) \\
&= ac + ad + bc + bd - \tfrac{1}{2}ac - \tfrac{1}{2}bd \\
&\quad - \tfrac{1}{2}ac - bc - bc - \tfrac{1}{2}bd \\
&= \cancel{ac} + ad + \cancel{bc} + \cancel{bd} - \cancel{\tfrac{1}{2}ac} - \cancel{\tfrac{1}{2}bd} \\
&\quad - \cancel{\tfrac{1}{2}ac} - \cancel{bc} - bc - \cancel{\tfrac{1}{2}bd} \\
&= ad - bc \\
&= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{aligned}
$$


y
x
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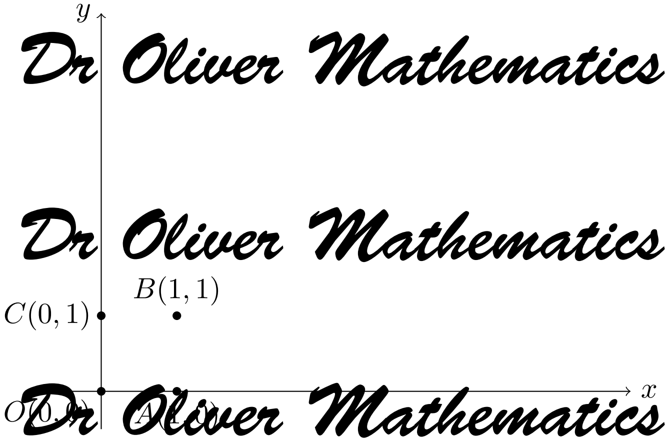
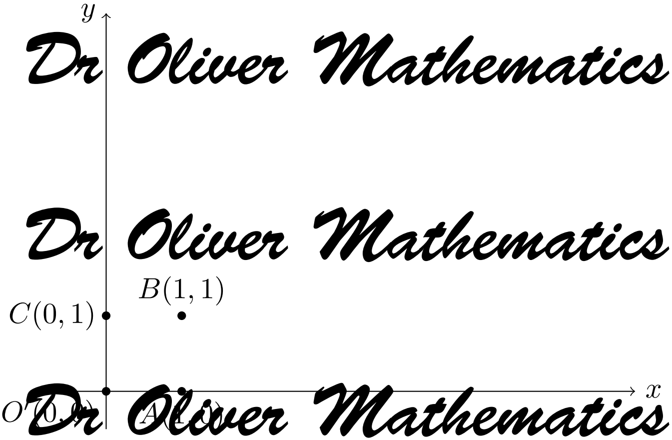
B(1, 1)
C(0, 1)
O(0, 0)
A(1, 0)

