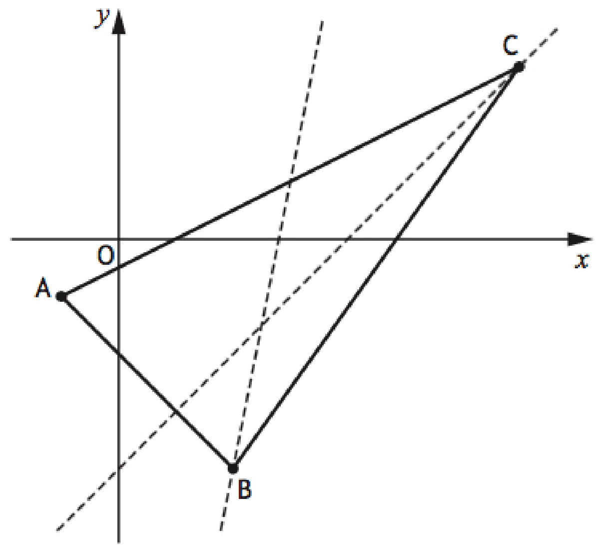
Dr Oliver Mathematics
Mathematics: Higher
2022 Paper 2: Calculator
1 hour 30 minutes

The total number of marks available is 65.
You must write down all the stages in your working.

1. Triangle ABC has vertices $A(-1, -1)$, $B(2, -4)$, and $C(7, 3)$.



(a) Find the equation of the altitude through C. (3)

Solution

$$\text{Gradient of } AB = \frac{-4 - (-1)}{2 - (-1)} = \frac{-3}{3} = -1$$

which means the perpendicular has gradient

$$-\frac{1}{-1} = 1.$$

Now, the equation of the altitude through C is $y = x + c$, for some constant c. Using $C(7, 3)$,

$$3 = 7 + c \Rightarrow c = -4.$$

Hence, the equation of the altitude through C is

$$\underline{\underline{y = x - 4}}. \quad (1)$$

(b) Find the equation of the median through B. (3)

Solution
The midpoint of AC is

$$\left(\frac{-1+7}{2}, \frac{-1+3}{2}\right) = D(3, 1).$$

Now,

$$\begin{aligned}\text{gradient of } BD &= \frac{-4-1}{2-3} \\ &= \frac{-5}{-1} \\ &= 5\end{aligned}$$

and the equation of the median through B is $y = 5x + d$, for some constant d. Using $B(2, -4)$,

$$-4 = 5(2) + d \Rightarrow d = -14.$$

Hence, the equation of the median through B is

$$\underline{\underline{y = 5x - 14}}. \quad (2)$$

(c) Determine the coordinates of the point of intersection of the altitude through C and the median through B. (2)

Solution
(1) = (2):

$$\begin{aligned}x - 4 = 5x - 14 &\Rightarrow 4x = 10 \\ &\Rightarrow x = 2\tfrac{1}{2} \\ &\Rightarrow y = -1\tfrac{1}{2},\end{aligned}$$

hence, the coordinates are $\underline{\underline{(2\tfrac{1}{2}, -1\tfrac{1}{2})}}$.

2. The equation (3)

$$2x^2 - 8x + (4 - p) = 0$$

has two real and distinct roots.

Determine the range of values for p.

Solution

$$\begin{aligned} b^2 - 4ac > 0 &\Rightarrow (-8)^2 - 4(2)(4-p) > 0 \\ &\Rightarrow 64 - 8(4-p) > 0 \\ &\Rightarrow 8(4-p) < 64 \\ &\Rightarrow 4 - p < 8 \\ &\Rightarrow -p < 4 \\ &\Rightarrow \underline{\underline{p > -4}}. \end{aligned}$$

3. (a) Express (4)

$$4\sin x + 5\cos x$$

in the form

$$k\sin(x + a),$$

where $k > 0$ and $0 < a < 2\pi$.

Solution

$$k\sin(x+a) \equiv k\sin x\cos a + R\cos x + \sin a$$

so

$$k\cos a = 4 \text{ and } R\sin a = 5.$$

Now,

$$\begin{aligned} k &= \sqrt{k^2} \\ &= \sqrt{k^2(\sin^2 a + \cos^2 a)} \\ &= \sqrt{k^2\sin^2 a + k^2\cos^2 a} \\ &= \sqrt{(k\sin a)^2 + (k\cos a)^2} \\ &= \sqrt{5^2 + 4^2} \\ &= \sqrt{41} \end{aligned}$$

and

$$\begin{aligned}\tan a &= \frac{k \sin a}{k \cos a} \\ &= \tfrac{5}{4} \\ a &= 0.896\,055\,384\,6 \text{ (FCD)}.\end{aligned}$$

Hence,

$$4 \sin x + 5 \cos x = \underline{\underline{\sqrt{41} \sin(x + 0.896\ldots)}}.$$

(b) Hence solve (3)

$$4 \sin x + 5 \cos x = 5.5$$

for $0 \leqslant x < 2\pi$.

