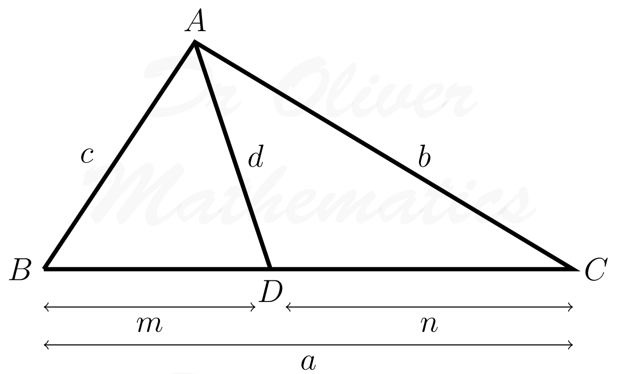
Dr Oliver Mathematics
Stewart's Theorem

In this note, we will prove Stewart's Theorem.



- ABC is a triangle.
- $BC = a$, $AC = b$, and $AB = c$.
- We fix a point, D, somewhere on BC where $BD = m$ and $DC = n$.

Then

$$\boxed{mb^2 + nc^2 = a(d^2 + mn).}$$

So, how do we prove it? Well, we first do a bit of trigonometry:

$$\begin{aligned}\cos(180-\theta)^\circ &= \cos 180^\circ \cos\theta^\circ + \sin 180^\circ \sin\theta^\circ \\ &= (-1)\cos\theta^\circ + (0)\sin\theta^\circ \\ &= -\cos\theta^\circ\end{aligned}$$

which means

$$\cos\theta^\circ = -\cos(180-\theta)^\circ \quad (1).$$

Next, we now employ the cosine rule for $\triangle ABD$:

$$c^2 = m^2 + d^2 - 2md\cos ADB \Rightarrow nc^2 = m^2n + d^2n - 2mnd\cos ADB \quad (2)$$

and for $\triangle ACD$:

$$b^2 = n^2 + d^2 - 2dn\cos ADC \Rightarrow mb^2 = mn^2 + d^2m - 2dmn\cos ADC \quad (3).$$

But hold on:

$$\cos ADC = -\cos ADB$$

and

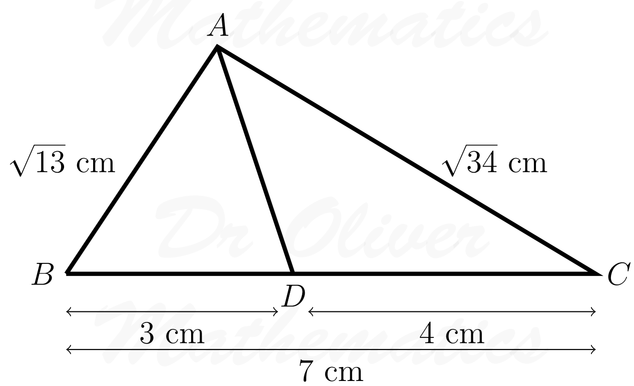
$$mb^2 = mn^2 + d^2m + 2dmn \cos ADB \quad (4).$$

We add (2)+(4):

$$\begin{aligned}
mb^2 + nc^2 &= (mn^2 + d^2m - 2dmn \cos ADC) + (m^2n + d^2n - 2mnd \cos ADB) \\
&= (mn^2 + d^2m + 2dmn \cos ADB) + (m^2n + d^2n - 2mnd \cos ADB) \\
&= (mn^2 + d^2m) + (m^2n + d^2n) \\
&= mn^2 + m^2n + d^2m + d^2n \\
&= mn(n + m) + d^2(m + n) \\
&= (m + n)(mn + d^2) \\
&= a(d^2 + mn),
\end{aligned}$$

as required.

Example 1
$\triangle ABC$ is a triangle.



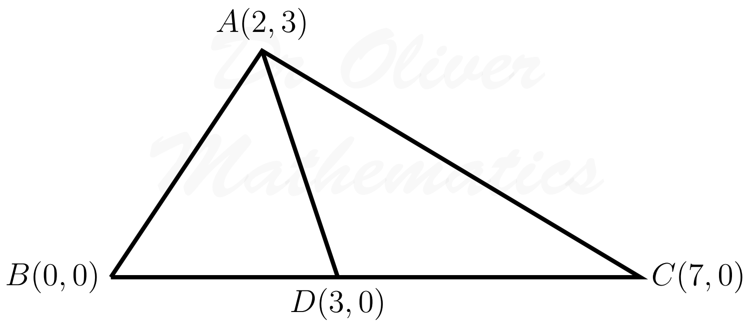
- $AB = \sqrt{13}$ cm.
- $AC = \sqrt{34}$ cm.
- $BD = 3$ cm.
- $DC = 4$ cm.
- $BD = 7$ cm.
- The diagram is not drawn accurately.

