Dr Oliver Mathematics

$$A = \frac{1}{6}|a - a'|(\beta - \alpha)^3$$

In this note, we will investigate a neat trick using only two parabolas and the resulting area:

$$\boxed{A = \tfrac{1}{6}|a - a'|(\beta - \alpha)^3.}$$

(This is a companion note to $A = \frac{1}{6}|a|(\beta - \alpha)^3$ so make sure you have read it first.)

1 Introduction

We will do an example.

Example 1
Two parabolas are drawn:

$$y = x^2 + x - 4 \text{ and } y = -x^2 + 2x + 2,$$

as shown in Figure 1.

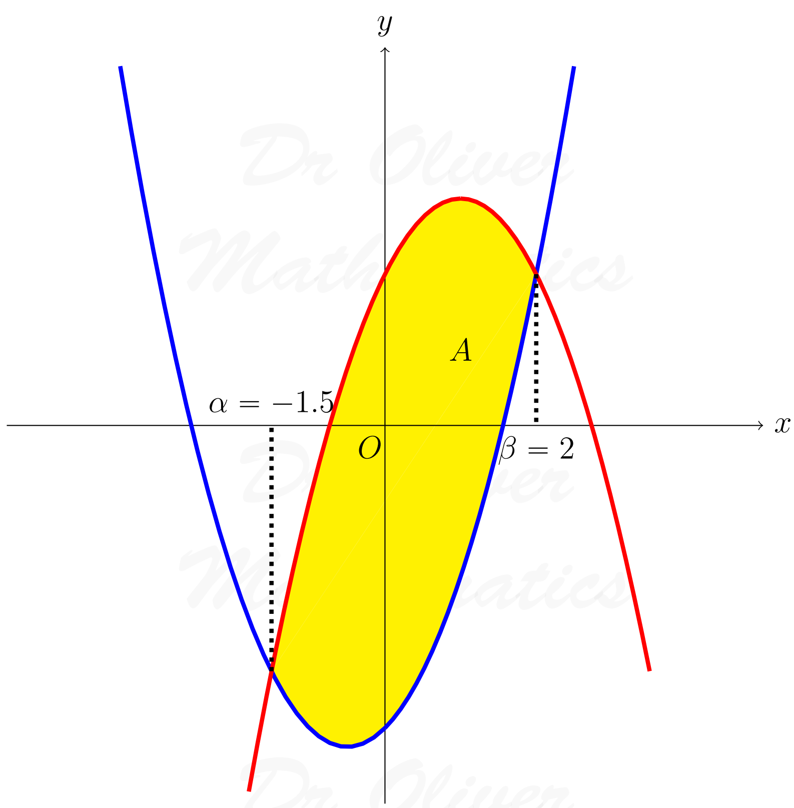

Figure 1: $y = x^2 + x - 4$ and $y = -x^2 + 2x + 2$

They intersect at $x = -1.5$ and $x = 2$.

Use calculus to find exact area of A.

Solution 1a
Now,

$$\begin{aligned}
\text{area under the curve} &= \text{area under the red curve} - \text{area under the blue curve} \\
&= \int_{-1.5}^{2} [(-x^2 + 2x + 2) - (x^2 + x - 4)]\,\mathrm{d}x \\
&= \int_{-1.5}^{2} (-2x^2 + x + 6)\,\mathrm{d}x \\
&= \left[-\tfrac{2}{3}x^3 + \tfrac{1}{2}x^2 + 6x\right]_{x=-1.5}^{2} \\
&= (-\tfrac{16}{3} + 2 + 12) - (\tfrac{9}{4} + \tfrac{9}{8} - 9) \\
&= \underline{\underline{14\tfrac{7}{24}}}.
\end{aligned}$$

So far, so what?

Solution 1b
There is another method.

Compare

$$y = ax^2 + bx + c \text{ with } y = x^2 + x - 4$$

and

$$y = a'x^2 + b'x + c' \text{ with } y = -x^2 + 2x + 2,$$

where the parabolas intersect at α and β, $\alpha < \beta$.

$$\boxed{A = \tfrac{1}{6}|a - a'|(\beta - \alpha)^3.}$$

Well, $a = 1$, $a' = -1$, $\alpha = -1.5$, and $\beta = 2$:

$$\begin{aligned}
A &= \tfrac{1}{6}|1 - (-1)|[2 - (-1.5)]^3 \\
&= \tfrac{1}{6}(2)(3.5^3) \\
&= \tfrac{1}{3}(42\tfrac{7}{8}) \\
&= \underline{\underline{14\tfrac{7}{24}}}.
\end{aligned}$$

2 The Theory

Consider the parabola

$$y = ax^2 + bx + c$$

and

$$y = a'x^2 + b'x + c'$$

which crosses the x-axis at $x = \alpha$ and $x = \beta$, where $\alpha < \beta$, and which has a shaded area of A.

Now,

$$\begin{aligned} ax^2 + bx + c = a'x^2 + b'x + c' &\Rightarrow ax^2 + bx + c - a'x^2 - b'x - c' = 0 \\ &\Rightarrow (a - a')x^2 + (b - b')x + (c - c') = 0 \\ &\Rightarrow (a - a')(x - \alpha)(x - \beta) = 0 \end{aligned}$$

and

$$\begin{aligned} A &= \int_{\alpha}^{\beta} |(a - a')(x - \alpha)(x - \beta)| \, \mathrm{d}x \\ &= |(a - a')| \int_{\alpha}^{\beta} |(x - \alpha)(x - \beta)| \, \mathrm{d}x \\ &= |(a - a')| \times \tfrac{1}{6}(\beta - \alpha)^3 \\ &= |\tfrac{1}{6}|a - a'|(\beta - \alpha)^3, \end{aligned}$$

as required.

Hence,

$$\boxed{A = \tfrac{1}{6}|a - a'|(\beta - \alpha)^3.}$$

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