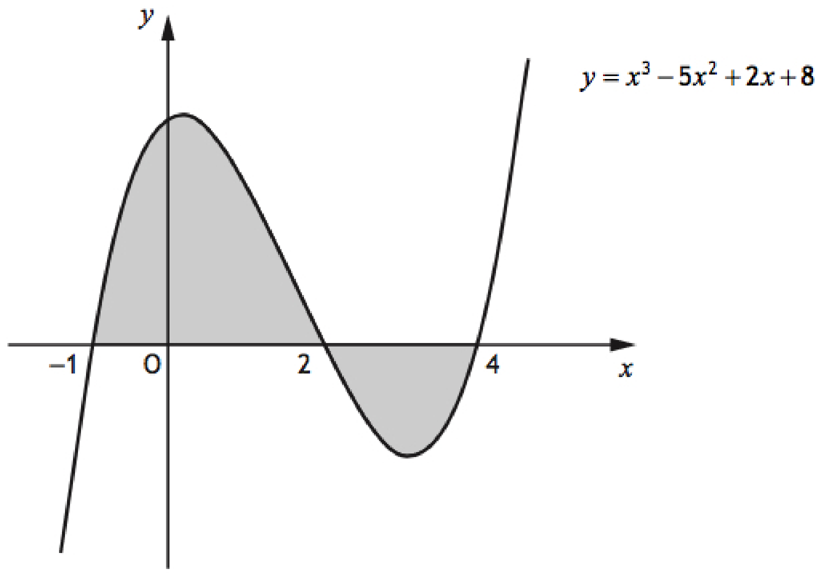
Solution

$$\begin{aligned}4 \sin x + 5 \cos x = 5.5 &\Rightarrow \sqrt{41} \sin(x + 0.896\ldots) = 5.5 \\ &\Rightarrow \sin(x + 0.896\ldots) = \frac{5.5}{\sqrt{41}} \\ &\Rightarrow x + 0.896\ldots = 1.033\,226\,702,\ 2.108\,365952 \text{ (FCD)} \\ &\Rightarrow x = 0.137\,171\,317\,4,\ 1.212\,310\,567 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 0.137,\ 1.21 \text{ (3 sf)}}}.\end{aligned}$$

4. The graph shown has equation

$$y = x^3 - 5x^2 + 2x + 8.$$

The total shaded area is bounded by the curve and the x-axis.



(a) Calculate the shaded area above the x-axis. (4)

Solution

$$\begin{aligned}\text{Area} &= \int_{-1}^{2} (x^3 - 5x^2 + 2x + 8)\,\mathrm{d}x \\ &= \left[\tfrac{1}{4}x^4 - \tfrac{5}{3}x^3 + x^2 + 8x\right]_{x=-1}^{2} \\ &= \left(4 - \tfrac{40}{3} + 4 + 16\right) - \left(\tfrac{1}{4} + \tfrac{5}{3} + 1 - 8\right) \\ &= \underline{\underline{15\tfrac{3}{4}}}.\end{aligned}$$

(b) Hence calculate the total shaded area. (3)

Solution

$$\begin{aligned}\int_{2}^{4} (x^3 - 5x^2 + 2x + 8)\,\mathrm{d}x &= \left[\tfrac{1}{4}x^4 - \tfrac{5}{3}x^3 + x^2 + 8x\right]_{x=2}^{4} \\ &= \left(64 - \tfrac{320}{3} + 16 + 32\right) - \left(4 - \tfrac{40}{3} + 4 + 16\right) \\ &= -5\tfrac{1}{3}.\end{aligned}$$

Now, as the area lies below the x-axis for $2 \leqslant x \leqslant 4$,

$$\text{total shaded area} = 15\tfrac{3}{4} + 5\tfrac{1}{3} = \underline{\underline{21\tfrac{1}{12}}}.$$

5. Functions f and g are given by

$$\mathrm{f}(x) = x^2 - 2 \text{ and } \mathrm{g}(x) = 3x + 5,\ x \in \mathbb{R}.$$