Find AD.

Solution 1

$$\begin{aligned} mb^2 + nc^2 = a(d^2 + mn) &\Rightarrow 3 \times (\sqrt{34})^2 + 4 \times (\sqrt{13})^2 = 7(d^2 + 3 \times 4) \\ &\Rightarrow 3 \times 34 + 4 \times 13 = 7(d^2 + 12) \\ &\Rightarrow 102 + 52 = 7(d^2 + 12) \\ &\Rightarrow 154 = 7(d^2 + 12) \\ &\Rightarrow 22 = d^2 + 12 \\ &\Rightarrow d^2 = 10 \\ &\Rightarrow \underline{\underline{d = \sqrt{10} \text{ cm}}}. \end{aligned}$$

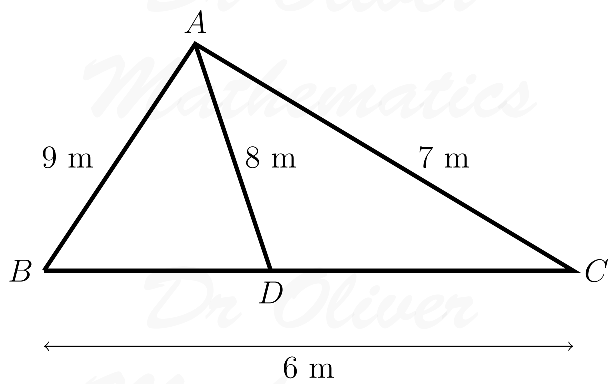
And that is exactly the solution is got when I prepared this example:



and

$$\begin{aligned} AD &= \sqrt{(3-2)^2 + (0-3)^2} \\ &= \sqrt{1^2 + (-3)^2} \\ &= \sqrt{1+9} \\ &= \sqrt{10}. \end{aligned}$$

Example 2
$\triangle ABC$ is a triangle.



- $AB = 9$ m.
- $AC = 7$ m.
- $BC = 6$ m.
- The diagram is not drawn accurately.

Find BD, to 3 significant figures.

Solution 2
Let $BD = x$ and then $DC = (6 - x)$. Now,

$$\begin{aligned} & mb^2 + nc^2 = a(d^2 + mn) \\ \Rightarrow\quad & (6-x) \times 9^2 + x \times 7^2 = 6\left[8^2 + x(6-x)\right] \\ \Rightarrow\quad & 81(6-x) + 49x = 6\left[64 + 6x - x^2\right] \\ \Rightarrow\quad & 486 - 81x + 49x = 384 + 36x - 6x^2 \\ \Rightarrow\quad & 6x^2 - 68x + 102 = 0 \end{aligned}$$

$a = 6$, $b = -68$, and $c = 102$:

$$\begin{aligned} \Rightarrow\quad & x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \Rightarrow\quad & x = \frac{-(-68) \pm \sqrt{(-68)^2 - 4 \times 6 \times 102}}{2 \times 6} \\ \Rightarrow\quad & x = \frac{68 \pm \sqrt{2\,176}}{12} \\ \Rightarrow\quad & x = 1.779\,365\,403,\ 9.553\,967\,93 \text{ (FCD)}. \end{aligned}$$

But, $x \neq 9.553\ldots$ (why?) so

$$\underline{\underline{x = 1.80 \text{ m (3 sf)}}}.$$

Returning to our statement, $mb^2 + nc^2 = a(d^2 + mn)$, we can re-write as follows:

$$\begin{aligned} mb^2 + nc^2 = a(d^2 + mn) &\Rightarrow mb^2 + nc^2 = ad^2 + amn \\ &\Rightarrow bmb + cnc = dad + man \\ &\Rightarrow man + dad = bmb + cnc, \end{aligned}$$

which gives us the mnemonic "a man and his dad put a bomb in the sink"!