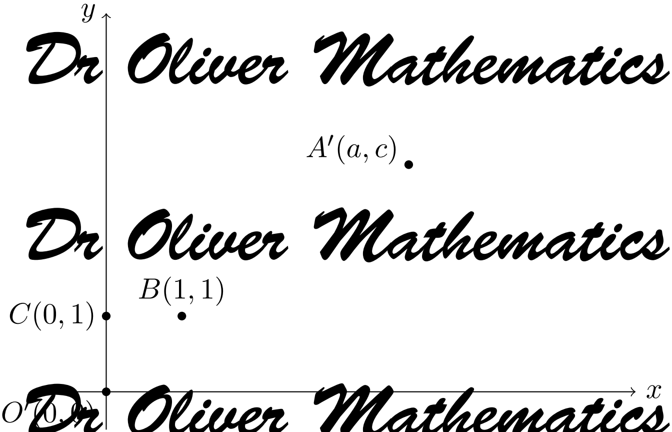
# Another Possibility



# Another Possibility



# Another Possibility





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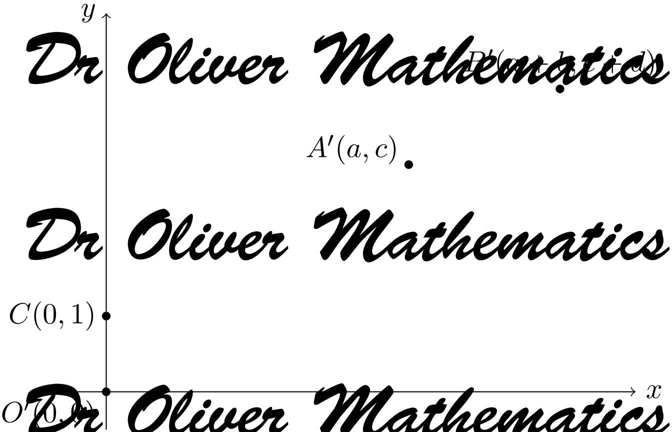
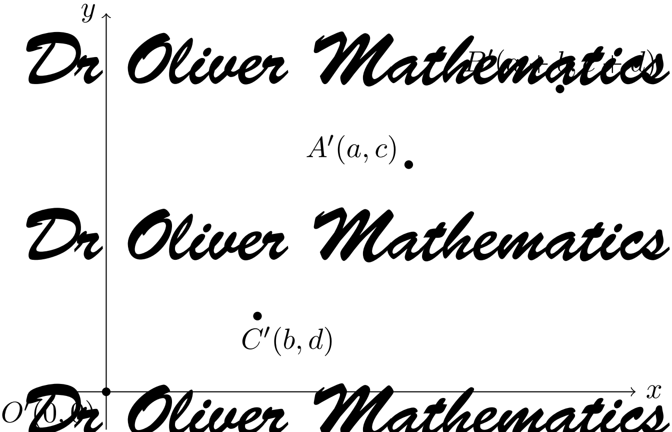
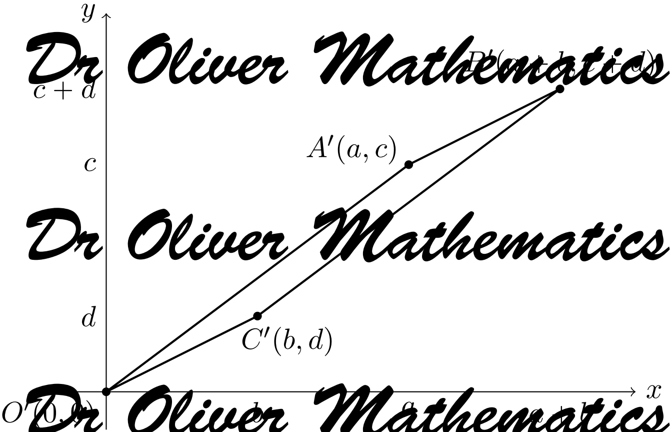
# Another Possibility