(a) Find expressions for

(i) $\mathrm{f}(\mathrm{g}(x))$, (2)

Solution

$$\begin{aligned}\mathrm{f}(\mathrm{g}(x)) &= \mathrm{f}(3x+5) \\ &= (3x+5)^2 - 2\end{aligned}$$

$\times$	$3x$	$+5$
$3x$	$9x^2$	$+15x$
$+5$	$+15x$	$+25$

$$\begin{aligned}&= (9x^2 + 15x + 25) - 2 \\ &= \underline{\underline{9x^2 + 30x + 23}}.\end{aligned}$$

(ii) $\mathrm{g}(\mathrm{f}(x))$. (1)

Solution

$$\begin{aligned}\mathrm{g}(\mathrm{f}(x)) &= \mathrm{g}(x^2 - 2) \\ &= 3(x^2 - 2) + 5 \\ &= (3x^2 - 6) + 5 \\ &= \underline{\underline{3x^2 - 1}}.\end{aligned}$$

(b) Determine the range of values of x for which (4)

$$\mathrm{f}(\mathrm{g}(x)) < \mathrm{g}(\mathrm{f}(x)).$$

Solution

$$\begin{aligned} f(g(x)) < g(f(x)) &\Rightarrow 9x^2 + 30x + 23 < 3x^2 - 1 \\ &\Rightarrow 6x^2 + 30x + 24 < 0 \\ &\Rightarrow 6(x^2 + 5x + 4) < 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +5 \\ \text{multiply to:} & +4 \end{array}\right\} +1,\ +4$$

$$\Rightarrow 6(x+1)(x+4) < 0$$

	$x < -4$	$x = -4$	$-4 < x < -1$	$x = -1$	$x > -1$
$x + 4$	$-$	0	$+$	$+$	$+$
$x + 1$	$-$	$-$	$-$	0	$+$
$(x+4)(x+1)$	$+$	0	$-$	0	$+$

$$\Rightarrow \underline{\underline{-4 < x < -1}}.$$

6. A curve with equation $y = f(x$ is such that (5)

$$\frac{dy}{dx} = 1 - \frac{3}{x^2},$$

where $x > 0$.

The curve passes through the point $(3, 6)$.

Express y in terms of x.

Solution

$$\begin{aligned} \frac{dy}{dx} = 1 - \frac{3}{x^2} &\Rightarrow \frac{dy}{dx} = 1 - 3x^{-2} \\ &\Rightarrow y = x + 3x^{-1} + c, \end{aligned}$$

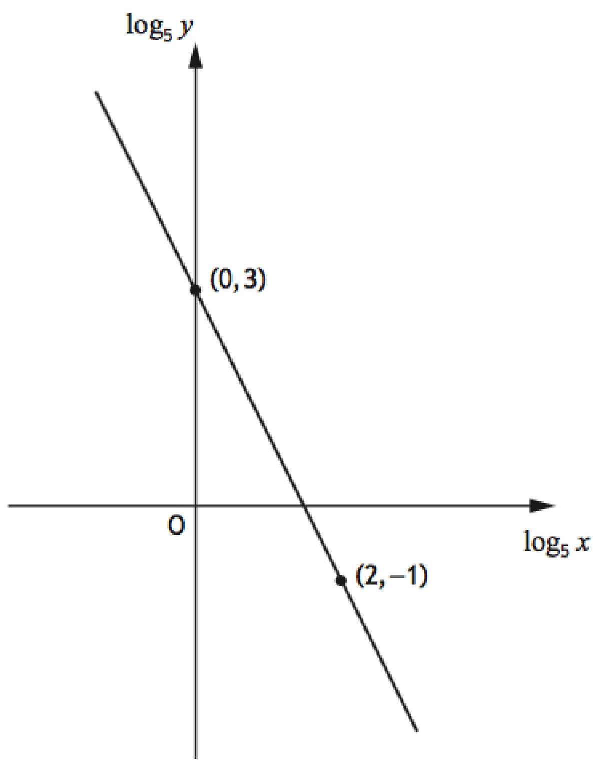
for some constant c. Now,

$$\begin{aligned} x = 3, y = 6 &\Rightarrow 6 = 3 + 3(3^{-1}) + c \\ &\Rightarrow 6 = 3 + 1 + c \\ &\Rightarrow c = 2, \end{aligned}$$

and hence

$$\underline{\underline{y = x + 3x^{-1} + 2}}.$$

7. Two variables, x and y, are connected by the equation $y = kx^n$. (5)
The graph of $\log_5 y$ against $\log_5 x$ is a straight line as shown.



Find the values of k and n.

Solution

$$\text{Gradient} = \frac{3-(-1)}{0-2} = \frac{4}{-2} = -2,$$

so

$$\log_5 y = -2\log_5 x + c,$$

for some constant c. Now,

$$\log_5 x = 0, \log_5 y = 3 \Rightarrow 3 = -2(0) + c \Rightarrow c = 3,$$

so

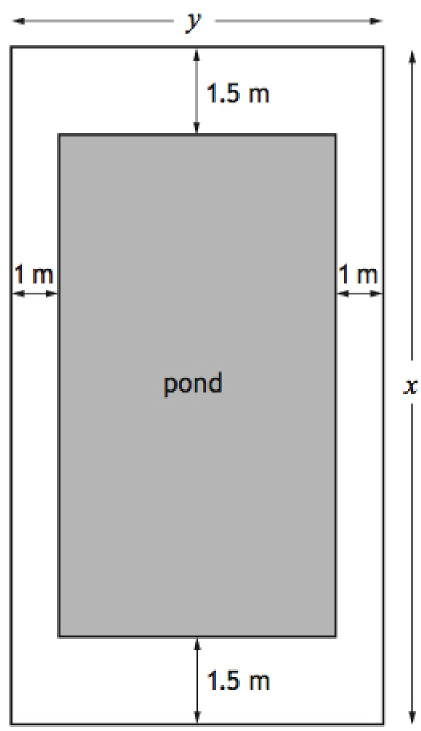
$$\begin{aligned}\log_5 y = -2\log_5 x + 3 &\Rightarrow \log_5 y + 2\log_5 x = 3\\ &\Rightarrow \log_5 y + \log_5 x^2 = 3\\ &\Rightarrow \log_5(yx^2) = 3\\ &\Rightarrow yx^2 = 5^3\\ &\Rightarrow yx^2 = 125\\ &\Rightarrow \underline{\underline{y = 125x^{-2}}};\end{aligned}$$

hence, $\underline{\underline{k = 125}}$ and $\underline{\underline{n = -2}}$.

8. A rectangular plot consists of a rectangular pond surrounded by a path.

   The length and breadth of the plot are x metres and y metres respectively.

   The path is 1.5 metres wide at the ends of the pond and 1 metre wide along the other sides as shown.



The total area of the **pond and path together** is 150 square metres.

(a) Show that the area of the pond, A square metres, is given by (3)

$$A(x) = 156 - 2x - \frac{450}{x}.$$

Solution

Now,

$$xy = 150 \Rightarrow y = \frac{150}{x}$$

and

$$\begin{aligned} A &= (x-3)(y-2) \\ &= (x-3)\left(\frac{150}{x} - 2\right) \\ &= 150 - 2x - \frac{450}{x} + 6 \\ &= \underline{\underline{156 - 2x - \frac{450}{x}}}, \end{aligned}$$

as required.

(b) Determine the maximum area of the pond. (6)

Solution

$$A(x) = 156 - 2x - \frac{450}{x} \Rightarrow A(x) = 156 - 2x - 450x^{-1}$$
$$\Rightarrow A'(x) = -2 + 450x^{-2}$$
$$\Rightarrow A''(x) = -900x^{-3}$$

and

$$A'(x) = 0 \Rightarrow -2 + 450x^{-2} = 0$$
$$\Rightarrow 450x^{-2} = 2$$
$$\Rightarrow 450 = 2x^2$$
$$\Rightarrow 225 = x^2$$
$$\Rightarrow x = 15$$
$$\Rightarrow A = 156 - 2(15) + \frac{450}{15}$$
$$\Rightarrow \underline{\underline{A = 96 \text{ m}^2}}.$$

Is this a maximum? Well,

$$A''(15) = -\frac{900}{15^3} = -\tfrac{4}{15} < 0,$$

and, hence, this a <u>maximum</u>.

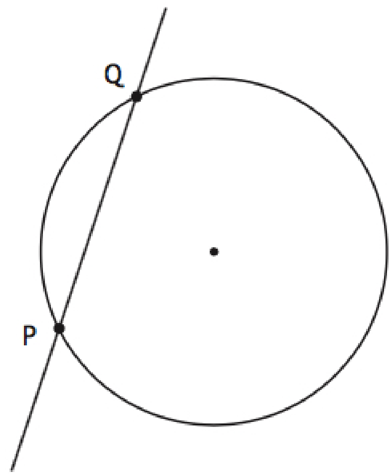
9. The line

$$y = 3x + 7$$

intersects the circle

$$x^2 + y^2 - 4x - 6y - 7 = 0$$

at the points P and Q.