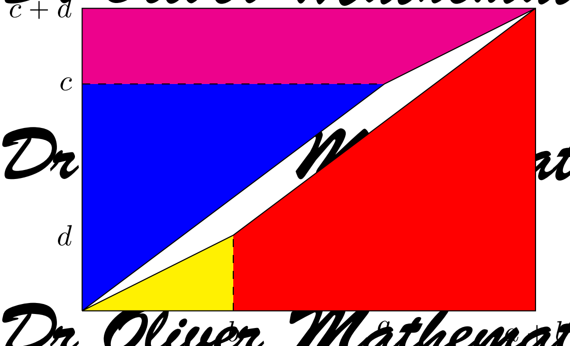


# Another Possibility

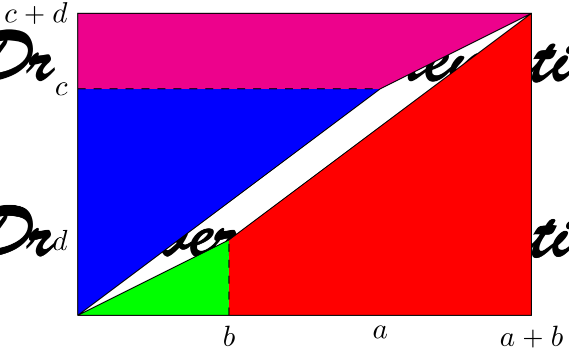



y
B'(a + b, c + d)
c + d
A'(a, c)
c
d
C'(b, d)
x
O(0, 0)
b
a
a + b

# Another Possibility

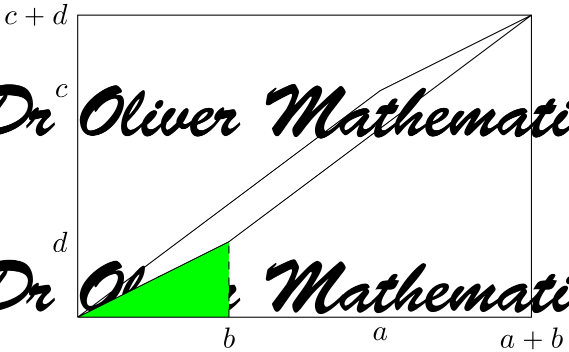


# The Area Scale Factor: the second possibility

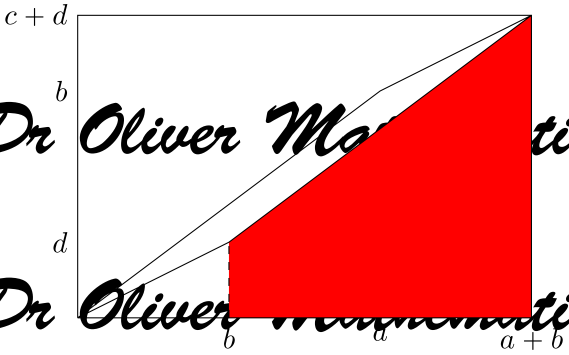



Area of the rectangle $= (a+b)(c+d)$

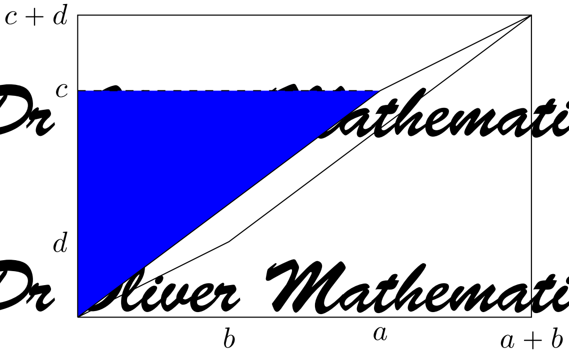
# The Area Scale Factor: the second possibility




$$\begin{aligned} \text{Green area} &= \text{area of a triangle} \\ &= \frac{1}{2} \times b \times d \\ &= \frac{1}{2}bd \end{aligned}$$




$$
\begin{aligned}
\text{Red area} &= \text{area of a trapezium} \\
&= \tfrac{1}{2} \times a \times [d + (c + d)] \\
&= \tfrac{1}{2} a\,(c + 2d)
\end{aligned}
$$




$$\begin{aligned} \text{Blue area} &= \text{area of a triangle} \\ &= \tfrac{1}{2} \times a \times c \\ &= \tfrac{1}{2}ac \end{aligned}$$




$$\begin{aligned}\text{Magenta area} &= \text{area of a trapezium} \\ &= \tfrac{1}{2} \times d \times [a + (a + b)] \\ &= \tfrac{1}{2} d \, (2a + b)\end{aligned}$$

$$\begin{aligned}
\text{Area of } O'A'B'C' &= (a+b)(c+d) - \tfrac{1}{2}bd - \tfrac{1}{2}ac \\
&\quad - \tfrac{1}{2}a(c+2d) - \tfrac{1}{2}d(2a+b) \\
&= ac + ad + bc + bd - \tfrac{1}{2}bd - \tfrac{1}{2}ac \\
&\quad - \tfrac{1}{2}ac - ad - ad - \tfrac{1}{2}bd \\
&= \cancel{ac} + \cancel{ad} + bc + \cancel{bd} - \cancel{\tfrac{1}{2}bd} - \cancel{\tfrac{1}{2}ac} \\
&\quad - \cancel{\tfrac{1}{2}ac} - \cancel{ad} - ad - \cancel{\tfrac{1}{2}bd} \\
&= bc - ad \\
&= -(ad - bc) \\
&= -\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{aligned}$$

# Volume Scale Factor



In the same way that a $2 \times 2$ matrix will transform the unit square into a parallelogram a $3 \times 3$ matrix will transform the unit cube into a parallelepiped. Recall that, if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are the three vectors that determine the sides of the parallelepiped, then the volume of the parallelepiped is given by

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$$|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|$$

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# Volume Scale Factor

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & l \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ d \\ g \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & l \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ e \\ h \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & l \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ f \\ l \end{bmatrix}$$

# Volume Scale Factor

$$\begin{bmatrix} b \\ e \\ h \end{bmatrix} \times \begin{bmatrix} c \\ f \\ l \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b & e & h \\ c & f & l \end{vmatrix}$$

$$= (el - fh)\mathbf{i} - (bl - ch)\mathbf{j} + (bf - ce)\mathbf{k}$$

and hence

$$\begin{bmatrix} a \\ d \\ g \end{bmatrix} \cdot \begin{bmatrix} b \\ e \\ h \end{bmatrix} \times \begin{bmatrix} c \\ f \\ l \end{bmatrix} = a(el - fh) - d(bl - ch) + g(bf - ce)$$

$$= a(el - fh) - b(dl - fg) + c(dh - eg)$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & l \end{vmatrix}$$