(a) Find the coordinates of P and Q. (5)

Solution

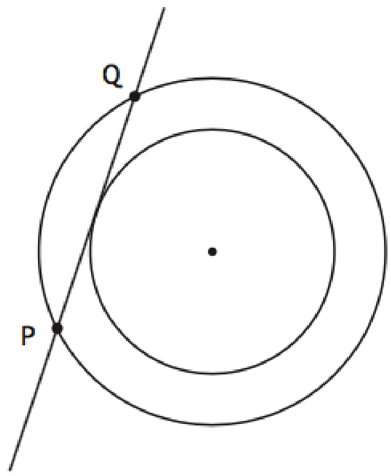
$$
\begin{aligned}
& x^2 + y^2 - 4x - 6y - 7 = 0 \\
\Rightarrow\ & x^2 + (3x+7)^2 - 4x - 6(3x+7) - 7 = 0
\end{aligned}
$$

$\times$	$3x$	$+7$
$3x$	$9x^2$	$+21x$
$+7$	$+21x$	$+42$

$$
\begin{aligned}
\Rightarrow\ & x^2 + (9x^2 + 42x + 49) - 4x - 18x - 42 - 7 = 0 \\
\Rightarrow\ & 10x^2 + 20x = 0 \\
\Rightarrow\ & 10x(x+2) = 0 \\
\Rightarrow\ & x = -2 \text{ or } x = 0 \\
\Rightarrow\ & y = 1 \text{ or } y = 7;
\end{aligned}
$$

the coordinates are $\underline{\underline{P(-2, 1)}}$ and $\underline{\underline{Q(0, 7)}}$.

PQ is a tangent to a second, smaller circle.
This circle is concentric with the first.



(b) Determine the equation of the smaller circle. (4)

Solution

$$\begin{aligned} x^2 + y^2 - 4x - 6y - 7 = 0 &\Rightarrow x^2 - 4x + y^2 - 6y = 7 \\ &\Rightarrow x^2 - 4x + 4 + y^2 - 6y + 9 = 7 + 4 + 9 \\ &\Rightarrow (x-2)^2 + (y-3)^2 = 20. \end{aligned}$$

The midpoint of PQ lies on the circle (why?) and

$$\left(\frac{-2+0}{2}, \frac{1+7}{2}\right) = (-1, 4).$$

Finally, the equation of the smaller circle is

$$\begin{aligned} x = -\tfrac{1}{2}, y = 3\tfrac{1}{2} &\Rightarrow (x-2)^2 + (y-3)^2 = (-1-2)^2 + (4-3)^2 \\ &\Rightarrow (x-2)^2 + (y-3)^2 = (-3)^2 + (1)^2 \\ &\Rightarrow \underline{\underline{(x-2)^2 + (y-3)^2 = 10}}. \end{aligned}$$

10. The heptathlon is an athletics contest made up of seven events.
Athletes score points for each event.
In the 200 metres event, the points are calculated using the formula

$$P = 4.990\,87(42.5 - T)^{1.81},$$

where P is the number of points awarded, and T is the athlete's time, in seconds.

(a) Calculate how many points would be awarded for a time of 24.55 seconds in the 200 metres event. (1)

Solution

$$\begin{aligned} T = 24.55 &\Rightarrow P = 4.990\,87(42.5 - 24.55)^{1.81} \\ &\Rightarrow P = 929.036\,800\,7 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{P = 929 \text{ points (nearest whole number)}}}. \end{aligned}$$

In the long jump event, the points are calculated using the formula

$$P = 0.188\,807(D - 210)^k,$$

where P is the number of points awarded, D is the distance jumped, in centimetres, and k is a constant.

(b) Given that 850 points are awarded for a jump of 600 cm, calculate the value of k. (4)

Solution

$$\begin{aligned} P = 850, D = 600 &\Rightarrow 850 = 0.188\,807(600 - 210)^k \\ &\Rightarrow 390^k = 4\,501.951\,728 \\ &\Rightarrow \log_{10} 390^k = \log_{10} 4\,501.951\,728 \\ &\Rightarrow k \log_{10} 390 = \log_{10} 4\,501.951\,728 \\ &\Rightarrow k = \frac{\log_{10} 4\,501.951\,728}{\log_{10} 390} \\ &\Rightarrow k = 1.409\,999\,899 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{k = 1.41 \text{ (3 sf)}}}. \end{aligned}$$